

# A mixed finite element method and solution multiplicity for Coulomb frictional contact

Riad HASSANI <sup>1</sup>, Patrick HILD <sup>2</sup>, Ioan IONESCU <sup>3\*</sup>, Nour-Dine SAKKI <sup>3</sup>

<sup>1</sup>*Laboratoire de Géophysique Interne et Tectonophysique,  
Université de Savoie / CNRS UMR 5559,  
73376 Le Bourget du Lac, France.*

<sup>2</sup>*Laboratoire de Mathématiques de Besançon,  
Université de Franche-Comté / CNRS UMR 6623,  
16 route de Gray, 25030 Besançon, France.*

<sup>3</sup>*Laboratoire de Mathématiques,  
Université de Savoie / CNRS UMR 5127,  
73376 Le Bourget du Lac, France.*

## Abstract

This paper is concerned with the discrete contact problem governed by Coulomb's friction law. We propose and study a new technique using mixed finite elements with two multipliers in order to determine numerically critical friction coefficients for which multiple solutions to the friction problem exist. The framework is based on eigenvalue problems and it allows to exhibit non-uniqueness cases involving an infinity of solutions located on a continuous branch. The theory is illustrated with several computations which clearly show the accuracy of the proposed method.

*Keywords* : Coulomb friction, contact problem, mixed finite elements, eigenvalue problem, solution multiplicity.

## 1. Introduction

Friction is one of the most basic phenomena arising in mechanics. The work in this paper is concerned with an investigation of the well-known Coulomb friction model in static or quasi-static elasticity (see [6, 16, 9]). Although quite simple in its formulation, the Coulomb friction law shows great mathematical difficulties which have not allowed a complete understanding of the model. In continuum elastostatics, only

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\*Corresponding author. Tel: +33.4.79.75.86.42; fax: +33.4.79.75.81.42. E-mail addresses: hassani@univ-savoie.fr (R. Hassani), hild@math.univ-fcomte.fr (P. Hild), ionescu@univ-savoie.fr (I. Ionescu), nsakk@univ-savoie.fr (N.-D. Sakki)

existence results for small friction are established (see [18, 15, 7]). The corresponding finite element problem admits always a solution which is unique provided that the friction coefficient is lower than a critical value vanishing when the discretization parameter tends to zero (see [8, 9]). In [13] an elementary example involving one finite element shows that the problem can admit one, multiple or an infinity of solutions located on a continuous branch and that the number of solutions can eventually decrease when the friction coefficient increases. Such an example in the finite element context completes the results using truss elements in the static or quasi-static cases (see [14, 2, 17, 3]).

Our aim in this paper is to propose and to study a framework for the finite element problem based on the ideas introduced for the continuous model in [10, 11] in order to obtain explicit examples of non-uniqueness. Our method involves finite element eigenvalue problems written in a mixed form. We show that the real eigenvalues of the latter problem are precisely critical friction coefficients for which multiple solutions to the Coulomb frictional contact problem exist. The loss of uniqueness for a specific friction coefficient has to be analyzed in the context of a varying friction coefficient during the quasi-static slip.

In section 2, we recall the continuous model which is discretized using mixed finite elements. An eigenvalue problem is introduced in section 3 and we prove that if a real eigenvalue exists then the problem is open to non-uniqueness. More precisely, if the friction coefficient has a critical value then there exist an infinity of solutions located on a continuous branch. Section 4 is concerned with some analytical calculus of eigenvalues on elementary finite element meshes. In the case of a single finite element mesh, the eigenvalue (i.e., the critical friction coefficient) is a bifurcation point. In section 5, the computations with arbitrary meshes and different finite elements clearly show that the convergence of the discrete eigenvalue problem is quite satisfactory independently of the degree and the type of the elements. Moreover, we observe numerically that there always exist at least a real limit for some discrete eigenvalues as the discretization parameter vanishes. Such a limit depends only on the geometry of the material, the partition of the boundary of the body into Dirichlet, Neumann and frictional contact conditions and on the Poisson ratio. Further computations show that such limits can be very small on specific geometries. Practically we explain how a simple non-uniqueness example can be always constructed using a critical friction coefficient.

## 2. The continuous and the discrete problems

### 2.1. The continuous problem

We consider the deformation of an elastic body occupying, in the initial unconstrained configuration a domain  $\Omega$  in  $\mathbb{R}^2$  where plane strain assumptions are assumed. The Lipschitz boundary  $\partial\Omega$  of  $\Omega$  consists of  $\Gamma_D, \Gamma_N$  and  $\Gamma_C$  where the measure of  $\Gamma_D$  does not vanish. The body  $\Omega$  is submitted to given displacements  $\mathbf{U}$  on  $\Gamma_D$  and subjected to surface traction forces  $\mathbf{F}$  on  $\Gamma_N$ ; the body forces are denoted  $\mathbf{f}$ . In the

initial configuration, the part  $\Gamma_C$  is a straight line segment considered as the candidate contact surface on a rigid foundation for the sake of simplicity which means that the contact zone cannot enlarge during the deformation process. The contact is assumed to be frictional and the stick, slip and separation zones on  $\Gamma_C$  are not known in advance. We denote by  $\mu > 0$  the given friction coefficient on  $\Gamma_C$ . The unit outward normal and tangent vectors of  $\partial\Omega$  are  $\mathbf{n} = (n_1, n_2)$  and  $\mathbf{t} = (-n_2, n_1)$  respectively.

