A priori and a posteriori error analyses in the study of viscoelastic problems [1]

J.R. Fernández [a,*] P. Hild [b]

[a] *Departamento de Matemática Aplicada, Universidade de Santiago de Compostela Facultade de Matemáticas, Campus Universitario Sur s/n 15782 Santiago de Compostela, Spain*

[b] *Laboratoire de Mathématiques de Besançon, Université de Franche-Comté UMR CNRS 6623, 16 route de Gray, 25030 Besançon, France Phone: +33381666349. Fax: +3381666623. E-mail address: patrick.hild@univ-fcomte.fr*

Abstract

In this work, the numerical approximation of a viscoelastic problem is studied. A fully discrete scheme is introduced by using the finite element method to approximate the spatial variable and an Euler scheme to discretize time derivatives. Then, two numerical analyses are presented. First, a priori estimates are proved from which the linear convergence of the algorithm is derived under suitable regularity conditions. Secondly, an a posteriori error analysis is provided extending some preliminary results obtained in the study of the heat equation. Upper and lower error bounds are obtained.

Key words: viscoelasticity, fully discrete approximations, a posteriori error estimates, a priori error estimates, finite elements
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1 Introduction

Viscoelastic materials have been studied in the last thirty years from both mathematical and engineering point of views. These are so interesting because

* Corresponding author: Phone: +34981563100 (Ext. 23244). Fax: +34982285926. E-mail address: jramon@usc.es

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many metals or crystals can be modelled using viscoelasticity theory. One of the most famous is the well-known Kelvin-Voigt viscoelastic constitutive law.

Since the first results provided by [9], many works dealing with mathematical problems including viscoelastic materials have been published (see, for instance, [4,8,10,11,13–15]). Moreover, recently this kind of materials have been considered in contact problems (see [12] and the references cited therein for the quasistatic case or, for example, [5] for the dynamical one).

In this paper, we will provide both a priori and a posteriori error analyses for the study of a viscoelastic problem. First, the a priori analysis is performed using some ideas already employed in [1] for the case including the contact with a deformable or rigid obstacle. As far as we know, the a priori error estimates result, Theorem 4.1, was not published yet. Secondly, an a posteriori error analysis is provided extending some arguments already applied in the study of the heat equation (see, e.g., [16,17,19]), some parabolic equations ([2]) or the Stokes equation ([3]).

The paper is structured as follows. In Section 2, the mechanical model and its variational formulation are described following the notation and assumptions introduced in [12]. Then, a fully discrete scheme is introduced in Section 3, by using the finite element method to approximate the spatial variable and an Euler scheme to discretize the time derivatives. In Section 4, an a priori error analysis is performed employing some arguments developed in the study of viscoelastic contact problems. Finally, extending some results obtained in the study of the heat equation, an a posteriori error analysis is done in Section 5, providing an upper bound for the error, Theorem 5.1, and a lower bound, Theorem 5.2.

2 Mechanical problem and its variational formulation

In this section, we present a brief description of the model (details can be found in [12]).

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, denote a domain occupied by a viscoelastic body with a smooth boundary $\Gamma = \partial\Omega$ decomposed into two disjoint parts Γ_D and Γ_F such that meas $(\Gamma_D) > 0$. Moreover, let $[0, T]$, $T > 0$, be the time interval of interest and denote by $\boldsymbol{\nu}$ the unit outer normal vector to Γ (see Fig. 1).

Let $\boldsymbol{x} \in \Omega$ and $t \in [0, T]$ be the spatial and time variables, respectively, and, in order to simplify the writing, we do not indicate the dependence of the functions on $\boldsymbol{x}$ and t. Moreover, a dot above a variable represents the derivative with respect to the time variable.

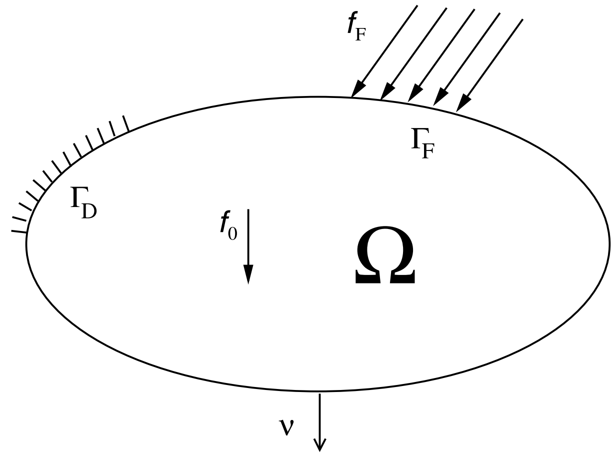

Fig. 1. Physical setting: a viscoelastic body.

Let $\boldsymbol{u}$ denote the displacement field, $\boldsymbol{\sigma}$ the stress tensor and $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))_{i,j=1}^{d}$ the linearized strain tensor given by

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

The body is assumed viscoelastic and satisfying the following constitutive law (see, for instance, [9]),

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}), \tag{1}$$

where $\mathcal{A} = (a_{ijkl})$ and $\mathcal{B} = (b_{ijkl})$ are, respectively, the fourth-order viscous and elastic tensors, and we denote $\mathcal{A}\boldsymbol{\varepsilon} = a_{ijkl}\varepsilon_{kl}$ and $\mathcal{B}\boldsymbol{\varepsilon} = b_{ijkl}\varepsilon_{kl}$.

We turn now to describe the boundary conditions.

On the boundary part Γ_D we assume that the body is clamped and thus the displacement field vanishes there (and so $\boldsymbol{u} = \boldsymbol{0}$ on $\Gamma_D \times (0, T)$). Moreover, we assume that a density of traction forces, denoted by $\boldsymbol{f}_F$, acts on the boundary part Γ_F; i.e.

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_F \quad \text{on} \quad \Gamma_F \times (0, T).$$

Denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ and by "$\cdot$" and $\|\cdot\|$ the inner product and the Euclidean norms on $\mathbb{R}^d$ and $\mathbb{S}^d$.

The mechanical problem of the quasistatic deformation of a viscoelastic body is then written as follows.

Problem P. *Find a displacement field* $\boldsymbol{u} : \Omega \times (0, T) \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times (0, T) \to \mathbb{S}^d$ *such that,*

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad in \quad \Omega \times (0,T), \tag{2}$$
$$-\mathrm{Div}\,\boldsymbol{\sigma} = \boldsymbol{f}_0 \quad in \quad \Omega \times (0,T), \tag{3}$$
$$\boldsymbol{u} = \boldsymbol{0} \quad on \quad \Gamma_D \times (0,T), \tag{4}$$
$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_F \quad on \quad \Gamma_F \times (0,T), \tag{5}$$
$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad in \quad \Omega. \tag{6}$$

Here, $\boldsymbol{u}_0$ represents an initial condition for the displacement field, and $\boldsymbol{f}_0$ denotes the density of body forces. Moreover, we notice that equilibrium equation (3) does not include the acceleration term because the problem is assumed quasistatic.

In order to obtain the variational formulation of Problem P, let us denote by $H = [L^2(\Omega)]^d$ and construct the variational spaces V and Q as follows,

$$V = \{\boldsymbol{w} \in [H^1(\Omega)]^d \,;\, \boldsymbol{w} = \boldsymbol{0} \quad \text{on} \quad \Gamma_D\},$$
$$Q = \{\boldsymbol{\tau} = (\tau_{ij})_{i,j=1}^d \in [L^2(\Omega)]^{d\times d} \;;\; \tau_{ij} = \tau_{ji},\; i,j = 1,\ldots,d\}.$$

We will make the following assumptions on the problem data.

