A RESIDUAL A POSTERIORI ERROR ESTIMATOR FOR ELASTO-VISCOPLASTICITY

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Abstract. The numerical approximation of an elasto-viscoplastic problem is considered in this paper. Fully discrete approximations are obtained by using the finite element method to approximate the spatial variable and the forward Euler scheme to discretize time derivatives. We first recall an a priori estimate result from which the linear convergence of the algorithm is derived under suitable regularity conditions. Then, an a posteriori error analysis is provided. Upper and lower error bounds are obtained.

Key words. Elasto-viscoplasticity, fully discrete approximations, a posteriori error estimates, finite elements.

1. Introduction

Elasto-viscoplastic materials are very common in the real life because some types of rocks and metals can be modelled using a rate-type viscoplastic law. As noticed in [9], these materials allow both creep and relaxation phenomena.

In this work, we will consider a semilinear elasto-viscoplastic constitutive law introduced in [5] and already studied, from both mathematical and numerical point of views, by Ionescu and Sofonea (see the monograph [9] and the references cited therein). In particular, fully discrete approximations were considered in [6], where a priori estimates were obtained for an explicit Euler scheme. In this paper, this problem is revisited and a posteriori error analysis is performed in the study of that elasto-viscoplastic problem. This is done extending some arguments already applied in the study of the heat equation (see, e.g., [10, 11, 13]), some parabolic equations ([1]) or the Stokes equation ([2]). Recently, contact problems involving this kind of materials were studied (see the monograph [7] and the numerous references cited therein), and this work can be seen as a first step to deal with this interesting kind of contact problems (see [8] for an early study in the linear elasticity case).

The paper is structured as follows. In Section 2, the mechanical model and its variational formulation are described following the notation and assumptions introduced in [7]. Then, fully discrete approximations are provided in Section 3, by using the finite element method to approximate the spatial variable and the forward Euler scheme to discretize the time derivatives. In Section 4, an a priori error analysis obtained in [6] is recalled. Finally, using some results obtained in the study of the heat equation, an a posteriori error analysis is done in Section 5, providing an upper bound for the error, Theorem 5.1, and a lower bound, Theorem 5.2.

2. Mechanical problem and its variational formulation

In this section, we present a brief description of the elasto-viscoplastic model (details can be found in [5, 9]).

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, denote a domain occupied by an elasto-viscoplastic body with a smooth boundary $\Gamma = \partial\Omega$ decomposed into two disjoint parts Γ_D and Γ_F such that meas $(\Gamma_D) > 0$. Moreover, let $[0, T]$, $T > 0$, be the time interval of interest and denote by $\boldsymbol{\nu}$ the unit outer normal vector to Γ.

Let $\boldsymbol{x} \in \Omega$ and $t \in [0, T]$ be the spatial and time variables, respectively, and, in order to simplify the writing, we do not indicate the dependence of the functions on $\boldsymbol{x}$ and t. Moreover, a dot above a variable represents the derivative with respect to the time variable.

Let us denote by $\boldsymbol{u} = (u_i)_{i=1}^d$, $\boldsymbol{\sigma} = (\sigma_{ij})_{i,j=1}^d$ and $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))_{i,j=1}^d$ the displacement field, the stress tensor and the linearized strain tensor, respectively. We recall that

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

The body is assumed elasto-viscoplastic and satisfying the following rate-type semi-linear constitutive law (see [5, 9]),

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})), \tag{1}$$

where $\mathcal{E}$ and $\mathcal{G}$ denote the fourth-order elastic tensor and the viscoplastic function, respectively.

We turn now to describe the boundary conditions.

On the boundary part Γ_D we assume that the body is clamped and thus the displacement field vanishes there (and so $\boldsymbol{u} = \boldsymbol{0}$ on $\Gamma_D \times (0, T)$). Moreover, we assume that a density of traction forces, denoted by $\boldsymbol{f}_F$, acts on the boundary part Γ_F; i.e.

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_F \quad \text{on} \quad \Gamma_F \times (0, T).$$

Denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ and by "$\cdot$" and $|\cdot|$ the inner product and the Euclidean norms on $\mathbb{R}^d$ and $\mathbb{S}^d$.

The mechanical problem of the quasistatic deformation of an elasto-viscoplastic body is then written as follows.

Problem P. *Find a displacement field* $\boldsymbol{u} : \Omega \times (0, T) \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times (0, T) \to \mathbb{S}^d$ *such that,*

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})) \quad \text{in} \quad \Omega \times (0, T), \tag{2}$$

$$-\mathrm{Div}\,\boldsymbol{\sigma} = \boldsymbol{f}_0 \quad \text{in} \quad \Omega \times (0, T), \tag{3}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_D \times (0, T), \tag{4}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_F \quad \text{on} \quad \Gamma_F \times (0, T), \tag{5}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \quad \text{in} \quad \Omega. \tag{6}$$

Here, $\boldsymbol{u}_0$ and $\boldsymbol{\sigma}_0$ represent initial conditions for the displacement field and the stress tensor, respectively, and $\boldsymbol{f}_0$ denotes the density of body forces. Moreover, we notice that equilibrium equation (3) does not include the acceleration term because the problem is assumed quasistatic.

In order to obtain the variational formulation of Problem P, let $H = [L^2(\Omega)]^d$ and we define the following variational spaces:

$$V = \{\boldsymbol{w} \in [H^1(\Omega)]^d \,;\, \boldsymbol{w} = \boldsymbol{0} \quad \text{on} \quad \Gamma_D\},$$
$$Q = \{\boldsymbol{\tau} = (\tau_{ij})_{i,j=1}^d \in [L^2(\Omega)]^{d\times d} \;;\; \tau_{ij} = \tau_{ji}, \;\; i, j = 1, \ldots, d\}.$$

The following assumptions are required on the problem data.

The elastic tensor $\mathcal{E}(\boldsymbol{x}) = (e_{ijkl}(\boldsymbol{x}))_{i,j,k,l=1}^{d} : \boldsymbol{\tau} \in \mathbb{S}^d \to \mathcal{E}(\boldsymbol{x})(\boldsymbol{\tau}) \in \mathbb{S}^d$ satisfies:

(7)
(a) $e_{ijkl} = e_{klij} = e_{jikl}$ for $i,j,k,l = 1,\ldots,d$.
(b) $e_{ijkl} \in L^\infty(\Omega)$ for $i,j,k,l = 1,\ldots,d$.
(c) There exists $m_{\mathcal{E}} > 0$ such that $\mathcal{E}(\boldsymbol{x})\boldsymbol{\tau}\cdot\boldsymbol{\tau} \geq m_{\mathcal{E}}\,|\boldsymbol{\tau}|^2$ $\forall\,\boldsymbol{\tau} \in \mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega$.