The contact problem with Coulomb's friction law consists of finding the displacement field  $\mathbf{u} : \Omega \rightarrow \mathbb{R}^2$  satisfying (2.1)–(2.6):

$$\operatorname{div} \boldsymbol{\sigma}(\mathbf{u}) + \mathbf{f} = \mathbf{0} \quad \text{in } \Omega, \quad (2.1)$$

$$\boldsymbol{\sigma}(\mathbf{u}) = \mathcal{C} \boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in } \Omega, \quad (2.2)$$

$$\mathbf{u} = \mathbf{U} \quad \text{on } \Gamma_D, \quad (2.3)$$

$$\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} = \mathbf{F} \quad \text{on } \Gamma_N. \quad (2.4)$$

The notation  $\boldsymbol{\sigma}(\mathbf{u}) : \Omega \rightarrow \mathcal{S}_2$  represents the stress tensor field lying in  $\mathcal{S}_2$ , the space of second order symmetric tensors on  $\mathbb{R}^2$ . The linearized strain tensor field is  $\boldsymbol{\varepsilon}(\mathbf{u}) = (\nabla \mathbf{u} + \nabla^T \mathbf{u})/2$  and  $\mathcal{C}$  is the fourth order symmetric and elliptic tensor of linear elasticity.

Afterwards we adopt the following notation for any displacement field  $\mathbf{u}$  and for any density of surface forces  $\boldsymbol{\sigma}(\mathbf{u})\mathbf{n}$  defined on  $\Gamma_C$ :

$$\mathbf{u} = u_n \mathbf{n} + u_t \mathbf{t} \quad \text{and} \quad \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} = \sigma_n(\mathbf{u})\mathbf{n} + \sigma_t(\mathbf{u})\mathbf{t}.$$

On  $\Gamma_C$ , the three conditions representing unilateral contact are given by

$$u_n \leq 0, \quad \sigma_n(\mathbf{u}) \leq 0, \quad \sigma_n(\mathbf{u}) u_n = 0, \quad (2.5)$$

and the Coulomb friction law is summarized by the following conditions:

$$\begin{cases} u_t = u_t^r \implies |\sigma_t(\mathbf{u})| \leq \mu |\sigma_n(\mathbf{u})|, \\ u_t \neq u_t^r \implies \sigma_t(\mathbf{u}) = -\mu |\sigma_n(\mathbf{u})| \frac{u_t - u_t^r}{|u_t - u_t^r|}, \end{cases} \quad (2.6)$$

where  $\mathbf{u}^r$  is the reference displacement and  $u_t - u_t^r$  is the slip. Two choices of  $\mathbf{u}^r$  are more used in literature. The first one is  $\mathbf{u}^r \equiv \mathbf{0}$  for the static case. The second one is used in the incremental formulation of a quasi-static process (see [4]). Indeed, if  $\Delta t$  is the time step then  $\mathbf{u}$  stands for  $\mathbf{u}((i+1)\Delta t)$ ,  $\mathbf{u}^r = \mathbf{u}(i\Delta t)$  and  $\mathbf{f}, \mathbf{F}, \mathbf{U}$  have to be replaced by  $\mathbf{f}((i+1)\Delta t), \mathbf{F}((i+1)\Delta t), \mathbf{U}((i+1)\Delta t)$ .

The variational formulation of problem (2.1)–(2.6) in its mixed form consists of finding  $(\mathbf{u}, \lambda_n, \lambda_t) \in \mathbf{U}_{ad} \times M_n \times M_t(-\mu\lambda_n) = \mathbf{U}_{ad} \times \mathbf{M}(-\mu\lambda_n)$  which satisfy:

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) - \int_{\Gamma_C} \lambda_n v_n \, d\Gamma - \int_{\Gamma_C} \lambda_t v_t \, d\Gamma = L(\mathbf{v}), & \forall \mathbf{v} \in \mathbf{V}, \\ \int_{\Gamma_C} (\nu_n - \lambda_n) u_n \, d\Gamma + \int_{\Gamma_C} (\nu_t - \lambda_t)(u_t - u_t^r) \, d\Gamma \geq 0, & \forall (\nu_n, \nu_t) \in \mathbf{M}(-\mu\lambda_n), \end{cases} \quad (2.7)$$

where  $\mathbf{M}(-\mu\lambda_n) = M_n \times M_t(-\mu\lambda_n)$  is defined next. We set

$$M_n = \left\{ \nu; \nu \in H^{-\frac{1}{2}}(\Gamma_C), \nu \leq 0 \text{ on } \Gamma_C \right\},$$

and, for any  $g \in -M_n$

$$M_t(g) = \left\{ \nu; \nu \in H^{-\frac{1}{2}}(\Gamma_C), -g \leq \nu \leq g \text{ on } \Gamma_C \right\},$$

where  $H^{-\frac{1}{2}}(\Gamma_C)$  is the dual space of  $H^{\frac{1}{2}}(\Gamma_C)$  (see [1]) and the inequality conditions incorporated in the definitions of  $M_n$  and  $M_t(g)$  have to be understood in the dual sense.

In (2.7), the standard notations are adopted

$$a(\mathbf{u}, \mathbf{v}) = \int_{\Omega} (\mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u})) : \boldsymbol{\varepsilon}(\mathbf{v}) \, d\Omega, \quad L(\mathbf{v}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\Omega + \int_{\Gamma_N} \mathbf{F} \cdot \mathbf{v} \, d\Gamma,$$

for any  $\mathbf{u}$  and  $\mathbf{v}$  in the Sobolev space  $(H^1(\Omega))^2$ . In these definitions the notations  $\cdot$  and  $:$  represent the canonical inner products in  $\mathbb{R}^2$  and  $\mathcal{S}_2$  respectively.