The viscosity tensor $\mathcal{A}(\boldsymbol{x}) = (a_{ijkl}(\boldsymbol{x}))_{i,j,k,l=1}^d : \boldsymbol{\tau} \in \mathbb{S}^d \to \mathcal{A}(\boldsymbol{x})(\boldsymbol{\tau}) \in \mathbb{S}^d$ satisfies:

$$\begin{aligned}
&\text{(a) } a_{ijkl} = a_{klij} = a_{jikl} \quad \text{for} \quad i,j,k,l = 1,\ldots,d.\\
&\text{(b) } a_{ijkl} \in L^\infty(\Omega) \quad \text{for} \quad i,j,k,l = 1,\ldots,d.\\
&\text{(c) There exists } m_{\mathcal{A}} > 0 \text{ such that } \mathcal{A}(\boldsymbol{x})\boldsymbol{\tau}\cdot\boldsymbol{\tau} \geq m_{\mathcal{A}}\,\|\boldsymbol{\tau}\|^2\\
&\qquad \forall\,\boldsymbol{\tau} \in \mathbb{S}^d,\ \text{a.e. } \boldsymbol{x} \in \Omega.
\end{aligned} \tag{7}$$

The elastic tensor $\mathcal{B}(\boldsymbol{x}) = (b_{ijkl}(\boldsymbol{x}))_{i,j,k,l=1}^d : \boldsymbol{\tau} \in \mathbb{S}^d \to \mathcal{B}(\boldsymbol{x})(\boldsymbol{\tau}) \in \mathbb{S}^d$ satisfies:

$$\begin{aligned}
&\text{(a) } b_{ijkl} = b_{klij} = b_{jikl} \quad \text{for} \quad i,j,k,l = 1,\ldots,d.\\
&\text{(b) } b_{ijkl} \in L^\infty(\Omega) \quad \text{for} \quad i,j,k,l = 1,\ldots,d.\\
&\text{(c) There exists } m_{\mathcal{B}} > 0 \text{ such that } \mathcal{B}(\boldsymbol{x})\boldsymbol{\tau}\cdot\boldsymbol{\tau} \geq m_{\mathcal{B}}\,\|\boldsymbol{\tau}\|^2\\
&\qquad \forall\,\boldsymbol{\tau} \in \mathbb{S}^d,\ \text{a.e. } \boldsymbol{x} \in \Omega.
\end{aligned} \tag{8}$$

The following regularity is assumed on the density of volume forces and tractions:

$$\boldsymbol{f}_0 \in C([0,T];H), \quad \boldsymbol{f}_F \in C([0,T];[L^2(\Gamma_F)]^d). \tag{9}$$

Finally, we assume that the initial displacement satisfies

$$\boldsymbol{u}_0 \in [H^2(\Omega)]^d. \tag{10}$$

Using Riesz' theorem, from (9) we can define the element $\boldsymbol{f}(t) \in V$ given by

$$(\boldsymbol{f}(t), \boldsymbol{w})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{w}\, d\boldsymbol{x} + \int_{\Gamma_F} \boldsymbol{f}_F(t) \cdot \boldsymbol{w}\, d\Gamma \quad \forall \boldsymbol{w} \in V,$$

and then $\boldsymbol{f} \in C([0,T]; V)$.

Plugging (2) into (3) and using the previous boundary conditions, applying a Green's formula we derive the following variational formulation of Problem P in terms of the displacement field $\boldsymbol{u}(t)$.

Problem VP. *Find a displacement field $\boldsymbol{u} : [0,T] \to V$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$ and for a.e. $t \in (0,T)$,*

$$(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q = (\boldsymbol{f}(t), \boldsymbol{w})_V \quad \forall \boldsymbol{w} \in V. \tag{11}$$

The existence of a unique weak solution to Problem VP has been considered in many works. For instance, proceeding as in [12] in the case without contact boundary conditions, we deduce the following.

Theorem 2.1 *Let assumptions (7)-(10) hold. Therefore, there exists a unique solution to Problem VP. Moreover, this solution has the regularity*

$$\boldsymbol{u} \in C^1([0,T]; V).$$

3 Fully discrete approximations

In this section, we now introduce a finite element algorithm to approximate solutions to Problem VP.

The discretization of Problem VP is done as follows. First, we assume that Ω is a polyhedral domain and we consider a finite dimensional space $V^h \subset V$, approximating the variational space V, given by

$$V^h = \{\boldsymbol{w}^h \in [C(\overline{\Omega})]^d \ ; \ \boldsymbol{w}^h_{|Tr} \in [P_1(Tr)]^d \quad Tr \in \mathcal{T}^h, \quad \boldsymbol{w}^h = \boldsymbol{0} \text{ on } \Gamma_D\}, \tag{12}$$

where $P_1(Tr)$ represents the space of polynomials of global degree less or equal to one in Tr and we denote by $(\mathcal{T}^h)_{h>0}$ a regular family of triangulations of $\overline{\Omega}$, compatible with the partition of the boundary $\Gamma = \partial\Omega$ into Γ_D and Γ_F; i.e. the finite element space V^h is composed of continuous and piecewise affine functions. Let h_{Tr} be the diameter of an element $Tr \in \mathcal{T}^h$ and let $h = \max_{Tr \in \mathcal{T}^h} h_{Tr}$ denote the spatial discretization parameter. Moreover, we assume that the discrete initial condition, denoted by $\boldsymbol{u}_0^h$, is given by

$$\boldsymbol{u}_0^h = \Pi^h \boldsymbol{u}_0, \tag{13}$$

where $\Pi^h : [C(\overline{\Omega})]^d \to V^h$ is the standard finite element interpolation operator (see, e.g., [6]).

To discretize the time derivatives, we consider a uniform partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \ldots < t_N = T$, and let k be the time step size, $k = T/N$. For a continuous function $f(t)$, let $f_n = f(t_n)$ and for a sequence $\{w_n\}_{n=0}^N$ we let $\delta w_n = (w_n - w_{n-1})/k$ be its corresponding divided differences.

Therefore, using an Euler scheme, we obtain the following fully discrete approximation of Problem VP.

Problem VP^{hk}. *Find a discrete displacement field $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset V^h$ such that $\boldsymbol{u}_0^{hk} = \boldsymbol{u}_0^h$ and for all $n = 1, \ldots, N$,*

$$(\mathcal{A}\boldsymbol{\varepsilon}(\delta \boldsymbol{u}_n^{hk}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q = (\boldsymbol{f}_n, \boldsymbol{w}^h)_V \quad \forall \boldsymbol{w}^h \in V^h. \tag{14}$$

Using Lax-Milgram Lemma, it is easy to obtain the following theorem which states the existence of a unique discrete solution $\boldsymbol{u}^{hk} \subset V^h$ to Problem VP^{hk}.

Theorem 3.1 *Let assumptions (7)-(10) hold. Therefore, there exists a unique solution to Problem VP^{hk}.*

We notice that this Euler scheme is more appropriate than the implicit one because it avoids the use of a fixed-point algorithm in the general case of nonlinear constitutive functions (see [12]).

4 An a priori estimate

In this section, we present a description of an a priori error estimates for Problem VP^{hk}. It is based on the arguments employed in [1] and we refer the reader there for details.

We have the following.

Theorem 4.1 *Let assumptions (7)-(10) hold. Let us denote by $\boldsymbol{u}$ and $\boldsymbol{u}^{hk}$ the respective solutions to problems VP and VP^{hk}. Therefore, there exists a positive constant $c > 0$, independent of the discretization parameters h and k but depending on the continuous solution $\boldsymbol{u}$ and the problem data, such that for all $\{\boldsymbol{w}_n^h\}_{n=0}^N \subset V^h$,*

$$\begin{aligned} \max_{0\le n\le N} \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 \le c\Big(& \max_{1\le n\le N} \|\boldsymbol{u}_n - \boldsymbol{w}_n^h\|_V^2 + \max_{1\le n\le N} \|\dot{\boldsymbol{u}}_n - \delta \boldsymbol{u}_n\|_V^2 \\ & + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + k^2 + \frac{1}{k}\sum_{n=1}^{N-1} \|\boldsymbol{u}_n - \boldsymbol{w}_n^h - (\boldsymbol{u}_{n+1} - \boldsymbol{w}_{n+1}^h)\|_V^2 \Big), \end{aligned} \tag{15}$$

where we employed the notation $\delta \boldsymbol{u}_n = (\boldsymbol{u}_n - \boldsymbol{u}_{n-1})/k$.