The viscoplastic function $\mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathcal{G}(\boldsymbol{x})(\boldsymbol{\tau}) \in \mathbb{S}^d$ satisfies:

(8)
(a) There exists $L_{\mathcal{G}} > 0$ such that $|\mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1) - \mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2)| \leq L_{\mathcal{G}}\left(|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| + |\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2|\right)$ for all $\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2,\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2 \in \mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega$.
(b) The function $\boldsymbol{x} \to \mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon})$ is measurable.
(c) The mapping $\boldsymbol{x} \to \mathcal{G}(\boldsymbol{x},\boldsymbol{0},\boldsymbol{0})$ belongs to Q.

The following regularity is assumed on the density of volume forces and tractions:

$$\boldsymbol{f}_0 \in C^1([0,T];H), \quad \boldsymbol{f}_F \in C^1([0,T];[L^2(\Gamma_F)]^d). \tag{9}$$

Using Riesz' theorem, from (9) we can define the element $\boldsymbol{f}(t) \in V$ given by

$$(\boldsymbol{f}(t),\boldsymbol{w})_V = \int_\Omega \boldsymbol{f}_0(t)\cdot\boldsymbol{w}\,d\boldsymbol{x} + \int_{\Gamma_F} \boldsymbol{f}_F(t)\cdot\boldsymbol{w}\,d\Gamma \quad \forall \boldsymbol{w} \in V,$$

and then $\boldsymbol{f} \in C^1([0,T];V)$.

Finally, we assume that the initial displacement and stress fields satisfy the following regularity and compatibility conditions,

$$\begin{aligned} &\boldsymbol{u}_0 \in V, \quad \boldsymbol{\sigma}_0 \in Q, \\ &(\boldsymbol{\sigma}_0,\boldsymbol{\varepsilon}(\boldsymbol{u}_0))_Q = (\boldsymbol{f}(0),\boldsymbol{u}_0)_V. \end{aligned} \tag{10}$$

Using the previous boundary conditions and applying Green's formula, we obtain the following variational formulation of Problem P.

Problem VP. *Find a displacement field* $\boldsymbol{u} : [0,T] \to V$ *and a stress field* $\boldsymbol{\sigma} : [0,T] \to Q$ *such that* $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0$ *and for a.e.* $t \in (0,T)$,

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{u}(t))), \tag{11}$$

$$(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{w}))_Q = (\boldsymbol{f}(t),\boldsymbol{w})_V \quad \forall \boldsymbol{w} \in V. \tag{12}$$

The existence of a unique weak solution to Problem VP has been considered in [9]. The following theorem, which establishes the existence of a unique solution to Problem VP, was proved there by using Banach's fixed point theorem.

Theorem 2.1. *Let assumptions (7)-(10) hold. Therefore, there exists a unique solution to Problem VP such that* $\boldsymbol{u} \in C^1([0,T];V)$, $\boldsymbol{\sigma} \in C^1([0,T];Q)$.

3. Fully discrete approximations

In this section, we now introduce a finite element algorithm to approximate solutions to Problem VP.

The discretization of Problem VP is done as follows. First, we assume that Ω is a polyhedral domain and we consider the finite dimensional spaces $V^h \subset V$ and $Q^h \subset Q$, approximating variational spaces V and Q, respectively, and given by

$$V^h = \{\boldsymbol{w}^h \in [C(\overline{\Omega})]^d \; ; \; \boldsymbol{w}^h_{|Tr} \in [P_1(Tr)]^d \;\; Tr \in \mathcal{T}^h, \quad \boldsymbol{w}^h = \boldsymbol{0} \text{ on } \Gamma_D\}, \tag{13}$$

$$Q^h = \{\boldsymbol{\tau}^h \in Q \; ; \; \boldsymbol{\tau}^h_{|Tr} \in [P_0(Tr)]^{d\times d} \;\; Tr \in \mathcal{T}^h\}, \tag{14}$$

where $P_q(Tr)$, $q = 0,1$, represents the space of polynomials of global degree less or equal to q in Tr and we denote by $\mathcal{T}^h$ a triangulation of $\overline{\Omega}$ compatible with the partition of the boundary $\Gamma = \partial\Omega$ into Γ_D and Γ_F; i.e. the finite element space V^h is composed of continuous and piecewise affine functions and the finite element space Q^h is made of piecewise constant functions. Here, $h > 0$ is the spatial discretization parameter. Moreover, we assume that the discrete initial conditions, denoted by $\boldsymbol{u}_0^h$ and $\boldsymbol{\sigma}_0^h$, are given by

$$\boldsymbol{u}_0^h = \Pi_{V^h}\boldsymbol{u}_0, \quad \boldsymbol{\sigma}_0^h = \Pi_{Q^h}\boldsymbol{\sigma}_0, \tag{15}$$

where $\Pi_{V^h} : [C(\overline{\Omega})]^d \to V^h$ and $\Pi_{Q^h} : Q \to Q^h$ are the standard finite element L^2-projection operator onto V^h and the L^2-projection operator onto the finite element space Q^h, respectively (see, e.g., [3]).

To discretize the time derivatives, we consider a uniform partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \ldots < t_N = T$ and let k be the time step size, $k = T/N$. For a continuous function $f(t)$, let $f_n = f(t_n)$ and for a sequence $\{w_n\}_{n=0}^N$ we let $\delta w_n = (w_n - w_{n-1})/k$ be its corresponding divided differences.

In order to simplify the writing and the calculations, we assume, without loss of generality, that $\mathcal{G}(Q^h, Q^h) \subset Q^h$. It is straightforward to extend the results presented in the next two sections to more general situations by using the operator Π_{Q^h}.

Therefore, using a hybrid combination of backward and forward Euler schemes, we obtain the following fully discrete approximation of Problem VP.

Problem VP^{hk}. *Find a discrete displacement field $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset V^h$ and a discrete stress field $\boldsymbol{\sigma}^{hk} = \{\boldsymbol{\sigma}_n^{hk}\}_{n=0}^N \subset Q^h$ such that $\boldsymbol{u}_0^{hk} = \boldsymbol{u}_0^h$, $\boldsymbol{\sigma}_0^{hk} = \boldsymbol{\sigma}_0^h$ and for all $n = 1, \ldots, N$,*

$$\delta\boldsymbol{\sigma}_n^{hk} = \mathcal{E}\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_n^{hk}) + \mathcal{G}(\boldsymbol{\sigma}_{n-1}^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk})), \tag{16}$$

$$(\boldsymbol{\sigma}_n^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q = (\boldsymbol{f}_n, \boldsymbol{w}^h)_V \quad \forall \boldsymbol{w}^h \in V^h. \tag{17}$$

Using Lax-Milgram lemma, it is easy to obtain the following theorem which states the existence of a unique discrete solution $\boldsymbol{u}^{hk} \subset V^h$ and $\boldsymbol{\sigma}^{hk} \subset Q^h$ to Problem VP^{hk}.

Theorem 3.1. *Let assumptions (7)-(10) hold. Therefore, there exists a unique solution to Problem VP^{hk}.*

4. An a priori error analysis

In this section, we recall an a priori error estimate for Problem VP^{hk}, which was derived in [6] for a particular case of the viscoplastic function $\mathcal{G}$ (see also [7] for a recent proof in the case including contact and the general constitutive law (1)).