In (2.7),  $\mathbf{V}$  and  $\mathbf{U}_{ad}$  denote following sets of displacement fields :

$$\mathbf{V} = \left\{ \mathbf{v} \in (H^1(\Omega))^2; \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D \right\}, \quad \mathbf{U}_{ad} = \left\{ \mathbf{v} \in (H^1(\Omega))^2; \mathbf{v} = \mathbf{U} \text{ on } \Gamma_D \right\}.$$

It is easy to see that if  $(\mathbf{u}, \lambda_n, \lambda_t)$  is a solution of (2.7), then  $\lambda_n = \sigma_n(\mathbf{u})$  and  $\lambda_t = \sigma_t(\mathbf{u})$ .

## 2.2. Finite element approximation

The body  $\Omega$  is discretized by using a family of triangulations  $(\mathcal{T}_h)_h$  made of finite elements of degree  $k \geq 1$  where  $h > 0$  is the discretization parameter representing the greatest diameter of a triangle in  $\mathcal{T}_h$ . The set approximating  $\mathbf{V}$  becomes:

$$\mathbf{V}_h = \left\{ \mathbf{v}_h; \mathbf{v}_h \in (C(\overline{\Omega}))^2, \mathbf{v}_h|_T \in (P_k(T))^2 \, \forall T \in \mathcal{T}_h, \mathbf{v}_h = \mathbf{0} \text{ on } \Gamma_D \right\},$$

where  $C(\overline{\Omega})$  stands for the space of continuous functions on  $\overline{\Omega}$  and  $P_k(T)$  represents the space of polynomial functions of degree  $k$  on  $T$ . Let us mention that we focus on the discrete problem and that any discussion concerning the convergence of the finite element problem towards the continuous model is out of the scope of this paper.

Let the notation  $\mathbf{U}_h$  stand for a convenient approximation of  $\mathbf{U}$  on  $\Gamma_D$ . On the boundary of  $\Omega$ , we still keep the notation  $\mathbf{v}_h = v_{hn}\mathbf{n} + v_{ht}\mathbf{t}$  for every  $\mathbf{v}_h \in \mathbf{V}_h$  and we denote by  $(T_h)_h$  the family of monodimensional meshes on  $\Gamma_C$  inherited by  $(\mathcal{T}_h)_h$ . Set

$$W_h = \left\{ \nu; \nu = \mathbf{v}_h|_{\Gamma_C} \cdot \mathbf{n}, \mathbf{v}_h \in \mathbf{V}_h \right\},$$

which is included in the space of continuous functions on  $\Gamma_C$  which are piecewise of degree  $k$  on  $(T_h)_h$  and coincides with the latter space when  $\overline{\Gamma}_C \cap \overline{\Gamma}_N = \emptyset$ .

We denote by  $p$  the dimension of  $W_h$  and by  $\psi_i, 1 \leq i \leq p$  the corresponding canonical finite element basis functions of degree  $k$ . For all  $\nu \in W_h$  we shall denote by  $F(\nu) = (F_i(\nu))_{1 \leq i \leq p}$  the generalized loads at the nodes of  $\Gamma_C$ :

$$F_i(\nu) = \int_{\Gamma_C} \nu \psi_i, \quad \forall 1 \leq i \leq p.$$

We next introduce the sets of Lagrange multipliers:

$$M_{hn} = \left\{ \nu; \nu \in W_h, F_i(\nu) \leq 0, \quad \forall 1 \leq i \leq p \right\}$$

and, for any  $g \in -M_{hn}$

$$M_{ht}(g) = \left\{ \nu; \nu \in W_h, |F_i(\nu)| \leq F_i(g), \quad \forall 1 \leq i \leq p \right\}.$$

Hence, the discrete problem issued from (2.7) becomes: find  $(\mathbf{u}_h, \lambda_{hn}, \lambda_{ht}) \in \mathbf{U}_{ad,h} \times M_{hn} \times M_{ht}(-\mu\lambda_{hn}) = \mathbf{U}_{ad,h} \times \mathbf{M}_h(-\mu\lambda_{hn})$  such that

$$\left\{ \begin{array}{l} a(\mathbf{u}_h, \mathbf{v}_h) - \int_{\Gamma_C} \lambda_{hn} v_{hn} d\Gamma - \int_{\Gamma_C} \lambda_{ht} v_{ht} d\Gamma = L(\mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ \int_{\Gamma_C} (\nu_{hn} - \lambda_{hn}) u_{hn} d\Gamma + \int_{\Gamma_C} (\nu_{ht} - \lambda_{ht})(u_{ht} - u_t^r) d\Gamma \geq 0, \\ \forall (\nu_{hn}, \nu_{ht}) \in \mathbf{M}_h(-\mu\lambda_{hn}). \end{array} \right. \quad (2.8)$$

where

$$\mathbf{U}_{ad,h} = \left\{ \mathbf{v}_h; \mathbf{v}_h \in (C(\bar{\Omega}))^2, \mathbf{v}_h|_T \in (P_k(T))^2 \quad \forall T \in \mathcal{T}_h, \mathbf{v}_h = \mathbf{U}_h \text{ on } \Gamma_D \right\}$$

and  $\mathbf{U}_h$  denotes a convenient approximation of  $\mathbf{U}$  on  $\Gamma_D$ .