Proof. First, we take $\boldsymbol{w} = \boldsymbol{w}^h \in V$ in (11) at time $t = t_n$ and we substract it to (14) to obtain

$$(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n - \delta \boldsymbol{u}_n^{hk}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q = 0 \quad \forall \boldsymbol{w}^h \in V^h.$$

Therefore,

$$\begin{aligned}&(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n - \delta \boldsymbol{u}_n^{hk}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}))_Q \\ &\quad = (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n - \delta \boldsymbol{u}_n^{hk}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{w}^h))_Q \quad \forall \boldsymbol{w}^h \in V^h.\end{aligned}$$

Keeping in mind that

$$(\mathcal{A}\boldsymbol{\varepsilon}(\delta \boldsymbol{u}_n - \delta \boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}))_Q \geq \frac{m_{\mathcal{A}}}{2k}(\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 - \|\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk}\|_V^2),$$

by using assumptions (7)-(10) and applying several times the inequality

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2, \quad a, b, \epsilon \in \mathbb{R},\ \epsilon > 0, \tag{16}$$

we find that,

$$\begin{aligned}\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 &\leq ck\Big(\|\dot{\boldsymbol{u}}_n - \delta \boldsymbol{u}_n\|_V^2 + \|\boldsymbol{u}_n - \boldsymbol{w}^h\|_V^2 + \|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\|_V^2 \\ &\quad + (\mathcal{A}\boldsymbol{\varepsilon}(\delta \boldsymbol{u}_n - \delta \boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{w}^h))_Q\Big) + \|\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk}\|_V^2 \quad \forall \boldsymbol{w}^h \in V^h.\end{aligned}$$

By induction it follows that

$$\begin{aligned}\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 &\leq ck\sum_{j=1}^n \Big(\|\dot{\boldsymbol{u}}_j - \delta \boldsymbol{u}_j\|_V^2 + \|\boldsymbol{u}_j - \boldsymbol{w}_j^h\|_V^2 + \|\boldsymbol{u}_{j-1} - \boldsymbol{u}_{j-1}^{hk}\|_V^2 \\ &\quad + \|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}\|_V^2 + (\mathcal{A}\boldsymbol{\varepsilon}(\delta \boldsymbol{u}_j - \delta \boldsymbol{u}_j^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{u}_j - \boldsymbol{w}_j^h))_Q\Big) + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2\end{aligned} \tag{17}$$

for all $\boldsymbol{w}^h = \{\boldsymbol{w}_j^h\}_{j=0}^n \subset V^h$.

Taking into account the estimate (see [1] for details),

$$\begin{aligned}&\sum_{j=1}^n (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_j - \boldsymbol{u}_j^{hk} - (\boldsymbol{u}_{j-1} - \boldsymbol{u}_{j-1}^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{u}_j - \boldsymbol{w}_j^h))_Q \\ &\qquad = (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{w}_n^h))_Q + (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_0 - \boldsymbol{u}_0^h), \boldsymbol{\varepsilon}(\boldsymbol{u}_1 - \boldsymbol{w}_1^h))_Q \\ &\qquad\qquad + \sum_{j=1}^{n-1} (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{u}_j - \boldsymbol{w}_j^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_{j+1} - \boldsymbol{w}_{j+1}^h))_Q \\ &\qquad \leq \epsilon \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 + c\|\boldsymbol{u}_n - \boldsymbol{w}_n^h\|_V^2 + c\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + \|\boldsymbol{u}_1 - \boldsymbol{w}_1^h\|_V^2 \\ &\qquad\qquad + \sum_{j=1}^{n-1} \|\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\|_V \|\boldsymbol{u}_j - \boldsymbol{w}_j^h - (\boldsymbol{u}_{j+1} - \boldsymbol{w}_{j+1}^h)\|_V,\end{aligned}$$

where $\epsilon > 0$ is a parameter assumed to be small enough.

We will use the following lemma which represents a discrete version of Gronwall's lemma (see [12] for details).

Lemma 4.2 *Assume that* $\{g_n\}_{n=0}^N$ *and* $\{e_n\}_{n=0}^N$ *are two sequences of nonnegative real numbers satisfying, for a positive constant* $c > 0$ *independent of* g_n *and* e_n*,*

$$\begin{aligned} &e_0 \leq c g_0, \\ &e_n \leq c g_n + c \sum_{j=1}^{n} k e_{j-1}, \quad n = 1, \dots, N, \end{aligned}$$

where k *is a positive constant. Then,*

$$\max_{0 \leq n \leq N} e_n \leq C \max_{0 \leq n \leq N} g_n,$$

where $C = c(1 + cTe^{cT})$ *and* $T = Nk$*.*

From estimates (17), keeping in mind the regularity $\boldsymbol{u} \in C^1([0,T];V)$ and using Lemma 4.2 with $e_n = \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2$, $g_0 = e_0 = \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2$ and g_n the remaining terms, we deduce (15). ∎

We notice that the above error estimates are the basis for the analysis of the convergence rate of the algorithm. Hence, under additional regularity assumptions we obtain the linear convergence of the algorithm that we state in the following.

Corollary 4.3 *Let assumptions of Theorem 4.1 hold. Under the additional regularity conditions*

$$\boldsymbol{u} \in H^2(0,T;V) \cap H^1(0,T;[H^2(\Omega)]^d),$$

there exists a positive constant $c > 0$*, independent of the discretization parameters* h *and* k*, such that*

$$\max_{0 \leq n \leq N} \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V \leq c(h + k). \tag{18}$$

The proof of the above corollary is obtained by using the well-known result on the approximation by finite elements and the finite element interpolation operator Π^h (see [6]),

$$\begin{aligned} &\inf_{w_n^h \in V^h} \|\boldsymbol{u}_n - \boldsymbol{w}_n^h\|_V \leq ch\|\boldsymbol{u}_n\|_{[H^2(\Omega)]^d} \leq ch\|\boldsymbol{u}\|_{H^1(0,T;[H^2(\Omega)]^d)}, \\ &\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V \leq ch\|\boldsymbol{u}_0\|_{[H^2(\Omega)]^d} \leq ch\|\boldsymbol{u}\|_{H^1(0,T;[H^2(\Omega)]^d)}, \end{aligned}$$

an straightforward estimate implies that

$$\max_{1\le n\le N} \|\dot{\boldsymbol{u}}_n - \delta \boldsymbol{u}_n\|_V \le ck\|\boldsymbol{u}\|_{H^2(0,T;V)},$$

and, finally, by applying the following estimate (see [1]),

$$\frac{1}{k}\sum_{n=1}^{N-1} \|\boldsymbol{u}_n - \boldsymbol{w}_n^h - (\boldsymbol{u}_{n+1} - \boldsymbol{w}_{n+1}^h)\|_V^2 \le ch^2\|\boldsymbol{u}\|^2_{H^1(0,T;[H^2(\Omega)]^d)}.$$