We have the following.

Theorem 4.1. *Let assumptions (7)-(10) hold. Let us denote by $(\boldsymbol{u}, \boldsymbol{\sigma})$ and $(\boldsymbol{u}^{hk}, \boldsymbol{\sigma}^{hk})$ the respective solutions to problems VP and VP^{hk}. Therefore, there exists a positive constant $c > 0$, independent of the discretization parameters h and k, such that for all $\{\boldsymbol{w}_n^h\}_{n=0}^N \subset V^h$,*

$$\begin{aligned}\max_{0\le n\le N} \{\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 + \|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk}\|_Q^2\} &\le c\Big(\max_{0\le n\le N} \|\boldsymbol{u}_n - \boldsymbol{w}_n^h\|_V^2 \\ &+ \max_{0\le n\le N} I_{\mathcal{G}_n} + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + \|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h\|_Q^2\Big),\end{aligned} \tag{18}$$

where the integration error $I_{\mathcal{G}_n}$ is given by

$$I_{\mathcal{G}_n} = \left\| \int_0^{t_n} \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds - \sum_{j=1}^n k\mathcal{G}(\boldsymbol{\sigma}_{j-1}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1})) \right\|_Q^2.$$

Proof. First, we integrate the ordinary differential equation (11) between 0 and t_n to obtain

$$\boldsymbol{\sigma}_n = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \int_0^{t_n} \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds. \tag{19}$$

Then, we rewrite equation (16) in the form,

$$\boldsymbol{\sigma}_n^{hk} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}) + \boldsymbol{\sigma}_0^h - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h) + k\sum_{j=1}^{n} \mathcal{G}(\boldsymbol{\sigma}_{j-1}^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk})). \tag{20}$$

Plugging (19) into (12), for $\boldsymbol{w} = \boldsymbol{w}^h \in V^h \subset V$, and (20) into (17) and subtracting them, we have

$$\Big(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}) + \int_0^{t_n} \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds - k\sum_{j=1}^{n} \mathcal{G}(\boldsymbol{\sigma}_{j-1}^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk}))$$
$$+\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0 - \boldsymbol{u}_0^h), \boldsymbol{\varepsilon}(\boldsymbol{w}^h)\Big)_Q = 0 \quad \forall \boldsymbol{w}^h \in V^h.$$

Therefore,

$$\Big(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}) + \int_0^{t_n} \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds - k\sum_{j=1}^{n} \mathcal{G}(\boldsymbol{\sigma}_{j-1}^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk}))$$
$$+\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0 - \boldsymbol{u}_0^h), \boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\Big)_Q$$
$$= \Big(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}) + \int_0^{t_n} \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds - k\sum_{j=1}^{n} \mathcal{G}(\boldsymbol{\sigma}_{j-1}^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk}))$$
$$+\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0 - \boldsymbol{u}_0^h), \boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{w}^h)\Big)_Q = 0 \quad \forall \boldsymbol{w}^h \in V^h.$$

Keeping in mind that

$$(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}))_Q \geq m_{\mathcal{E}}\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2,$$

by using assumptions (7)-(10) and applying several times the inequality

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon}b^2, \quad a, b, \epsilon \in \mathbb{R},\ \epsilon > 0, \tag{21}$$

it follows that,

$$\begin{aligned}\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 \leq c\Big(&\|\boldsymbol{u}_n - \boldsymbol{w}^h\|_V^2 + I_{\mathcal{G}_n} + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + \|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h\|_Q^2 \\ &+k\sum_{j=1}^{n}[\|\boldsymbol{\sigma}_{j-1} - \boldsymbol{\sigma}_{j-1}^{hk}\|_Q^2 + \|\boldsymbol{u}_{j-1} - \boldsymbol{u}_{j-1}^{hk}\|_V^2]\Big) \quad \forall \boldsymbol{w} \in V^h.\end{aligned} \tag{22}$$

Finally, let us estimate the numerical error on the stress field. Substracting (19) and (20) we easily find that

$$\begin{aligned}\|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk}\|_Q \leq c\Big(&\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V + \|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h\|_Q + I_{\mathcal{G}_n} \\ &+k\sum_{j=1}^{n}[\|\boldsymbol{\sigma}_{j-1} - \boldsymbol{\sigma}_{j-1}^{hk}\|_Q^2 + \|\boldsymbol{u}_{j-1} - \boldsymbol{u}_{j-1}^{hk}\|_V^2]\Big).\end{aligned} \tag{23}$$

Combining now (22) and (23) and using a discrete version of Gronwall's inequality (see [7]), we deduce (18). □

We notice that the above error estimates are the basis for the analysis of the convergence rate of the algorithm. Hence, under additional regularity assumptions, we obtain the linear convergence of the algorithm that we state in the following.

Corollary 4.2. *Let the assumptions of Theorem 4.1 hold. Under the additional regularity conditions*

$$\boldsymbol{u} \in C([0,T];[H^2(\Omega)]^d), \quad \boldsymbol{\sigma}_0 \in [H^1(\Omega)]^{d\times d},$$

there exists a positive constant $c > 0$, independent of the discretization parameters h and k, such that

$$\max_{0\le n\le N} \{\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V + \|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk}\|_Q\} \le c(h+k). \tag{24}$$

The proof of the above corollary is obtained by using the well-known result on the approximation by finite elements (see [3]),

$$\max_{w_n^h\in V^h} \|\boldsymbol{u}_n - \boldsymbol{w}_n^h\|_V \le ch\|\boldsymbol{u}\|_{C([0,T];[H^2(\Omega)]^d)},$$

an straightforward estimate implies that

$$\max_{0\le n\le N} I_{\mathcal{G}_n} \le ck[\|\boldsymbol{u}\|_{C^1([0,T];V)} + \|\boldsymbol{\sigma}\|_{C^1([0,T];Q)}],$$

and, finally, by using the definition of the operators Π_{V^h} and Π_{Q^h}:

$$\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V \le ch\|\boldsymbol{u}_0\|_{[H^2(\Omega)]^d}, \quad \|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h\|_Q \le ch\|\boldsymbol{\sigma}_0\|_{[H^1(\Omega)]^{d\times d}}.$$