Let  $U_n = (U_n)_i$ ,  $U_t = (U_t)_i$  and  $U_t^r = (U_t^r)_i$ ,  $1 \leq i \leq p$  denote the vectors of components the nodal values on  $\Gamma_C$  of  $u_{hn}, u_{ht}$  and  $u_t^r$  respectively. It can be easily checked (see [5]) that the vector formulation of the frictional contact conditions incorporated in the inequality of (2.8) are:

$$F_i(\lambda_{hn}) \leq 0, \quad (U_n)_i \leq 0, \quad F_i(\lambda_{hn}) (U_n)_i = 0, \quad 1 \leq i \leq p, \quad (2.9)$$

$$|F_i(\lambda_{ht})| \leq -\mu F_i(\lambda_{hn}), \quad F_i(\lambda_{ht})(U_t - U_t^r)_i \leq 0, \quad 1 \leq i \leq p, \quad (2.10)$$

$$|F_i(\lambda_{ht})| < -\mu F_i(\lambda_{hn}) \implies (U_t - U_t^r)_i = 0, \quad 1 \leq i \leq p. \quad (2.11)$$

**Proposition 2.1** *For any positive  $\mu$ , there exists a solution to Coulomb's discrete frictional contact problem (2.8).*

**Proof.** See [5], Proposition 3.2.  $\square$

### 3. A finite element eigenvalue approach for solution multiplicity

Let us consider a solution  $(\mathbf{u}_h, \lambda_{hn}, \lambda_{ht}) \in \mathbf{V}_h \times \mathbf{M}_h(-\mu\lambda_{hn})$  of the discrete Coulomb frictional contact problem (2.8). Then we denote by  $I_f$ ,  $I_s$  and  $I_c$  the set of nodes of

$\Gamma_C$  which are currently free (separated from the rigid foundation), the set of nodes of  $\Gamma_C$  which are stuck to the rigid foundation, and the set of nodes of  $\Gamma_C$  which are currently in contact but are candidate to slip, respectively. In other words, if  $p = \dim(W_h)$  denotes the number of nodes belonging to  $\Gamma_C$ , we can write

$$\begin{aligned} I_f &= \left\{ i \in [1, p]; (U_n)_i < 0 \right\}, \\ I_s &= \left\{ i \in [1, p]; (U_n)_i = 0, |F_i(\lambda_{ht})| < -\mu F_i(\lambda_{hn}) \right\}, \\ I_c &= \left\{ i \in [1, p]; (U_n)_i = 0, |F_i(\lambda_{ht})| = -\mu F_i(\lambda_{hn}) \right\}. \end{aligned}$$

Henceforth, we assume that all the nodes of  $I_c$  are slipping (not necessarily in the same direction), i.e.,

$$(U_t - U_t^r)_i \neq 0, \quad \forall i \in I_c, \quad (3.1)$$

and we denote by

$$\gamma_i = \frac{(U_t - U_t^r)_i}{|(U_t - U_t^r)_i|}, \quad \forall i \in I_c,$$

the sign of the slip at node number  $i$  of  $I_c$ . Next we consider the following eigenvalue problem:

**Eigenvalue problem.** Find the eigenvalue  $\alpha_h \in \mathbb{C}$  and the corresponding eigenfunction(s)  $(\mathbf{0}, 0, 0) \neq (\varphi_h, \theta_{hn}, \theta_{ht}) \in \mathbf{V}_h \times W_h \times W_h$  such that

$$\left\{ \begin{array}{l} a(\varphi_h, \mathbf{v}_h) - \int_{\Gamma_C} \theta_{hn} v_{hn} d\Gamma - \int_{\Gamma_C} \theta_{ht} v_{ht} d\Gamma = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ (\Phi_n)_i = (\Phi_t)_i = 0, \quad \forall i \in I_s, \\ F_i(\theta_{hn}) = F_i(\theta_{ht}) = 0, \quad \forall i \in I_f, \\ (\Phi_n)_i = 0, \quad F_i(\theta_{ht}) = \alpha_h F_i(\theta_{hn}) \gamma_i, \quad \forall i \in I_c, \end{array} \right. \quad (3.2)$$

where  $\Phi_n$  and  $\Phi_t$  denote the vectors of the normal and tangential components, respectively, of  $\varphi_h$  on  $\Gamma_C$ .

**Proposition 3.1** *Let  $p^0$  be the number of nodes belonging to  $I_c$ . Then problem (3.2) admits exactly  $p^0$  eigenvalues  $\alpha_h$  and eigenfunctions  $(\varphi_h, \theta_{hn}, \theta_{ht})$ .*

**Proof.** We number as follows the basis functions of  $\mathbf{V}_h$ : the normal displacement basis functions on  $\Gamma_C$  from 1 to  $p$  (those corresponding to  $I_c$  from 1 to  $p^0$ ), the tangential displacement basis functions on  $\Gamma_C$  from  $p+1$  to  $2p$  (those corresponding to  $I_c$  from  $p+1$  to  $p+p^0$ ) and the basis functions of interior nodes from  $2p+1$  to  $m = \dim(\mathbf{V}_h)$ . Let us mention that the first equation in (3.2) can be written as follows

$$K\Phi - \begin{pmatrix} F(\theta_{hn}) \\ F(\theta_{ht}) \\ 0 \end{pmatrix} = 0$$

where  $K$  denotes the stiffness matrix of order  $m$  and  $\Phi$  denotes the vector associated with  $\varphi_h$ .