5 A posteriori error estimates

In this section, we will use the finite element spaces and the notations introduced in the previous two sections. Moreover, throughout this section, we will assume that the mesh of the domain $\overline{\Omega}$ may change during the time, and so, for any $0 < h < 1$ and for any $n = 0, 1, \ldots, N$, let $\mathcal{T}^{hn}$ be a mesh of $\overline{\Omega}$ composed of closed elements Tr with diameter h_{Tr} less than h. We will also assume that, for each $n = 1, \ldots, N$, the mesh $\{(t_{n-1}, t_n) \times Tr\,;\, Tr \in \mathcal{T}^{hn}\}$ is regular in the sense of [6] and that $\mathcal{T}^{h(n-1)} \subset \mathcal{T}^{hn}$. Thus, for any $n = 1, \ldots, N$ and for any $Tr \in \mathcal{T}^{hn}$, let h_{Tr} (respectively ρ_{Tr}) be the diameter of the smallest (resp. largest) ball containing (resp. contained in) $(t_{n-1}, t_n) \times Tr$. Therefore, there exists a positive constant β such that

$$\frac{h_{Tr}}{\rho_{Tr}} \le \beta \quad \forall Tr \in \mathcal{T}^{hn},\ n = 0, 1, \ldots, N.$$

In order to simplify the writing and the calculations, in this section we assume that $\boldsymbol{f}_F = \boldsymbol{0}$ and therefore $(\boldsymbol{f}, \boldsymbol{w})_V = (\boldsymbol{f}, \boldsymbol{w})_H$ for all $\boldsymbol{w} \in V$, where $\boldsymbol{f} = \boldsymbol{f}_0 \in C([0,T]; H)$. It is straightforward to extend the results presented below to more general situations.

Finally, the notation $a \lesssim b$ means that there exists a positive constant c independent of a and b (and of the time and space discretization parameters) such that $a \le c\, b$.

Let us define the continuous and piecewise linear approximation in time given by

$$\boldsymbol{u}^{h\tau}(\boldsymbol{x}, t) = \frac{t - t_{n-1}}{k}\boldsymbol{u}_n^{hk}(\boldsymbol{x}) + \frac{t_n - t}{k}\boldsymbol{u}_{n-1}^{hk}(\boldsymbol{x}) \quad t_{n-1} \le t \le t_n, \quad \boldsymbol{x} \in \overline{\Omega}.$$

Since $\dot{\boldsymbol{u}}^{h\tau} = \delta \boldsymbol{u}_n^{hk}$, we can write variational equation (14) in the following equivalent form, for $n = 1, \ldots, N$,

$$(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q = (\boldsymbol{f}_n, \boldsymbol{w}^h)_H \quad \forall \boldsymbol{w}^h \in V^h,\ t_{n-1} \le t \le t_n.$$

According to [19], let us define the residual $R(\boldsymbol{u}^{h\tau}) \in L^2(0,T;V')$ as follows,

$$\langle R(\boldsymbol{u}^{h\tau}), \boldsymbol{w}\rangle_{V'\times V} = (\boldsymbol{f}, \boldsymbol{w})_H - (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}^{h\tau}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q$$

for all $\boldsymbol{w} \in V$ and $t \in [0,T]$, and decompose it into the temporal residual $R_\tau(\boldsymbol{u}^{h\tau}) \in L^2(0,T;V')$ given by

$$\langle R_\tau(\boldsymbol{u}^{h\tau}), \boldsymbol{w}\rangle_{V'\times V} = (\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}^{hk}_{n-1} - \boldsymbol{u}^{h\tau}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q \quad \text{on} \quad (t_{n-1}, t_n], \tag{19}$$

for all $\boldsymbol{w} \in V$, and into the spatial residual $R_h(\boldsymbol{u}^{h\tau}) \in L^2(0,T;V')$ defined as

$$\langle R_h(\boldsymbol{u}^{h\tau}), \boldsymbol{w}\rangle_{V'\times V} = (\boldsymbol{f}_{h\tau}, \boldsymbol{w})_H - (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}^{hk}_{n-1}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q \quad \text{on} \quad (t_{n-1}, t_n]$$

for all $\boldsymbol{w} \in V$, where we used the notation $\boldsymbol{f}_{h\tau}$ for the function which is piecewise constant on the time intervals and which, on each interval $(t_{n-1}, t_n]$, is equal to the L^2-projection of $\boldsymbol{f}_n$ onto the finite element space V^h.

Obviously, we have $R(\boldsymbol{u}^{h\tau}) = \boldsymbol{f} - \boldsymbol{f}_{h\tau} + R_\tau(\boldsymbol{u}^{h\tau}) + R_h(\boldsymbol{u}^{h\tau})$.

First, let us estimate the spatial residual. From its definition, it follows that

$$\langle R_h(\boldsymbol{u}^{h\tau}), \boldsymbol{w}^h\rangle_{V'\times V} = 0 \quad \forall \boldsymbol{w}^h \in V^h.$$

Hence, for each $\boldsymbol{w} \in V$, let us define by $\boldsymbol{w}^h = \Pi^h_C \boldsymbol{w}$, where Π^h_C is the Clément's interpolant on the triangulation $\mathcal{T}^{hn}$ (see [7]). We recall that this operator satisfies:

$$\|\boldsymbol{w} - \Pi^h_C \boldsymbol{w}\|_{[L^2(Tr)]^d} \le c h_{Tr} \|\boldsymbol{w}\|_{[H^1(\Delta Tr)]^d}, \tag{20}$$

$$\|\boldsymbol{w} - \Pi^h_C \boldsymbol{w}\|_{[L^2(E)]^d} \le c h_E^{1/2} \|\boldsymbol{w}\|_{[H^1(\Delta Tr)]^d}, \tag{21}$$

where c is a positive constant which depends on the given constant β, ΔTr denotes the set of elements having a common vertex, edge or face with Tr, E represents a point, an edge or a face of Tr and h_E denotes the size of the edge or face E.

Integrating in Ω and using Green's formula, we find that

$$\begin{aligned}\langle R_h(\boldsymbol{u}^{h\tau}), \boldsymbol{w}\rangle_{V'\times V} = \sum_{Tr\in\mathcal{T}^{hn}} \Big(&\int_{Tr} \mathrm{Div}(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}^{hk}_{n-1})) \cdot \boldsymbol{w}\, d\boldsymbol{x} \\ &+ \int_{Tr} \boldsymbol{f}_{h\tau} \cdot \boldsymbol{w}\, d\boldsymbol{x} - \sum_{E\in\mathcal{E}^{hn}_{Tr}} \int_E [(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}^{hk}_{n-1}))\boldsymbol{\nu}_E] \cdot \boldsymbol{w}\, d\boldsymbol{x}\Big),\end{aligned}$$

where $\mathcal{E}^{hn}_{Tr}$ is the set of interior points, edges or faces of the element Tr, and $[\boldsymbol{\tau\nu}]$ denotes the jump of $\boldsymbol{\tau\nu}$ across the point, edge or face E.

Therefore, using properties (20) and (21) for operator Π_C^h it follows that

$$
\begin{aligned}
\langle R_h(\boldsymbol{u}^{h\tau}), \boldsymbol{w}\rangle_{V'\times V} &= \langle R_h(\boldsymbol{u}^{h\tau}), \boldsymbol{w} - \Pi_C^h \boldsymbol{w}\rangle_{V'\times V} \\
&\lesssim \sum_{Tr\in\mathcal{T}^{hn}} \Big(h_{Tr} \|\boldsymbol{f}_{h\tau} + \mathrm{Div}(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}))\|_{[L^2(Tr)]^d} \|\boldsymbol{w}\|_{[H^1(\Delta Tr)]^d} \\
&\qquad + \sum_{E\in\mathcal{E}_{Tr}^{hn}} h_E^{1/2} \|[(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}))\boldsymbol{\nu}_E]\|_{[L^2(E)]^d} \|\boldsymbol{w}\|_{[H^1(\Delta Tr)]^d} \Big) \\
&\lesssim \left(\sum_{Tr\in\mathcal{T}^{hn}} h_{Tr}^2 \|\boldsymbol{f}_{h\tau} + \mathrm{Div}(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}))\|_{[L^2(Tr)]^d}^2 \right)^{1/2} \\
&\qquad\qquad \times \left(\sum_{Tr\in\mathcal{T}^{hn}} \|\boldsymbol{w}\|_{[H^1(\Delta Tr)]^d}^2 \right)^{1/2} \\
&\qquad + \Big(\sum_{E\in\mathcal{E}^{hn}} h_E \|[(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}))\boldsymbol{\nu}_E]\|_{[L^2(E)]^d}^2 \Big)^{1/2} \\
&\qquad\qquad \times \left(\sum_{Tr\in\mathcal{T}^{hn}} \|\boldsymbol{w}\|_{[H^1(\Delta Tr)]^d}^2 \right)^{1/2},
\end{aligned}
$$

where $\mathcal{E}^{hn}$ denotes the set of interior points, edges or faces that do not belong to Γ_D.