5. An a posteriori error analysis

In this section, we will use the finite element spaces and the notations introduced in the previous two sections. Moreover, throughout this section, we will assume that the mesh of the domain Ω may change during the time, and so, for any $0 < h < 1$ and for any $n = 0, 1, \ldots, N$, let $\mathcal{T}^{hn}$ be a mesh of $\overline{\Omega}$ composed of finite elements Tr with diameter less than h. We will also assume that, for each $n = 1, \ldots, N$, the mesh $\{(t_{n-1}, t_n) \times Tr\,;\, Tr \in \mathcal{T}^{hn}\}$ is regular in the sense of [3] and, to simplify the calculations, that $\mathcal{T}^{hn} \subset \mathcal{T}^{h(n-1)}$. Thus, for any $n = 1, \ldots, N$ and for any $Tr \in \mathcal{T}^{hn}$, let $h^n_{T_r}$ (respectively $\rho^n_{T_r}$) be the diameter of the smallest (resp. largest) ball containing (resp. contained in) $(t_{n-1}, t_n)\times Tr$. Therefore, there exists a positive constant β such that

$$\frac{h^n_{T_r}}{\rho^n_{T_r}} \le \beta \quad \forall Tr \in \mathcal{T}^{hn},\ n = 1, \ldots, N.$$

In order to simplify the writing and the calculations, in this section we assume that $\boldsymbol{f}_F = \boldsymbol{0}$ and therefore $(\boldsymbol{f}, \boldsymbol{w})_V = (\boldsymbol{f}, \boldsymbol{w})_H$, where $\boldsymbol{f} = \boldsymbol{f}_0 \in C([0,T];H)$. It is straightforward to extend the results presented below to more general situations. Moreover, the notation $a \lesssim b$ means that there exists a positive constant c independent of a and b (and of the discretization parameters) such that $a \le c\, b$. The notation $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$ hold simultaneously.

Let us define the continuous and piecewise linear approximations in time given by

$$\boldsymbol{u}^{h\tau}(\boldsymbol{x},t) = \frac{t-t_{n-1}}{k}\boldsymbol{u}_n^{hk}(\boldsymbol{x}) + \frac{t_n - t}{k}\boldsymbol{u}_{n-1}^{hk}(\boldsymbol{x}) \quad t_{n-1} \le t \le t_n, \quad \boldsymbol{x} \in \overline{\Omega},$$

$$\boldsymbol{\sigma}^{h\tau}(\boldsymbol{x},t) = \frac{t-t_{n-1}}{k}\boldsymbol{\sigma}_n^{hk}(\boldsymbol{x}) + \frac{t_n - t}{k}\boldsymbol{\sigma}_{n-1}^{hk}(\boldsymbol{x}) \quad t_{n-1} \le t \le t_n, \quad \boldsymbol{x} \in \overline{\Omega}.$$

Since $\dot{\boldsymbol{u}}^{h\tau} = \delta\boldsymbol{u}_n^{hk}$ and $\dot{\boldsymbol{\sigma}}^{h\tau} = \delta\boldsymbol{\sigma}_n^{hk}$, we can write discrete problem VP^{hk} in the following more general problem, for $n = 1, \ldots, N$,

$$\dot{\boldsymbol{\sigma}}^{h\tau} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau}) + \mathcal{G}(\boldsymbol{\sigma}_{n-1}^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk})), \tag{25}$$

$$(\boldsymbol{\sigma}^{h\tau}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q = (\boldsymbol{f}(t), \boldsymbol{w}^h)_H \quad \forall \boldsymbol{w}^h \in V^h,\ t_{n-1} \le t \le t_n. \tag{26}$$

Theorem 5.1. *Let assumptions (7)-(10) hold. Denote by $(\boldsymbol{u},\boldsymbol{\sigma})$ the solution to Problem VP and by $(\boldsymbol{u}^{h\tau},\boldsymbol{\sigma}^{h\tau})$ the continuous piecewise linear approximation of the solution to Problem VP^{hk}. Then*

$$\|\boldsymbol{u}-\boldsymbol{u}^{h\tau}\|_{C([0,T];V)}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}^{h\tau}\|_{C([0,T];Q)}\lesssim\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V+\|\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0^h\|_Q$$
$$+\sum_{n=1}^{N}k\eta_1^n+\max_{1\le n\le N}\max_{t\in[t_{n-1},t_n]}\eta_2^n(t)+\max_{1\le n\le N}\max_{t\in[t_{n-1},t_n]}\eta_3^n(t),$$

where the error estimators η_1^n, η_2^n and η_3^n are given by

$$\eta_1^n=\|\boldsymbol{\sigma}_n^{hk}-\boldsymbol{\sigma}_{n-1}^{hk}\|_Q+\|\boldsymbol{u}_n^{hk}-\boldsymbol{u}_{n-1}^{hk}\|_V,\tag{27}$$

$$\eta_2^n(t)=\left(\sum_{Tr\in\mathcal{T}^{hn}}|Tr|^2\|\boldsymbol{f}(t)\|^2_{[L^2(Tr)]^d}\right)^{1/2},\tag{28}$$

$$\eta_3^n(t)=\left(\sum_{Tr\in\mathcal{T}^{hn}}\sum_{E\in\mathcal{E}_{Tr}^{hn}}|E|\|[\boldsymbol{\sigma}^{h\tau}(t)\boldsymbol{\nu}_E]\|^2_{[L^2(E)]^d}\right)^{1/2},\tag{29}$$

and $\mathcal{E}_{Tr}^{hn}$ is the set of interior points, edges or faces of the element Tr, and $[\boldsymbol{\tau\nu}]$ denotes the jump of $\boldsymbol{\tau\nu}$ across the point, edge or face E.

Proof. First, let us estimate the error on the stress field. We then integrate (11) and (25) between t_{n-1} and $t\in(t_{n-1},t_n]$ to obtain

$$\boldsymbol{\sigma}(t)=\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))+\boldsymbol{\sigma}_{n-1}-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1})+\int_{t_{n-1}}^{t}\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,ds,$$
$$\boldsymbol{\sigma}^{h\tau}(t)=\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}^{h\tau}(t))+\boldsymbol{\sigma}_{n-1}^{hk}-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk})+\int_{t_{n-1}}^{t}\mathcal{G}(\boldsymbol{\sigma}_{n-1}^{hk},\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}))\,ds,$$

and therefore, by induction it follows that

$$\boldsymbol{\sigma}(t)=\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))+\boldsymbol{\sigma}_0-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0)+\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,ds$$
$$+\int_{t_{n-1}}^{t}\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,ds,\tag{30}$$

$$\boldsymbol{\sigma}^{h\tau}(t)=\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}^{h\tau}(t))+\boldsymbol{\sigma}_0^h-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h)+\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\mathcal{G}(\boldsymbol{\sigma}_{j-1}^{hk},\boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk}))\,ds$$
$$+\int_{t_{n-1}}^{t}\mathcal{G}(\boldsymbol{\sigma}_{n-1}^{hk},\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}))\,ds.\tag{31}$$