Now we consider the following problem which for a given  $r = (r_i)_i \in \mathbb{R}^{p^0}$  consists of finding the solution  $\mathcal{T}(r) = (V, X, Y) \in \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^p$  of the following algebraic system:

$$\begin{cases} KV - \begin{pmatrix} X \\ Y \\ 0 \end{pmatrix} = 0, \\ (V_n)_i = (V_t)_i = 0, & \forall i \in I_s, \\ X_i = Y_i = 0, & \forall i \in I_f, \\ (V_n)_i = 0, & Y_i = r_i, & \forall i \in I_c. \end{cases} \quad (3.3)$$

Let us show that for any  $r \in \mathbb{R}^{p^0}$  there always exists a unique solution to problem (3.3). The equations in (3.3) can be rewritten as follows (with obvious notations):

$$K \begin{pmatrix} 0 \\ 0 \\ V_n(I_f) \\ V_t(I_c) \\ 0 \\ V_t(I_f) \\ \hat{V} \end{pmatrix} = \begin{pmatrix} X(I_c) \\ X(I_s) \\ 0 \\ r \\ Y(I_s) \\ 0 \\ 0 \end{pmatrix}, \quad (3.4)$$

where  $K = K_{ij}$ ,  $1 \leq i, j \leq 7$ . The vectors  $V_n(I_f)$ ,  $V_t(I_c)$ ,  $V_t(I_f)$  and  $\hat{V}$  are the unique solutions of the symmetric positive definite system:

$$\begin{pmatrix} K_{33} & K_{34} & K_{36} & K_{37} \\ K_{43} & K_{44} & K_{46} & K_{47} \\ K_{63} & K_{64} & K_{66} & K_{67} \\ K_{73} & K_{74} & K_{76} & K_{77} \end{pmatrix} \begin{pmatrix} V_n(I_f) \\ V_t(I_c) \\ V_t(I_f) \\ \hat{V} \end{pmatrix} = \begin{pmatrix} 0 \\ r \\ 0 \\ 0 \end{pmatrix}.$$

The vectors  $X(I_c)$ ,  $X(I_s)$  and  $Y(I_s)$  are given by (3.4).

Let us consider the linear operator  $T : \mathbb{R}^{p^0} \rightarrow \mathbb{R}^{p^0}$  which associates to any  $r \in \mathbb{R}^{p^0}$  the vector  $q \in \mathbb{R}^{p^0}$  given by  $q_i = X_i \gamma_i$  for all  $1 \leq i \leq p^0$  and let us denote by  $\beta_i$  and  $b_i$  the  $p^0$  eigenvalues and eigenvectors of the operator  $T$ , i.e.,  $Tb_i = \beta_i b_i$ . Now it becomes straightforward that  $\alpha_h$  and  $(\varphi_h, \theta_{hn}, \theta_{ht})$  are solutions of (3.2) if and only if  $(\Phi, F(\theta_{hn}), F(\theta_{ht})) = \mathcal{T}(r)$  for some eigenvector  $r$  of  $T$  having  $1/\alpha_h$  as eigenvalue (note that  $\alpha_h = 0$  cannot be an eigenvalue in (3.2) and that the components of  $r$  are precisely those of  $F(\theta_{ht})$  on  $I_c$ ).  $\square$

**Remark 3.2** *Let us use the same numbering of the basis functions of  $\mathbf{V}_h \times W_h \times W_h$  as in the previous proof and let us suppose, for the sake of simplicity, that  $p^0 = p$  and*

$\gamma_i = 1$  (i.e.,  $I_s = I_f = \emptyset$  and  $(U_t - U_t^r)_i > 0$ ,  $\forall i \in [1, p]$ ). In this case, the eigenvalue problem (3.2) becomes:

$$\Phi = K^{-1} \begin{pmatrix} F(\theta_{hn}) \\ F(\theta_{ht}) \\ 0 \end{pmatrix} \quad \text{and} \quad (\Phi_n)_i = 0, \quad F_i(\theta_{ht}) = \alpha_h F_i(\theta_{hn}), \forall i \in [1, p], \quad (3.5)$$

which is equivalent to solve the following problem: find the eigenvalue  $-1/\alpha_h$  and the eigenvector  $F(\theta_{hn})$  satisfying

$$(\tilde{K}_{nn})^{-1} \tilde{K}_{nt} F(\theta_{hn}) = -\frac{1}{\alpha_h} F(\theta_{hn}). \quad (3.6)$$

where the following notation is adopted

$$K^{-1} = \begin{pmatrix} \tilde{K}_{nn} & \tilde{K}_{nt} & \tilde{K}_{ni} \\ \tilde{K}_{nt} & \tilde{K}_{tt} & \tilde{K}_{ti} \\ \tilde{K}_{ni} & \tilde{K}_{ti} & \tilde{K}_{ii} \end{pmatrix}. \quad (3.7)$$

Having at our disposal  $F(\theta_{hn})$  and  $\alpha_h$ , we see that  $F(\theta_{ht})$  and  $\Phi$  can be easily determined.

Using the eigenvalue problem (3.2) allows us to obtain sufficient conditions for the non-uniqueness of the solution  $(\mathbf{u}_h, \lambda_{hn}, \lambda_{ht})$  of (2.8). This is achieved in the following theorem.