Since $\left(\sum_{Tr\in\mathcal{T}^{hn}} \|\boldsymbol{w}\|_{[H^1(\Delta Tr)]^d}^2 \right)^{1/2} \lesssim \|\boldsymbol{w}\|_V$ and the element $\boldsymbol{w}$ was chosen arbitrarily we then conclude that, for any $t \in (t_{n-1}, t_n]$,

$$
\begin{aligned}
\|R_h(\boldsymbol{u}^{h\tau})\|_{V'} &\lesssim \left(\sum_{Tr\in\mathcal{T}^{hn}} h_{Tr}^2 \|\boldsymbol{f}_{h\tau} + \mathrm{Div}(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}))\|_{[L^2(Tr)]^d}^2 \right)^{1/2} \\
&\qquad + \Big(\sum_{E\in\mathcal{E}^{hn}} h_E \|[(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}))\boldsymbol{\nu}_E]\|_{[L^2(E)]^d}^2 \Big)^{1/2} \\
&\lesssim \Big\{ \sum_{Tr\in\mathcal{T}^{hn}} \Big(h_{Tr} \|\boldsymbol{f}_{h\tau} + \mathrm{Div}(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}))\|_{[L^2(Tr)]^d} \\
&\qquad + \sum_{E\in\mathcal{E}_{Tr}^{int}} h_E^{1/2} \|[(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}))\boldsymbol{\nu}_E]\|_{[L^2(E)]^d} \Big)^2 \Big\}^{1/2} \\
&= \eta_1^{hn}.
\end{aligned}
$$

As a consequence, we deduce that

$$
\begin{aligned}
\|R_h(\boldsymbol{u}^{h\tau})\|_{L^2(0,T;V')} &\lesssim \left(\sum_{n=1}^{N} k(\eta_1^{hn})^2\right)^{1/2} \\
&= \Bigg\{ \sum_{n=1}^{N} \sum_{Tr\in\mathcal{T}^{hn}} k\Big(h_{Tr} \|\boldsymbol{f}_{h\tau} + \mathrm{Div}(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}))\|_{[L^2(Tr)]^d} \\
&\qquad + \sum_{E\in\mathcal{E}_{Tr}^{int}} h_E^{1/2} \|[(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}))\boldsymbol{\nu}_E]\|_{[L^2(E)]^d}\Big)^2 \Bigg\}^{1/2} \\
&= \eta_1^h. \qquad (22)
\end{aligned}
$$

Let us bound now the time residual. From (19) we immediately have

$$
\|R_\tau(\boldsymbol{u}^{h\tau})\|_{V'} \lesssim \|\boldsymbol{u}_{n-1}^{hk} - \boldsymbol{u}^{h\tau}\|_V \quad \text{on} \quad (t_{n-1}, t_n],
$$

and therefore,

$$
\begin{aligned}
\|R_\tau(\boldsymbol{u}^{h\tau})\|_{L^2(0,T;V')} &\lesssim \left\{ \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|\boldsymbol{u}_{n-1}^{hk} - \boldsymbol{u}^{h\tau}\|_V^2 \, dt \right\}^{1/2} \\
&= \left\{ \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \left(\frac{t-t_{n-1}}{k}\right)^2 \|\boldsymbol{u}_n^{hk} - \boldsymbol{u}_{n-1}^{hk}\|_V^2 \, dt \right\}^{1/2} \\
&= \left\{ \sum_{n=1}^{N} \frac{k}{3} \|\boldsymbol{u}_n^{hk} - \boldsymbol{u}_{n-1}^{hk}\|_V^2 \right\}^{1/2} \\
&= \left\{ \sum_{n=1}^{N} \sum_{Tr\in\mathcal{T}^{hn}} \frac{k}{3} \|\boldsymbol{u}_n^{hk} - \boldsymbol{u}_{n-1}^{hk}\|_{[H^1(Tr)]^d}^2 \right\}^{1/2} \\
&= \left(\sum_{n=1}^{N} k(\eta_2^{hn})^2\right)^{1/2} \qquad (23) \\
&= \eta_2^h,
\end{aligned}
$$

where $\eta_2^{hn} = \dfrac{1}{\sqrt{3}} \|\boldsymbol{u}_n^{hk} - \boldsymbol{u}_{n-1}^{hk}\|_V$.

Now, combining (22) and (23) we obtain the following estimate for the residual:

$$
\|R(\boldsymbol{u}^{h\tau})\|_{L^2(0,T;V')} \lesssim \eta_1^h + \eta_2^h + \|\boldsymbol{f} - \boldsymbol{f}_{h\tau}\|_{L^2(0,T;V')}.
$$

Finally, let us prove a relation between the residual $R(\boldsymbol{u}^{h\tau})$ and the error $\boldsymbol{u} - \boldsymbol{u}^{h\tau}$. From the definition of the residual, it follows that

$$
(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}} - \dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}^{h\tau}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q = (R(\boldsymbol{u}^{h\tau}), \boldsymbol{w})_{V'\times V} \qquad (24)
$$

for all $\boldsymbol{w} \in V$ and $t \in (0, T]$.

If we take $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{u}^{h\tau}$ in the previous variational equation and we employ assumptions (7)-(8), by using the ellipticity of $\mathcal{B}$ and Young's inequality, we immediately get

$$(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}} - \dot{\boldsymbol{u}}^{h\tau}), \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}^{h\tau}))_Q \lesssim \|R(\boldsymbol{u}^{h\tau})\|_{V'}^2.$$

Taking into account that

$$\begin{aligned}(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}} - \dot{\boldsymbol{u}}^{h\tau}), \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}^{h\tau}))_Q &= \int_\Omega \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}} - \dot{\boldsymbol{u}}^{h\tau}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}^{h\tau}) d\boldsymbol{x} \\ &= (\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}} - \dot{\boldsymbol{u}}^{h\tau}), \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}^{h\tau}))_{\mathcal{A},Q} \\ &= \frac{1}{2}\frac{d}{dt}\|\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}^{h\tau})\|_{\mathcal{A},Q}^2 \quad \forall t \in (0,T],\end{aligned}$$

where $\|\cdot\|_{\mathcal{A},Q}$ represents the norm in the space Q associated to the positive definite fourth-order viscous tensor $\mathcal{A}$, from properties (7) it follows that the norms $\|\cdot\|_{\mathcal{A},Q}$ and $\|\cdot\|_Q$ are equivalent.