By subtracting now (30) and (31), we find that

$$\|\boldsymbol{\sigma}(t)-\boldsymbol{\sigma}^{h\tau}(t)\|_Q\lesssim\Big(\|\boldsymbol{u}(t)-\boldsymbol{u}^{h\tau}(t)\|_V+\|\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0^h\|_Q+\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V$$
$$+\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}[\|\boldsymbol{\sigma}(s)-\boldsymbol{\sigma}_{j-1}^{hk}\|_Q+\|\boldsymbol{u}(s)-\boldsymbol{u}_{j-1}^{hk}\|_V]\,ds$$
$$+\int_{t_{n-1}}^{t}[\|\boldsymbol{\sigma}(s)-\boldsymbol{\sigma}_{n-1}^{hk}\|_Q+\|\boldsymbol{u}(s)-\boldsymbol{u}_{n-1}^{hk}\|_V]\,ds\Big)\quad\forall t\in(t_{n-1},t_n].$$

Keeping in mind that

$$\|\boldsymbol{\sigma}(s)-\boldsymbol{\sigma}_{n-1}^{hk}\|_Q\le\|\boldsymbol{\sigma}(s)-\boldsymbol{\sigma}^{h\tau}(s)\|_Q+\|\boldsymbol{\sigma}^{h\tau}(s)-\boldsymbol{\sigma}_{n-1}^{hk}\|_Q,$$
$$\|\boldsymbol{u}(s)-\boldsymbol{u}_{n-1}^{hk}\|_V\le\|\boldsymbol{u}(s)-\boldsymbol{u}^{h\tau}(s)\|_V+\|\boldsymbol{u}^{h\tau}(s)-\boldsymbol{u}_{n-1}^{hk}\|_V,$$

$$\int_{t_{n-1}}^{t}\|\boldsymbol{\sigma}^{h\tau}(s)-\boldsymbol{\sigma}^{hk}_{n-1}\|_Q\,ds+\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\|\boldsymbol{\sigma}^{h\tau}(s)-\boldsymbol{\sigma}^{hk}_{j-1}\|_Q\,ds$$
$$\leq\sum_{j=1}^{n}k\|\boldsymbol{\sigma}^{hk}_j-\boldsymbol{\sigma}^{hk}_{j-1}\|_Q,$$
$$\int_{t_{n-1}}^{t}\|\boldsymbol{u}^{h\tau}(s)-\boldsymbol{u}^{hk}_{n-1}\|_V\,ds+\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\|\boldsymbol{u}^{h\tau}(s)-\boldsymbol{u}^{hk}_{j-1}\|_V\,ds$$
$$\leq\sum_{j=1}^{n}k\|\boldsymbol{u}^{hk}_j-\boldsymbol{u}^{hk}_{j-1}\|_V,$$

we immediately get

$$\begin{aligned}\|\boldsymbol{\sigma}(t)-\boldsymbol{\sigma}^{h\tau}(t)\|_Q\lesssim\Big(&\|\boldsymbol{u}(t)-\boldsymbol{u}^{h\tau}(t)\|_V+\|\boldsymbol{\sigma}_0-\boldsymbol{\sigma}^h_0\|_Q+\|\boldsymbol{u}_0-\boldsymbol{u}^h_0\|_V\\&+\int_0^t[\|\boldsymbol{\sigma}(s)-\boldsymbol{\sigma}^{h\tau}(s)\|_Q+\|\boldsymbol{u}(s)-\boldsymbol{u}^{h\tau}(s)\|_V]\,ds\\&+\sum_{j=1}^{n}k[\|\boldsymbol{\sigma}^{hk}_j-\boldsymbol{\sigma}^{hk}_{j-1}\|_Q+\|\boldsymbol{u}^{hk}_j-\boldsymbol{u}^{hk}_{j-1}\|_V]\Big)\quad\forall t\in(t_{n-1},t_n].\end{aligned}$$

Secondly, we estimate the numerical error on the displacement field. Then, we subtract equation (12) for $\boldsymbol{w}=\boldsymbol{w}^h\in V^h\subset V$ and equation (26) to obtain

$$(\boldsymbol{\sigma}-\boldsymbol{\sigma}^{h\tau},\boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q=0\quad\forall\boldsymbol{w}^h\in V^h.$$

Therefore,

$$(\boldsymbol{\sigma}-\boldsymbol{\sigma}^{h\tau},\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}^{h\tau}))_Q=(\boldsymbol{\sigma}-\boldsymbol{\sigma}^{h\tau},\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{w}^h))_Q\quad\forall\boldsymbol{w}^h\in V^h.\tag{32}$$

We consider the left-hand side of the previous equation. Using again equations (30) and (31) it leads to the following,

$$\begin{aligned}(\boldsymbol{\sigma}-\boldsymbol{\sigma}^{h\tau},\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}^{h\tau}))_Q&=(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}^{h\tau})),\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}^{h\tau}))_Q\\&+(\boldsymbol{\sigma}_0-\boldsymbol{\sigma}^h_0-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0-\boldsymbol{u}^h_0),\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}^{h\tau}))_Q\\&+\Big(\int_{t_{n-1}}^{t}[\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))-\mathcal{G}(\boldsymbol{\sigma}^{hk}_{n-1},\boldsymbol{\varepsilon}(\boldsymbol{u}^{hk}_{n-1}))]\,ds,\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}^{h\tau})\Big)_Q\\&+\sum_{j=1}^{n-1}\Big(\int_{t_{j-1}}^{t_j}[\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))-\mathcal{G}(\boldsymbol{\sigma}^{hk}_{j-1},\boldsymbol{\varepsilon}(\boldsymbol{u}^{hk}_{j-1}))]\,ds,\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}^{h\tau})\Big)_Q,\end{aligned}$$

and taking into account property (7) and the previous algebra, we have

$$\begin{aligned}&(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}^{h\tau})),\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}^{h\tau}))_Q\geq m_{\mathcal{E}}\|\boldsymbol{u}-\boldsymbol{u}^{h\tau}\|^2_V,\\&|(\boldsymbol{\sigma}_0-\boldsymbol{\sigma}^h_0-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0-\boldsymbol{u}^h_0),\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}^{h\tau}))_Q|\\&\qquad\lesssim(\|\boldsymbol{\sigma}_0-\boldsymbol{\sigma}^h_0\|_Q+\|\boldsymbol{u}_0-\boldsymbol{u}^h_0\|_V)\|\boldsymbol{u}-\boldsymbol{u}^{h\tau}\|_V,\end{aligned}$$

$$\begin{aligned}&\Big|\Big(\int_{t_{n-1}}^{t}[\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))-\mathcal{G}(\boldsymbol{\sigma}^{hk}_{n-1},\boldsymbol{\varepsilon}(\boldsymbol{u}^{hk}_{n-1}))]\,ds,\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}^{h\tau})\Big)_Q\Big|\\&\qquad+\Big|\Big(\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}[\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))-\mathcal{G}(\boldsymbol{\sigma}^{hk}_{j-1},\boldsymbol{\varepsilon}(\boldsymbol{u}^{hk}_{j-1}))]\,ds,\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}^{h\tau})\Big)_Q\Big|\\&\lesssim\Big(\int_0^t[\|\boldsymbol{u}(s)-\boldsymbol{u}^{h\tau}(s)\|_V+\|\boldsymbol{\sigma}(s)-\boldsymbol{\sigma}^{h\tau}(s)\|_Q]\,ds\\&\qquad+\sum_{j=1}^{n}k[\|\boldsymbol{\sigma}^{hk}_j-\boldsymbol{\sigma}^{hk}_{j-1}\|_Q+\|\boldsymbol{u}^{hk}_j-\boldsymbol{u}^{hk}_{j-1}\|_V]\Big)\|\boldsymbol{u}-\boldsymbol{u}^{h\tau}\|_V.\end{aligned}$$