**Theorem 3.3** *Let  $(\mathbf{u}_h, \lambda_{hn}, \lambda_{ht})$  be a solution of Coulomb's discrete frictional contact problem (2.8) with  $\mu > 0$  as friction coefficient. We assume that  $I_c \neq \emptyset$  and that (3.1) holds. Moreover we suppose that*

$$F_i(\lambda_{hn}) < 0, \quad \forall i \in I_c. \quad (3.8)$$

If  $\mu$  is an eigenvalue of (3.2) then the Coulomb's frictional contact problem (2.8) admits an infinity of solutions located on a continuous branch. More precisely, if we denote by  $(\varphi_h, \theta_{hn}, \theta_{ht})$  the corresponding eigenvector then there exists  $\delta_0 > 0$  such that  $(\mathbf{u}_h + \delta\varphi_h, \lambda_{hn} + \delta\theta_{hn}, \lambda_{ht} + \delta\theta_{ht})$  is solution of (2.8) for any  $\delta$  with  $|\delta| \leq \delta_0$ .

**Proof.** Let us firstly remark that

$$(\mathbf{u}_h + \delta\varphi_h, \lambda_{hn} + \delta\theta_{hn}, \lambda_{ht} + \delta\theta_{ht})$$

satisfies the equation in (2.8) for any  $\delta \in \mathbb{R}$ .

Next, we have to check that  $(\mathbf{u}_h + \delta\varphi_h, \lambda_{hn} + \delta\theta_{hn}, \lambda_{ht} + \delta\theta_{ht})$  verifies the frictional contact conditions in the inequality of (2.8) (or equivalently (2.9)–(2.11)) for a sufficiently small  $|\delta|$ . Let us recall that  $\Phi_n$  and  $\Phi_t$  denote the vectors of components the normal and tangential values respectively of  $\varphi_h$  on  $\Gamma_C$ . To simplify, we set  $X = F(\theta_{hn})$  and  $Y = F(\theta_{ht})$  (i.e., the generalized loads corresponding to  $\theta_{hn}$  and  $\theta_{ht}$  respectively).

Since  $F_i(\lambda_{hn}) < 0$  for all  $i \in I_c \cup I_s$  there exists  $\delta_a > 0$  such that  $F_i(\lambda_{hn}) + \delta X_i \leq 0$ , for all  $i \in I_c \cup I_s$  and  $|\delta| \leq \delta_a$ . Having in mind that  $F_i(\lambda_{hn}) = X_i = 0$  for  $i \in I_f$  we deduce that  $F_i(\lambda_{hn}) + \delta X_i \leq 0$  for all  $i \in I_c \cup I_s \cup I_f$ . The same technique can be used to prove that  $(U_n + \delta \Phi_n)_i \leq 0$  for a sufficiently small  $|\delta|$  and that  $(F_i(\lambda_{hn}) + \delta X_i)(U_n + \delta \Phi_n)_i = 0$ . Hence the conditions (2.9) hold.

According to the definition of  $I_s$  there exists  $\delta_b > 0$  such that  $|\delta| \leq \delta_b$  implies  $|F_i(\lambda_{ht})| < -\mu F_i(\lambda_{hn}) - \delta(|Y_i| + \mu|X_i|)$  for all  $i \in I_s$ . Therefore  $|F_i(\lambda_{ht}) + \delta Y_i| < -\mu(F_i(\lambda_{hn}) + \delta X_i)$  and  $(U_t + \delta \Phi_t)_i = (U_t^r)_i$  for all  $i \in I_s$ . So the conditions (2.10)–(2.11) are satisfied for  $i \in I_s$ .

From the definition of  $I_f$ , we deduce  $F_i(\lambda_{hn}) = F_i(\lambda_{ht}) = X_i = Y_i = 0$  for all  $i \in I_f$ . As a consequence (2.10) and (2.11) are fulfilled for  $i \in I_f$ .

It remains to show that (2.10) and (2.11) hold for  $i \in I_c$ . Since  $|F_i(\lambda_{ht})| = -\mu F_i(\lambda_{hn})$ , we deduce from the definition of  $I_c$  that  $F_i(\lambda_{ht}) = \mu F_i(\lambda_{hn}) \gamma_i$ ,  $\forall i \in I_c$ . Since  $Y_i = \mu X_i \gamma_i$  we have  $F_i(\lambda_{ht}) + \delta Y_i = \mu(F_i(\lambda_{hn}) + \delta X_i) \gamma_i$  for all  $i \in I_c$ . From (3.8), we get  $|F_i(\lambda_{ht}) + \delta Y_i| = -\mu(F_i(\lambda_{hn}) + \delta X_i)$  for  $|\delta| \leq \delta_c$  and  $i \in I_c$ . The definition of  $\gamma_i$  on  $I_c$  implies that there exists  $\delta_d > 0$  such that  $\gamma_i(U_t + \delta \Phi_t - U_t^r)_i = \gamma_i(U_t - U_t^r)_i + \delta \gamma_i(\Phi_t)_i > 0$  for  $|\delta| \leq \delta_d$  and  $i \in I_c$ .