Integrating in time between 0 and t the last expression, we find that

$$\begin{aligned}\|(\boldsymbol{u} - \boldsymbol{u}^{h\tau})(t)\|_V^2 &\lesssim \|\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}^{h\tau})(t)\|_{\mathcal{A},Q}^2 \\ &\lesssim \|R(\boldsymbol{u}^{h\tau})\|_{L^2(0,t;V')}^2 + \|\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}^{h\tau})(0)\|_{\mathcal{A},Q}^2 \\ &\lesssim \|R(\boldsymbol{u}^{h\tau})\|_{L^2(0,t;V')}^2 + \|(\boldsymbol{u} - \boldsymbol{u}^{h\tau})(0)\|_V^2 \\ &= \|R(\boldsymbol{u}^{h\tau})\|_{L^2(0,t;V')}^2 + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2,\end{aligned}$$

and therefore,

$$\|\boldsymbol{u} - \boldsymbol{u}^{h\tau}\|_{L^2(0,T;V)} \lesssim \|\boldsymbol{u} - \boldsymbol{u}^{h\tau}\|_{C([0,T];V)} \lesssim \|R(\boldsymbol{u}^{h\tau})\|_{L^2(0,T;V')} + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V.$$

Using again (24) with $\boldsymbol{w} = \dot{\boldsymbol{u}} - \dot{\boldsymbol{u}}^{h\tau}$, we obtain after similar calculations

$$\|\dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}^{h\tau}(t)\|_V^2 \lesssim \|R(\boldsymbol{u}^{h\tau})\|_{L^2(0,T;V')}^2 + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2.$$

Hence

$$\|\dot{\boldsymbol{u}} - \dot{\boldsymbol{u}}^{h\tau}\|_{L^2(0,T;V)} \lesssim \|R(\boldsymbol{u}^{h\tau})\|_{L^2(0,T;V')} + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V.$$

Summarizing the previous results, it leads to the following theorem which provides an upper bound for the error.

Theorem 5.1 *Let the assumptions of Theorem 2.1 hold. Denote by* $\boldsymbol{u}$ *and* $\boldsymbol{u}^{h\tau}$ *the solution to Problem VP and the continuous piecewise linear approximation of the solution to Problem VP*^{hk}*, respectively. If we denote by* $\eta = \sqrt{(\eta_1^h)^2 + (\eta_2^h)^2}$*, then we have*

$$\begin{aligned}\|\boldsymbol{u} - \boldsymbol{u}^{h\tau}\|_{C([0,T];V)} &+ \|\dot{\boldsymbol{u}} - \dot{\boldsymbol{u}}^{h\tau}\|_{L^2(0,T;V)} \\ &\lesssim \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V + \eta + h\|\boldsymbol{f} - \boldsymbol{f}_{h\tau}\|_{L^2(0,T;H)},\end{aligned} \tag{25}$$

where the error estimators η_1^h and η_2^h were defined in (22) and (23), respectively.

Finally, in the following theorem we prove a lower bound for these error estimators.

Theorem 5.2 *Let the assumptions of Theorem 5.1 hold. For all elements $Tr \in \mathcal{T}^{hn}$, the following local lower error bounds are obtained for $n = 1, \ldots, N$:*

$$\begin{aligned}\eta_{1Tr}^{hn} \lesssim \|\dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}^{h\tau}\|_{[H^1(\Delta Tr)]^d} + \|\boldsymbol{u}(t) - \boldsymbol{u}_{n-1}^{hk}\|_{[H^1(\Delta Tr)]^d} \\ + h_{Tr}\|\boldsymbol{f}(t) - \boldsymbol{f}_{h\tau}\|_{[L^2(\Delta Tr)]^d} \quad \text{for a.e. } t \in (t_{n-1}, t_n],\end{aligned}$$

where η_{1Tr}^{hn} is the local error in space given by

$$\begin{aligned}\eta_{1Tr}^{hn} = & h_{Tr}\|\boldsymbol{f}_{h\tau} + \mathrm{Div}(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}))\|_{[L^2(Tr)]^d} \\ & + \sum_{E\in\mathcal{E}_{Tr}^{hn}} h_E^{1/2}\|[(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}))\boldsymbol{\nu}_E]\|_{[L^2(E)]^d},\end{aligned}$$

and $\mathcal{E}_{Tr}^{hn}$ represents the set of interior points, edges or faces of Tr which do not belong to Γ_D.

If we denote by η^n the error estimator at time step n:

$$\eta^n = k^{1/2}((\eta_1^{hn})^2 + (\eta_2^{hn})^2)^{1/2},$$

then

$$\begin{aligned}\eta^n \lesssim \|\boldsymbol{u} - \boldsymbol{u}^{h\tau}\|_{L^2(t_{n-1},t_n;V)} + \|\dot{\boldsymbol{u}} - \dot{\boldsymbol{u}}^{h\tau}\|_{L^2(t_{n-1},t_n;V)} \\ + \|\boldsymbol{u} - \boldsymbol{u}_{n-1}^{hk}\|_{L^2(t_{n-1},t_n;V)} + h\|\boldsymbol{f} - \boldsymbol{f}_{h\tau}\|_{L^2(t_{n-1},t_n;H)}.\end{aligned} \tag{26}$$

Obviously, it follows that

$$\eta = \left(\sum_{n=1}^{N}(\eta^n)^2\right)^{1/2}.$$

Proof. From equation (24) we deduce, for any $t \in [0, T]$,

$$\|R(\boldsymbol{u}^{h\tau})\|_{V'} \lesssim \|\boldsymbol{u} - \boldsymbol{u}^{h\tau}\|_V + \|\dot{\boldsymbol{u}} - \dot{\boldsymbol{u}}^{h\tau}\|_V,$$

and therefore,

$$\|R(\boldsymbol{u}^{h\tau})\|_{L^2(t_1,t_2;V')} \lesssim \|\boldsymbol{u} - \boldsymbol{u}^{h\tau}\|_{L^2(t_1,t_2;V)} + \|\dot{\boldsymbol{u}} - \dot{\boldsymbol{u}}^{h\tau}\|_{L^2(t_1,t_2;V)},$$

for any t_1, t_2 in $[0, T]$. Next we bound η^n. We begin with the second term $k^{1/2}\|\boldsymbol{u}_n^{hk} - \boldsymbol{u}_{n-1}^{hk}\|_V$. We have, for any $t \in [t_{n-1}, t_n]$,

$$
\begin{aligned}
\left(\frac{t-t_{n-1}}{k}\right)^2 \|\boldsymbol{u}_n^{hk}-\boldsymbol{u}_{n-1}^{hk}\|_V^2 &= \|\boldsymbol{u}_{n-1}^{hk}-\boldsymbol{u}^{h\tau}\|_V^2 \\
&\lesssim (\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}-\boldsymbol{u}^{h\tau}),\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}-\boldsymbol{u}^{h\tau}))_Q \\
&= \langle R_\tau(\boldsymbol{u}^{h\tau}),\boldsymbol{u}_{n-1}^{hk}-\boldsymbol{u}^{h\tau}\rangle_{V'\times V} \\
&= \langle R(\boldsymbol{u}^{h\tau}),\boldsymbol{u}_{n-1}^{hk}-\boldsymbol{u}^{h\tau}\rangle_{V'\times V} - \langle R_h(\boldsymbol{u}^{h\tau}),\boldsymbol{u}_{n-1}^{hk}-\boldsymbol{u}^{h\tau}\rangle_{V'\times V} \\
&\quad -(\boldsymbol{f}-\boldsymbol{f}_{h\tau},\boldsymbol{u}_{n-1}^{hk}-\boldsymbol{u}^{h\tau})_{V'\times V}.
\end{aligned}
$$

Using Cauchy-Schwarz inequality and integrating the last expression from t_{n-1} to t_n we get

$$
\begin{aligned}
\frac{k}{3}\|\boldsymbol{u}_n^{hk}-\boldsymbol{u}_{n-1}^{hk}\|_V^2 \lesssim \Big(&\|R(\boldsymbol{u}^{h\tau})\|_{L^2(t_{n-1},t_n;V')} + \|R_h(\boldsymbol{u}^{h\tau})\|_{L^2(t_{n-1},t_n;V')} \\
&+\|\boldsymbol{f}-\boldsymbol{f}_{h\tau}\|_{L^2(t_{n-1},t_n;V')}\Big)\|\boldsymbol{u}_{n-1}^{hk}-\boldsymbol{u}^{h\tau}\|_{L^2(t_{n-1},t_n;V)}.
\end{aligned}
$$