Let $\boldsymbol{w} \in V$ and let Π_C^h be the Clément's interpolant on the triangulation $\mathcal{T}^{hn}$ (see [4]). We recall that this operator satisfies:

$$\|\boldsymbol{w} - \Pi_C^h \boldsymbol{w}\|_{[L^2(Tr)]^d} \lesssim |Tr| \|\boldsymbol{w}\|_{[H^1(\Delta Tr)]^d},$$
$$\|\boldsymbol{w} - \Pi_C^h \boldsymbol{w}\|_{[L^2(E)]^d} \lesssim |E|^{1/2} \|\boldsymbol{w}\|_{[H^1(\Delta Tr)]^d},$$

where ΔTr denotes the set of elements having a common vertex, edge or face with Tr, and E being a point, an edge or a face of Tr.

We consider now the right-hand side of equation (32) which equals to

$$(\boldsymbol{f}, \boldsymbol{u} - \boldsymbol{w}^h)_H - (\boldsymbol{\sigma}^{h\tau}, \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{w}^h))_Q.$$

Taking $\boldsymbol{w}^h = \boldsymbol{u}^{h\tau} + \Pi_C^h(\boldsymbol{u} - \boldsymbol{u}^{h\tau})$ in the previous expression, applying Green's formula on each triangle, and using the approximation properties of Π_C^h, we get

$$\begin{aligned}
&(\boldsymbol{f}, \boldsymbol{u} - \boldsymbol{w}^h)_H - (\boldsymbol{\sigma}^{h\tau}, \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{w}^h))_Q \\
&\quad = \sum_{Tr \in \mathcal{T}^{hn}} \int_{Tr} (\boldsymbol{f} + \operatorname{Div}(\boldsymbol{\sigma}^{h\tau})) \cdot (\boldsymbol{u} - \boldsymbol{u}^{h\tau} - \Pi_C^h(\boldsymbol{u} - \boldsymbol{u}^{h\tau}))\, dx \\
&\quad - \sum_{Tr \in \mathcal{T}^{hn}} \sum_{E \in \mathcal{E}_{Tr}^{hn}} \boldsymbol{\sigma}^{h\tau} \boldsymbol{\nu}_E \cdot (\boldsymbol{u} - \boldsymbol{u}^{h\tau} - \Pi_C^h(\boldsymbol{u} - \boldsymbol{u}^{h\tau})) \\
&\quad \lesssim \sum_{Tr \in \mathcal{T}^{hn}} \|\boldsymbol{f} + \operatorname{Div}(\boldsymbol{\sigma}^{h\tau})\|_{[L^2(Tr)]^d} \|\boldsymbol{u} - \boldsymbol{u}^{h\tau} - \Pi_C^h(\boldsymbol{u} - \boldsymbol{u}^{h\tau})\|_{[L^2(Tr)]^d} \\
&\quad + \sum_{Tr \in \mathcal{T}^{hn}} \sum_{E \in \mathcal{E}_{Tr}^{hn}} \|[\boldsymbol{\sigma}^{h\tau} \boldsymbol{\nu}_E]\|_{[L^2(E)]^d} \|\boldsymbol{u} - \boldsymbol{u}^{h\tau} - \Pi_C^h(\boldsymbol{u} - \boldsymbol{u}^{h\tau})\|_{[L^2(E)]^d} \\
&\quad \lesssim \Big(\sum_{Tr \in \mathcal{T}^{hn}} |Tr|^2 \|\boldsymbol{f} + \operatorname{Div}(\boldsymbol{\sigma}^{h\tau})\|^2_{[L^2(Tr)]^d} \Big)^{1/2} \Big(\sum_{Tr \in \mathcal{T}^{hn}} \|\boldsymbol{u} - \boldsymbol{u}^{h\tau}\|^2_{[H^1(\Delta Tr)]^d} \Big)^{1/2} \\
&\quad + \Big(\sum_{Tr \in \mathcal{T}^{hn}} \sum_{E \in \mathcal{E}_{Tr}^{hn}} |E| \|[\boldsymbol{\sigma}^{h\tau} \boldsymbol{\nu}_E]\|^2_{[L^2(E)]^d} \Big)^{1/2} \Big(\sum_{Tr \in \mathcal{T}^{hn}} \|\boldsymbol{u} - \boldsymbol{u}^{h\tau}\|^2_{[H^1(\Delta Tr)]^d} \Big)^{1/2} \\
&\quad \lesssim (\eta_2^n(t) + \eta_3^n(t)) \|\boldsymbol{u} - \boldsymbol{u}^{h\tau}\|_V,
\end{aligned}$$

where we take into account that $\operatorname{Div}(\boldsymbol{\sigma}^{h\tau}) = \boldsymbol{0}$ in Tr. Combining the previous estimates, we conclude that

$$\begin{aligned}
\|\boldsymbol{u}(t) - \boldsymbol{u}^{h\tau}(t)\|_V + \|\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}^{h\tau}(t)\|_Q &\lesssim \|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h\|_Q + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V \\
&+ \int_0^t [\|\boldsymbol{u}(s) - \boldsymbol{u}^{h\tau}(s)\|_V + \|\boldsymbol{\sigma}(s) - \boldsymbol{\sigma}^{h\tau}(s)\|_Q]\, ds \\
&+ \sum_{j=1}^n k[\|\boldsymbol{\sigma}_j^{hk} - \boldsymbol{\sigma}_{j-1}^{hk}\|_Q + \|\boldsymbol{u}_j^{hk} - \boldsymbol{u}_{j-1}^{hk}\|_V] + \eta_2^n(t) + \eta_3^n(t),
\end{aligned}$$

for all $t \in (t_{n-1}, t_n]$. Using Gronwall's inequality we find that

$$\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{u}^{h\tau}\|_{C([0,T];V)} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}^{h\tau}\|_{C([0,T];Q)} &\lesssim \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V + \|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h\|_Q \\
&+ \sum_{n=1}^N k\eta_1^n + \max_{1 \le n \le N} \max_{t \in [t_{n-1}, t_n]} \eta_2^n(t) + \max_{1 \le n \le N} \max_{t \in [t_{n-1}, t_n]} \eta_3^n(t),
\end{aligned}$$

which concludes the proof. □

Finally, we prove a lower bound for these error estimators that we provide in the following.