Consequently for any  $|\delta| \leq \delta_0 = \min(\delta_a, \delta_b, \delta_c, \delta_d)$  all the conditions (2.9)–(2.11) hold for  $(\mathbf{u}_h + \delta \boldsymbol{\varphi}_h, \lambda_{hn} + \delta \theta_{hn}, \lambda_{ht} + \delta \theta_{ht})$ . This completes the proof.  $\square$

**Remark 3.4 1.** *The statement in the theorem is a sufficient condition for non-uniqueness detecting an infinity of solutions located on a continuous branch. The technique developed in this paper does not allow us to find multiple solutions which are isolated as in [13].*

*2. The assumptions considered in the theorem require that the friction coefficient  $\mu$  is an eigenvalue in (3.2). The latter eigenvalue problem depends on the geometry (the domain  $\Omega$  and the distribution of the different types of boundaries  $\Gamma_D, \Gamma_N, \Gamma_C$ ), on the elastic properties incorporated in the operator  $\mathcal{C}$  (more precisely on the Poisson coefficient  $\nu$  for an isotropic elastic material) and on the finite element mesh (we will see in the section devoted to the numerical experiments that the mesh and the type of finite elements used have a little influence on the eigenvalues).*

*3. The positive eigenvalues represent critical friction coefficients for which the problem (2.8) is open to non-uniqueness. We will show in the section concerned with the numerical experiments that if (3.2) admits a positive eigenvalue then an example of non-uniqueness with an infinity of solutions can be explicitly constructed. This can be performed by choosing simple loads  $\mathbf{F}, \mathbf{f}$  and a zero reference displacement field  $\mathbf{u}^r$ . In fact the solution  $(\mathbf{u}_h, \lambda_{hn}, \lambda_{ht})$  of Coulomb's discrete frictional contact problem (2.8) for this particular friction coefficient  $\mu$  must satisfy (3.1) and (3.8).*

#### 4. Some elementary examples

In what follows, we consider the commonly used Hooke's constitutive law corresponding to homogeneous isotropic materials in (2.2):

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk}(\mathbf{u}) + 2G \varepsilon_{ij}(\mathbf{u}) \quad \text{in } \Omega, \quad (4.1)$$

where  $\lambda$  and  $G$  are the positive Lamé coefficients and  $\delta_{ij}$  denotes the Kronecker symbol. Note that  $\lambda = (E\nu)/((1 - 2\nu)(1 + \nu))$  and  $G = E/(2(1 + \nu))$  where  $E$  and  $\nu$  represent Young's modulus and Poisson's ratio, respectively.

It is easy to see that the only constitutive constant involved in the eigenvalue problem (3.2) is the Poisson ratio  $\nu$  and that the eigenvalues and eigenfunctions are independent of the Young modulus  $E$ .

Our aim in this section is to illustrate with simple examples the eigenvalue problem in (3.6). This means that we determine critical friction coefficients involving an infinity of solutions located on a continuous branch (with slip only in one direction).

#### 4.1. First example

Here we propose to determine explicitly the eigenvalues for the finite element mesh comprising one triangular element, depicted in Figure 1, and to exhibit a bifurcation point between the "stick solution" and a vertical branch where an infinity of solutions are located.

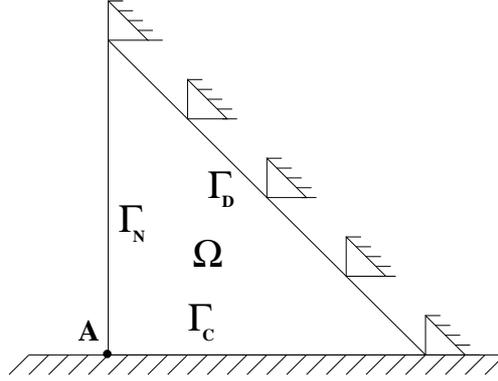


Figure 1: First example of an elementary finite element mesh

In this case  $I_c$  is reduced to the node  $A$ . The stiffness matrix becomes:

$$K = \frac{1}{2} \begin{pmatrix} \lambda + 3G & \lambda + G \\ \lambda + G & \lambda + 3G \end{pmatrix}.$$

Using the notations in (3.7) we get

$$(\tilde{K}_{nn})^{-1}(\tilde{K}_{nt}) = -\frac{\lambda + G}{\lambda + 3G}.$$

In this case there exists a unique eigenvalue  $(-1/\alpha_h)$  in (3.6). Obviously the unique critical friction coefficient denoted  $\mu_{cr} = \alpha_h$  is

$$\mu_{cr} = \frac{\lambda + 3G}{\lambda + G} = 3 - 4\nu.$$

Note that the friction coefficient  $\mu_{cr}$  depends in a linear way on  $\nu$ .

Let us determine the set of solutions. We have to consider a solution of (2.8) satisfying the equation:

$$K \begin{pmatrix} U_n \\ U_t \end{pmatrix} - \begin{pmatrix} F(\lambda_{hn}) \\ \mu F(\lambda_{hn}) \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \quad (4.2)$$

with  $U_n = 0$ ,  $U_t > 0$  and  $F(\lambda_{hn}) < 0$ . The notations  $F_1$  and  $F_2$  represent the forces corresponding to the surface loads on  $\Gamma_N$  in the horizontal and vertical directions, respectively. We suppose in the following that  $F_1/F_2 = (\lambda+G)/(\lambda+3G)$ , with  $F_2 > 0$ . Equation (4.2) becomes:

$$\begin{cases} \frac{1}{2}(\lambda + G)U_t - F(\lambda_{hn}) = F_1 \\ \frac{1}{2}(\lambda + 3G)U_t - \mu F(\lambda_{hn}) = F_2 \end{cases} \quad (4.3)$$

For  $\mu = \mu_{cr} = (\lambda + 3G)/(\lambda + G)$  we deduce that the system of equations (4.3) admits an infinity of solutions verifying:

$$\begin{cases} U_t \in \left(0, \frac{2F_1}{\lambda + G}\right), \\ F(\lambda_{hn}) = \frac{1}{2}(\lambda + G)U_t - F_1 \in (-F_1, 0). \end{cases}$$