Keeping in mind that

$$
\begin{aligned}
\|\boldsymbol{u}_{n-1}^{hk}-\boldsymbol{u}^{h\tau}\|_{L^2(t_{n-1},t_n;V)} &= \left(\int_{t_{n-1}}^{t_n} \|\boldsymbol{u}_{n-1}^{hk}-\boldsymbol{u}^{h\tau}\|_V^2\right)^{1/2} \\
&= \left(\int_{t_{n-1}}^{t_n} (\frac{t-t_{n-1}}{k})^2\|\boldsymbol{u}_n^{hk}-\boldsymbol{u}_{n-1}^{hk}\|_V^2\right)^{1/2} \\
&= \left(\frac{k}{3}\right)^{1/2} \|\boldsymbol{u}_n^{hk}-\boldsymbol{u}_{n-1}^{hk}\|_V,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\left(\frac{k}{3}\right)^{1/2} \|\boldsymbol{u}_n^{hk}-\boldsymbol{u}_{n-1}^{hk}\|_V &\lesssim \|R(\boldsymbol{u}^{h\tau})\|_{L^2(t_{n-1},t_n;V')} + \|R_h(\boldsymbol{u}^{h\tau})\|_{L^2(t_{n-1},t_n;V')} \\
&\quad +\|\boldsymbol{f}-\boldsymbol{f}_{h\tau}\|_{L^2(t_{n-1},t_n;V')} \\
&\lesssim \|\boldsymbol{u}-\boldsymbol{u}^{h\tau}\|_{L^2(t_{n-1},t_n;V)} + \|\dot{\boldsymbol{u}}-\dot{\boldsymbol{u}}^{h\tau}\|_{L^2(t_{n-1},t_n;V)} + \|R_h(\boldsymbol{u}^{h\tau})\|_{L^2(t_{n-1},t_n;V')} \\
&\quad +\|\boldsymbol{f}-\boldsymbol{f}_{h\tau}\|_{L^2(t_{n-1},t_n;V')} \\
&\lesssim \|\boldsymbol{u}-\boldsymbol{u}^{h\tau}\|_{L^2(t_{n-1},t_n;V)} + \|\dot{\boldsymbol{u}}-\dot{\boldsymbol{u}}^{h\tau}\|_{L^2(t_{n-1},t_n;V)} + k^{1/2}\eta_1^{hn} \\
&\quad +\|\boldsymbol{f}-\boldsymbol{f}_{h\tau}\|_{L^2(t_{n-1},t_n;V')}.
\end{aligned}
$$

From the properties of the $[L^2(\Omega)]^d$-projection operator, we have

$$\|\boldsymbol{f}-\boldsymbol{f}_{h\tau}\|_{V'} \leq h\|\boldsymbol{f}-\boldsymbol{f}_{h\tau}\|_H.$$

Thus, it only remains to bound $k^{1/2}\eta_1^{hn}$. Recalling that

$$\eta_1^{hn} = \Big(\sum_{Tr \in \mathcal{T}^{hn}} \Big(h_{Tr} \| \boldsymbol{f}_{h\tau} + \mathrm{Div}(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk})) \|_{[L^2(Tr)]^d}$$

$$+ \sum_{E \in \mathcal{E}_{Tr}^{int}} h_E^{1/2} \| [(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}))\boldsymbol{\nu}_E] \|_{[L^2(E)]^d} \Big)^2 \Big)^{1/2}$$

this is done in the following, when the estimate of the estimator η_{1Tr}^{hn} is obtained. Let w_{Tr} be the bubble function associated with the element Tr (for instance, in the two-dimensional setting, we have $w_{Tr} = \lambda_{a1}\lambda_{a2}\lambda_{a3}$, where λ_{ai}, $i = 1, 2, 3$ denote the barycentric coordinates and a_1, a_2 and a_3 are the three nodes of the element Tr). We notice that $w_{Tr} \in H_0^1(Tr)$. Let us define $\boldsymbol{w}_{Tr} \in [H_0^1(Tr)]^d$ which is constructed as $w_i = w_{Tr}$ for $i = 1, \ldots, d$.

It follows that the function $\boldsymbol{\psi}_{Tr} = \boldsymbol{w}_{Tr} \cdot (\boldsymbol{f}_{h\tau} + \mathrm{Div}(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk})))$ verifies (see [18]),

$$\| \boldsymbol{f}_{h\tau} + \mathrm{Div}(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk})) \|_{[L^2(Tr)]^d}^2 \lesssim \int_{Tr} (\boldsymbol{f}_{h\tau} - \boldsymbol{f}) \cdot \boldsymbol{\psi}_{Tr} \, d\boldsymbol{x}$$
$$+ \int_{Tr} (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}} - \dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_{n-1}^{hk})) \cdot \boldsymbol{\varepsilon}(\boldsymbol{\psi}_{Tr}) \, d\boldsymbol{x}.$$

Using an inverse inequality, it follows that

$$\| \boldsymbol{\varepsilon}(\boldsymbol{\psi}_{Tr}) \|_{[L^2(Tr)]^{d\times d}} \lesssim h_{Tr}^{-1} \| \boldsymbol{\psi}_{Tr} \|_{[L^2(Tr)]^d},$$

and therefore,

$$\begin{aligned} h_{Tr} \| \boldsymbol{f}_{h\tau} + \mathrm{Div}(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk})) \|_{[L^2(Tr)]^d} & \\ \lesssim \| \dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}^{h\tau}(t) \|_{[H^1(Tr)]^d} + h_{Tr} \| \boldsymbol{f}(t) - \boldsymbol{f}_{h\tau}(t) \|_{[L^2(Tr)]^d} & \\ + \| \boldsymbol{u}(t) - \boldsymbol{u}_{n-1}^{hk} \|_{[H^1(Tr)]^d}. & \end{aligned} \tag{27}$$

We turn now to estimate the second term of error estimator η_{1Tr}^{hn}. Proceeding in a similar way that in the previous estimate, let us consider the bubble function w_E associated with the point, edge or face E. Hence, taking now $\boldsymbol{w}_E = [w_E]^d$ we deduce that (see again [18]),

$$\| [(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}))\boldsymbol{\nu}_E] \|_{[L^2(E)]^d}^2 \lesssim \Big(\| \boldsymbol{f}(t) - \boldsymbol{f}_{h\tau}(t) \|_{[L^2(\Delta Tr)]^d}$$
$$+ h_E^{-1} (\| \dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}^{h\tau}(t) \|_{[H^1(\Delta Tr)]^d} + \| \boldsymbol{u}_{n-1}^{hk}(t) - \boldsymbol{u}(t) \|_{[H^1(\Delta Tr)]^d})$$
$$+ \| \boldsymbol{f}_{h\tau} + \mathrm{Div}(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}))) \|_{[L^2(\Delta Tr)]^d} \Big) \| \boldsymbol{\psi}_E \|_{[L^2(\Delta Tr)]^d},$$

where ΔTr stands for the set of elements of $\mathcal{T}^{hn}$ sharing the common point, edge or face E. From the definition of $\boldsymbol{w}_E$ we conclude that

$$
\begin{aligned}
h_E^{1/2} & \|[(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}))\boldsymbol{\nu}_E]\|_{[L^2(E)]^d} \\
& \lesssim h_E \|\boldsymbol{f}(t) - \boldsymbol{f}_{h\tau}(t)\|_{[L^2(\Delta Tr)]^d} \\
& \qquad + \|\dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}^{h\tau}(t)\|_{[H^1(\Delta Tr)]^d} + \|\boldsymbol{u}_{n-1}^{hk}(t) - \boldsymbol{u}(t)\|_{[H^1(\Delta Tr)]^d} \\
& \qquad + h_E \|\boldsymbol{f}_{h\tau} + \mathrm{Div}(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk})))\|_{[L^2(\Delta Tr)]^d} \\
& \lesssim h_E \|\boldsymbol{f}(t) - \boldsymbol{f}_{h\tau}(t)\|_{[L^2(\Delta Tr)]^d} \\
& \qquad + \|\dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}^{h\tau}(t)\|_{[H^1(\Delta Tr)]^d} + \|\boldsymbol{u}_{n-1}^{hk}(t) - \boldsymbol{u}(t)\|_{[H^1(\Delta Tr)]^d}.
\end{aligned}
$$