Theorem 5.2. *Let assumptions (7)-(10) hold. For all elements $Tr \in \mathcal{T}^{hn}$, the following local lower error bounds are obtained for $n = 1, \dots, N$:*

$$\begin{aligned}
\eta_{1Tr}^{hn} &\lesssim \|\boldsymbol{\sigma} - \boldsymbol{\sigma}^{h\tau}\|_{C([t_{n-1},t_n];[L^2(Tr)]^{d\times d})} + \|\boldsymbol{u} - \boldsymbol{u}^{h\tau}\|_{C([t_{n-1},t_n];[H^1(Tr)]^d)} \\
&\quad + \|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}\|_{[H^1(Tr)]^d} + \|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_{n-1}\|_{[L^2(Tr)]^{d\times d}}, \\
\eta_{2Tr}^{hn}(t) &\lesssim \|\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}^{h\tau}(t)\|_{[L^2(Tr)]^{d\times d}} \quad t \in (t_{n-1}, t_n], \\
\eta_{3Tr}^{hn}(t) &\lesssim \|\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}^{h\tau}(t)\|_{[L^2(\Delta Tr)]^{d\times d}} \quad t \in (t_{n-1}, t_n],
\end{aligned}$$

where we denote by η_{1Tr}^{hn}, η_{2Tr}^{hn} and η_{3Tr}^{hn} the local errors given by

$$\begin{aligned}
\eta_{1Tr}^{hn} &= \|\boldsymbol{\sigma}_n^{hk} - \boldsymbol{\sigma}_{n-1}^{hk}\|_{[L^2(Tr)]^{d\times d}} + \|\boldsymbol{u}_n^{hk} - \boldsymbol{u}_{n-1}^{hk}\|_{[H^1(Tr)]^d}, \\
\eta_{2Tr}^{hn}(t) &= |Tr| \|\boldsymbol{f}(t)\|_{[L^2(Tr)]^d}, \\
\eta_{3Tr}^{hn}(t) &= \Big(\sum_{E \in \mathcal{E}_{Tr}^{hn}} |E| \|[\boldsymbol{\sigma}^{h\tau}(t)\boldsymbol{\nu}_E]\|_{[L^2(E)]^d}^2 \Big)^{1/2}.
\end{aligned}$$

Obviously, we have

$$\begin{aligned}
\eta_1^n &\sim \Big(\sum_{Tr \in \mathcal{T}^{hn}} (\eta_{1Tr}^{hn})^2 \Big)^{1/2}, \\
\eta_2^n &= \Big(\sum_{Tr \in \mathcal{T}^{hn}} (\eta_{2Tr}^{hn})^2 \Big)^{1/2}, \\
\eta_3^n &= \Big(\sum_{Tr \in \mathcal{T}^{hn}} (\eta_{3Tr}^{hn})^2 \Big)^{1/2}.
\end{aligned}$$

Proof. First, let us bound the error estimator η_1^n. We have

$$\begin{aligned}
\eta_1^n &= \|\boldsymbol{\sigma}_n^{hk} - \boldsymbol{\sigma}_{n-1}^{hk}\|_Q + \|\boldsymbol{u}_n^{hk} - \boldsymbol{u}_{n-1}^{hk}\|_V \\
&\le \|\boldsymbol{\sigma}^{h\tau}(t_n) - \boldsymbol{\sigma}(t_n)\|_Q + \|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_{n-1}\|_Q + \|\boldsymbol{\sigma}(t_{n-1}) - \boldsymbol{\sigma}^{h\tau}(t_{n-1})\|_Q \\
&\quad + \|\boldsymbol{u}^{h\tau}(t_n) - \boldsymbol{u}(t_n)\|_V + \|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}\|_V + \|\boldsymbol{u}(t_{n-1}) - \boldsymbol{u}^{h\tau}(t_{n-1})\|_V \\
&\le \|\boldsymbol{\sigma}^{h\tau} - \boldsymbol{\sigma}\|_{C([t_{n-1},t_n];Q)} + \|\boldsymbol{u}^{h\tau} - \boldsymbol{u}\|_{C([t_{n-1},t_n];V)} \\
&\quad + \|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}\|_V + \|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_{n-1}\|_Q,
\end{aligned}$$

and therefore,

$$\begin{aligned}
(\eta_1^n)^2 \lesssim \sum_{Tr \in \mathcal{T}^{hn}} \Big(&\|\boldsymbol{\sigma}^{h\tau} - \boldsymbol{\sigma}\|_{C([t_{n-1},t_n];[L^2(Tr)]^{d\times d})}^2 + \|\boldsymbol{u}^{h\tau} - \boldsymbol{u}\|_{C([t_{n-1},t_n];[H^1(Tr)]^d)}^2 \\
&+ \|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}\|_{[H^1(Tr)]^d}^2 + \|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_{n-1}\|_{[L^2(Tr)]^{d\times d}}^2 \Big).
\end{aligned}$$

Proceeding in a similar way we also obtain that

$$\begin{aligned}
\eta_{1Tr}^{nh} &\lesssim \|\boldsymbol{\sigma}^{h\tau} - \boldsymbol{\sigma}\|_{C([t_{n-1},t_n];[L^2(Tr)]^{d\times d})}^2 + \|\boldsymbol{u}^{h\tau} - \boldsymbol{u}\|_{C([t_{n-1},t_n];[H^1(Tr)]^d)}^2 \\
&\quad + \|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}\|_{[H^1(Tr)]^d}^2 + \|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_{n-1}\|_{[L^2(Tr)]^{d\times d}}^2.
\end{aligned}$$

We estimate now η_2^n. Let w_{Tr} be the bubble function associated with the element Tr (for instance, in the two-dimensional setting, we have $w_{Tr} = \lambda_{a1}\lambda_{a2}\lambda_{a3}$, where λ_{ai}, $i = 1, 2, 3$, denote the barycentric coordinates and $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$ are the three nodes of the element Tr). We notice that $w_{Tr} \in H_0^1(Tr)$. Let us define $\boldsymbol{w}_{Tr} = (w_i)_{i=1}^d \in [H_0^1(Tr)]^d$ which is constructed as $w_i = w_{Tr}$ for $i = 1, \dots, d$.