Keeping in mind (27) and the previous estimate, we obtain, for all $Tr \in \mathcal{T}^{hn}$,

$$
\begin{aligned}
\eta_{1Tr}^{hn} &= h_{Tr} \|\boldsymbol{f}_{h\tau} + \mathrm{Div}(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}))\|_{[L^2(Tr)]^d} \\
& \qquad + \sum_{E \in \mathcal{E}_{Tr}^{hn}} h_E^{1/2} \|[(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}))\boldsymbol{\nu}_E]\|_{[L^2(E)]^d} \\
& \lesssim \|\dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}^{h\tau}(t)\|_{[H^1(\Delta Tr)]^d} + \|\boldsymbol{u}(t) - \boldsymbol{u}_{n-1}^{hk}\|_{[H^1(\Delta Tr)]^d} \\
& \qquad + h_{Tr} \|\boldsymbol{f}(t) - \boldsymbol{f}_{h\tau}(t)\|_{[L^2(\Delta Tr)]^d},
\end{aligned}
$$

and therefore,

$$
\eta_1^{hn} \lesssim \|\dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}^{h\tau}(t)\|_V + \|\boldsymbol{u}(t) - \boldsymbol{u}_{n-1}^{hk}\|_V + h_{Tr} \|\boldsymbol{f}(t) - \boldsymbol{f}_{h\tau}(t)\|_H.
$$

Thus, we find that

$$
\begin{aligned}
k^{1/2} \eta_1^{hn} \lesssim \|\dot{\boldsymbol{u}} - \dot{\boldsymbol{u}}^{h\tau}\|_{L^2(t_{n-1}, t_n; V)} + \|\boldsymbol{u} - \boldsymbol{u}_{n-1}^{hk}\|_{L^2(t_{n-1}, t_n; V)} \\
+ h \|\boldsymbol{f} - \boldsymbol{f}_{h\tau}\|_{L^2(t_{n-1}, t_n; H)},
\end{aligned}
$$

and, combining all these results and taking into account the definitions (22) and (23), it leads to the desired lower error bounds of η^n. ■

We observe that, from Theorem 5.2, we can prove a similar convergence order as provided in the a priori error analysis which we state in the following.

Corollary 5.3 *Let the assumptions of Theorem 5.2 hold. If the continuous solution has the regularity* $\boldsymbol{u} \in C^1([0,T];[H^2(\Omega)]^d)$ *and we assume that the density of volume forces satisfies* $\boldsymbol{f}_0 \in C([0,T];[H^1(\Omega)]^d)$*, we have*

$$
\eta \le c(h + k),
$$

for a positive constant c *which depends on the given data and the continuous solution* $\boldsymbol{u}$.

Proof. The proof of this corollary is obtained taking into account the following straightforward estimate

$$\|\boldsymbol{f} - \boldsymbol{f}_{h\tau}\|_{L^2(0,T;H)} \leq ch\|\boldsymbol{f}\|_{C([0,T];[H^1(\Omega)]^d)}.$$

Using estimates (18), under the required regularity we conclude that

$$\|\boldsymbol{u} - \boldsymbol{u}^{h\tau}\|_{C([0,T];V)} \leq c(h+k),$$

and similarly, we also have

$$\left(\sum_{n=1}^{N} \|\boldsymbol{u} - \boldsymbol{u}_{n-1}^{hk}\|^2_{L^2(t_{n-1},t_n;V)}\right)^{1/2} \leq c(h+k).$$

Finally, using again (24) we find that, for $n = 1, \ldots, N$,

$$(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}^{h\tau}(t)) + \mathcal{B}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t) - \boldsymbol{u}_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q = 0 \quad \forall \boldsymbol{w}^h \in V^h,\ t_{n-1} \leq t \leq t_n,$$

and therefore, since $\dot{\boldsymbol{u}}^{h\tau}(t) \in V^h$,

$$\begin{aligned}
&(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}^{h\tau}(t)) + \mathcal{B}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t) - \boldsymbol{u}_{n-1}^{hk}), \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}^{h\tau}(t)))_Q \\
&\quad = (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}^{h\tau}(t)) + \mathcal{B}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t) - \boldsymbol{u}_{n-1}^{hk}), \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t) - \boldsymbol{w}^h))_Q \quad \forall \boldsymbol{w}^h \in V^h,
\end{aligned}$$

for $t_{n-1} \leq t \leq t_n$. Using properties (7) and (8) and applying several times inequality (16), it follows that

$$\|\dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}^{h\tau}(t)\|_V^2 \leq c(\|\boldsymbol{u}(t) - \boldsymbol{u}_{n-1}^{hk}\|_V^2 + \|\dot{\boldsymbol{u}}(t) - \boldsymbol{w}^h\|_V^2) \quad \forall \boldsymbol{w}^h \in V^h,$$

from which, using the regularity condition $\dot{\boldsymbol{u}} \in C([0,T];[H^2(\Omega)]^d)$, we conclude that (see [6]),

$$\inf_{w^h \in V^h} \|\dot{\boldsymbol{u}}(t) - \boldsymbol{w}^h\|_V \leq c(h+k)\|\boldsymbol{u}\|_{H^1(0,T;[H^2(\Omega)]^d)}.$$

It implies the linear convergence. ■

Remark 5.4 *If we denote by η_{2Tr}^{hn} the local error in time given by*

$$\eta_{2Tr}^{hn} = \frac{1}{\sqrt{3}}\|\boldsymbol{u}_n^{hk} - \boldsymbol{u}_{n-1}^{hk}\|_{[H^1(Tr)]^d},$$

we obviously have the following local error estimate in time,

$$\eta_{2Tr}^{hn} \leq \frac{1}{\sqrt{3}}\|\boldsymbol{u}(t) - \boldsymbol{u}_n^{hk}\|_{[H^1(Tr)]^d} + \frac{1}{\sqrt{3}}\|\boldsymbol{u}(t) - \boldsymbol{u}_{n-1}^{hk}\|_{[H^1(Tr)]^d}.$$

Using this estimate in the proof of Theorem 5.2, it leads to the following local error estimate in space and time:

$$\begin{aligned}\eta_{Tr}^{hn} &= \left((\eta_{1Tr}^{hn})^2 + (\eta_{2Tr}^{hn})^2\right)^{1/2} \\ &\leq \|\dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}^{h\tau}\|_{[H^1(\Delta Tr)]^d} + \|\boldsymbol{u}(t) - \boldsymbol{u}_{n-1}^{hk}\|_{[H^1(\Delta Tr)]^d} + \|\boldsymbol{u}(t) - \boldsymbol{u}_n^{hk}\|_{[H^1(\Delta Tr)]^d} \\ &\qquad + h_{Tr}\|\boldsymbol{f}(t) - \boldsymbol{f}_{h\tau}(t)\|_{[L^2(\Delta Tr)]^d}.\end{aligned}$$

By adding these terms, we obtain a rougher bound of η^n than in (26) due to the presence of $\|\boldsymbol{u}(t) - \boldsymbol{u}_n^{hk}\|_{[H^1(\Delta Tr)]^d}$.

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