It is easy to check that the function $\boldsymbol{\psi}_{Tr} = \boldsymbol{w}_{Tr} \cdot \boldsymbol{f}$ verifies (see [12]),

$$\|\boldsymbol{f}\|_{[L^2(Tr)]^d}^2 \lesssim \int_{Tr} (\boldsymbol{\sigma} - \boldsymbol{\sigma}^{h\tau}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{\psi}_{Tr})\, d\boldsymbol{x}.$$

Using the inverse inequality, we find that

$$\|\boldsymbol{\varepsilon}(\boldsymbol{\psi}_{Tr})\|_{[L^2(Tr)]^{d\times d}} \lesssim |Tr|^{-1} \|\boldsymbol{\psi}_{Tr}\|_{[L^2(Tr)]^d},$$

and therefore,

$$\|\boldsymbol{f}\|_{[L^2(Tr)]^d} \lesssim |Tr|^{-1}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}^{h\tau}\|_{[L^2(Tr)]^{d\times d}}. \tag{33}$$

Thus, it only remains to estimate η_3^n. Proceeding in a similar way that in the previous estimate, let us consider the bubble function w_E associated with the point, edge or face E. Hence, taking now $\boldsymbol{w}_E=[w_E]^d$ we deduce that (see again [12]),

$$\|[\boldsymbol{\sigma}^{h\tau}\boldsymbol{\nu}_E]\|^2_{[L^2(E)]^d} \lesssim |E|^{-1}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}^{h\tau}\|_{[L^2(\Delta Tr)]^{d\times d}}\|\boldsymbol{\psi}_E\|_{[L^2(\Delta Tr)]^d},$$

where $\boldsymbol{\psi}_E=\boldsymbol{w}_E\cdot[\boldsymbol{\sigma}^{h\tau}\boldsymbol{\nu}_E]$ and ΔTr stands for the set of elements of $\mathcal{T}^{hn}$ sharing the common point, edge or face E. From the definition of $\boldsymbol{\psi}_E$, it follows that $\|\boldsymbol{\psi}_E\|_{[L^2(\Delta Tr)]^d} \lesssim |E|^{1/2}\|[\boldsymbol{\sigma}^{h\tau}\boldsymbol{\nu}_E]\|_{[L^2(E)]^d}$, and we conclude that

$$\|[\boldsymbol{\sigma}^{h\tau}\boldsymbol{\nu}_E]\|_{[L^2(E)]^d} \lesssim |E|^{-1/2}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}^{h\tau}\|_{[L^2(\Delta Tr)]^{d\times d}},$$

which implies, for all $Tr\in\mathcal{T}^{hn}$,

$$\Big(\sum_{E\in\mathcal{E}^{hn}_{Tr}}|E|\|[\boldsymbol{\sigma}^{h\tau}\boldsymbol{\nu}_E]\|^2_{[L^2(E)]^d}\Big)^{1/2} \lesssim \|\boldsymbol{\sigma}-\boldsymbol{\sigma}^{h\tau}\|_{[L^2(\Delta Tr)]^{d\times d}}.$$

Combining all these results and taking into account the definitions (27), (28) and (29), we obtain the desired lower error bounds. □

We observe that, from Theorem 5.2, we can prove a similar convergence order than in the a priori error analysis that we state in the following.

Corollary 5.3. *Let assumptions (7)-(10) hold. If the continuous solution has the following additional regularity:*

$$\boldsymbol{u}\in C([0,T];[H^2(\Omega)]^d),\quad \boldsymbol{\sigma}_0\in[H^1(\Omega)]^{d\times d},$$

we have

$$\sum_{n=1}^N k\eta_1^n+\|\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0^h\|_Q+\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V+\max_{1\le n\le N}\max_{t\in[t_{n-1},t_n]}\eta_2^n + \max_{1\le n\le N}\max_{t\in[t_{n-1},t_n]}\eta_3^n\le c(h+k),$$

for a positive constant c which depends on the given data and the continuous solution $(\boldsymbol{u},\boldsymbol{\sigma})$.

Proof. Using estimates (24), under the required regularity we conclude that

$$\|\boldsymbol{u}-\boldsymbol{u}^{h\tau}\|_{C([0,T];V)}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}^{h\tau}\|_{C([0,T];Q)}\le c(h+k), \tag{34}$$

which implies that

$$\max_{1\le n\le N}\max_{t\in[t_{n-1},t_n]}\eta_2^n+\max_{1\le n\le N}\max_{t\in[t_{n-1},t_n]}\eta_3^n\le c(h+k).$$

¿From the regularity $\boldsymbol{u}\in C^1([0,T];V)$ and $\boldsymbol{\sigma}\in C^1([0,T];Q)$ (see Theorem 2.1), we easily find that

$$\sum_{n=1}^N k\,[\|\boldsymbol{u}_n-\boldsymbol{u}_{n-1}\|_V+\|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_{n-1}\|_Q]\le ck,$$

and using again (34), it follows that

$$\sum_{n=1}^N k\eta_1^n\le c(h+k).$$

Finally, we only need to estimate the numerical error on the approximation of the initial conditions. From the definition of the finite element projection operator Π_{V^h} (see [3]) and the projection operator Π_{Q^h} we have,

$$\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V \leq ch\|\boldsymbol{u}_0\|_{[H^2(\Omega)]^d}, \quad \|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h\|_Q \leq ch\|\boldsymbol{\sigma}_0\|_{[H^1(\Omega)]^{d\times d}}.$$

This concludes the proof. □

Acknowledgments

The work of J.R. Fernández was partially supported by the Ministerio de Educación y Ciencia (Project MTM2006-13981) and the work of P. Hild was supported by l'Agence Nationale de la Recherche (Project ANR-05-JCJC-0182-01).

References

[1] A. Bergam, C. Bernardi and Z. Mghazli, A posteriori analysis of the finite element discretization of some parabolic equations, Math. Comp. 74(251) (2005) 1117–1138

[2] C. Bernardi and R. Verfürth, A posteriori error analysis of the fully discretized time-dependent Stokes equations, M2AN Math. Model. Numer. Anal. 38(3) (2004) 437–455.

[3] P.G. Ciarlet, The finite element method for elliptic problems, in: (P.G. Ciarlet and J.L. Lions Eds.), Handbook of Numerical Analysis, Vol. II (North Holland, 1991) 17–352.

[4] P. Clément, Approximation by finite element functions using local regularization, RAIRO Anal. Numér. 9 (1975) 77–84.

[5] N. Cristescu and I. Suliciu, Viscoplasticity, Mechanics of Plastic Solids, 5. Martinus Nijhoff Publishers, The Hague, 1982.

[6] I.R. Ionescu, Error estimates of an Euler method for a quasistatic elastic-visco-plastic problem, Z. Angew. Math. Mech. 70(3) (1990) 173–180.

[7] W. Han and M. Sofonea, Quasistatic contact problems in viscoelasticity and viscoplasticity, American Mathematical Society-International Press, 2002.

[8] P. Hild and S. Nicaise, Residual a posteriori error estimators for contact problems in elasticity, M2AN Math. Model. Numer. Anal. 41(5) (2007) 897–923.

[9] I.R. Ionescu and M. Sofonea, Functional and numerical methods in viscoplasticity, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993.

[10] S. Nicaise and N. Soualem, A posteriori error estimates for a nonconforming finite element discretization of the heat equation, M2AN Math. Model. Numer. Anal. 39(2) (2005) 319–348.

[11] M. Picasso, Adaptive finite elements for a linear parabolic problem, Comput. Methods Appl. Mech. Engrg. 167(3-4) (1998) 223–237.

[12] R. Verfürth, A review of a posteriori error estimation and adaptive mesh-refinement techniques, Wiley and Teubner, 1996.

[13] R. Verfürth, A posteriori error estimates for finite element discretizations of the heat equation, Calcolo 40(3) (2003) 195–212.

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