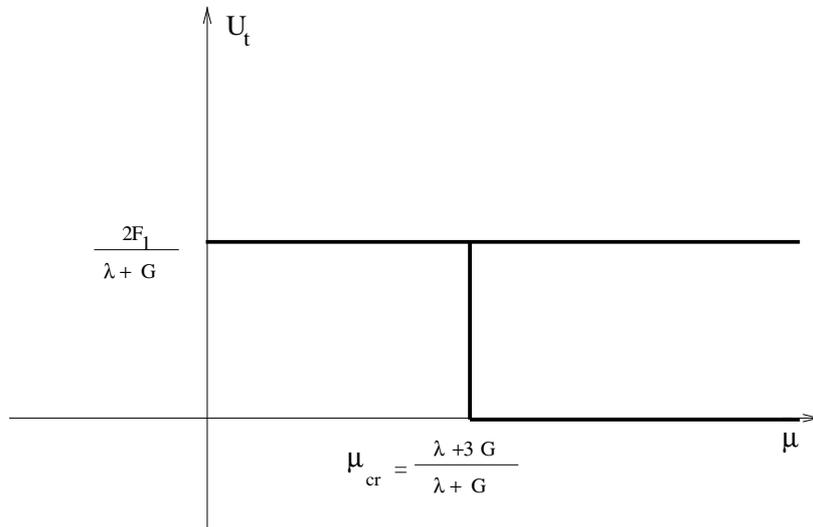


Figure 2: The bifurcation point  $\mu = \mu_{cr}$  between the "slip solution" and a vertical branch. (the problem admits an infinity of solutions)

This result corresponds precisely to an infinity of solutions located on a continuous branch which is represented in Figure 2. In other words, if  $\mu = \mu_{cr}$  then there exists an infinity of solutions to the problem (2.8). As it follows from [13] it can be easily checked that for all  $\mu \geq \mu_{cr}$  the "stick position"  $U_t = U_n = 0$  is a solution of (2.8).

Moreover when  $\mu > 0$  the slip solution  $U_t = 2F_1/(\lambda + G)$  solves (2.8). That means that the problem has one solution for  $\mu < \mu_{cr}$ , an infinity of solutions for  $\mu = \mu_{cr}$  and two (isolated) solutions for  $\mu > \mu_{cr}$ . The critical frictional coefficient  $\mu_{cr}$  corresponds to a bifurcation point (see Figure 2).

**4.2. Second example**

The next example is concerned with the square of Figure 3 meshed with 4 linear triangles. Here  $I_c = \{A, B\}$  and the number of degrees of freedom for the displacements is 6.

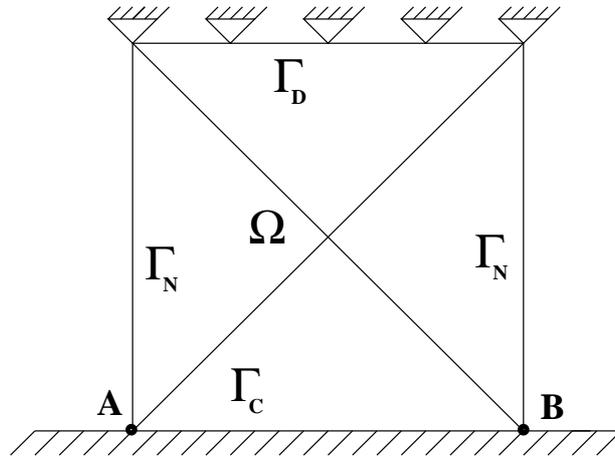


Figure 3: Second example of elementary finite element mesh

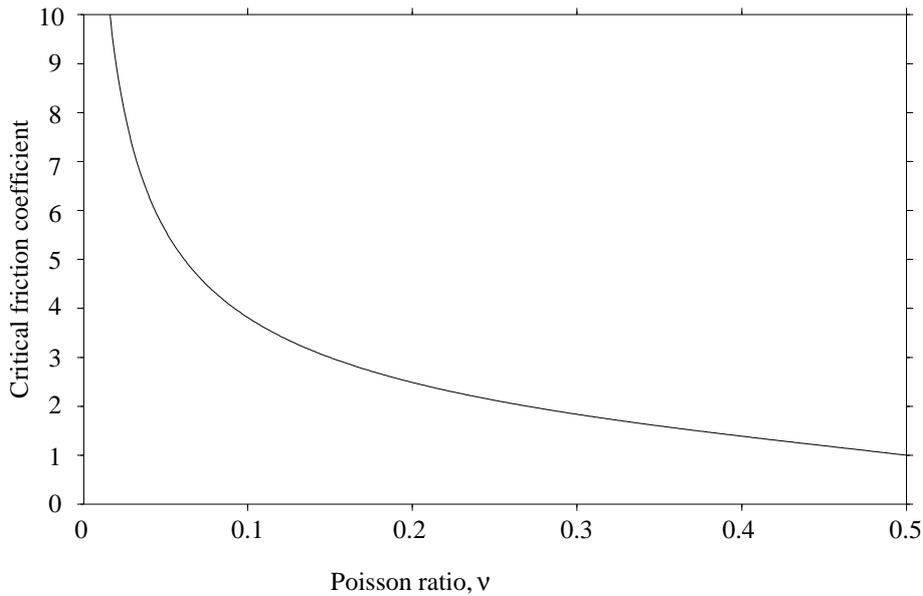


Figure 4: The behavior of the critical friction coefficient as a function of Poisson ratio  $\nu$  for the second elementary example.

The corresponding stiffness matrix is:

$$K = \frac{1}{2} \begin{pmatrix} \lambda + 3G & \frac{1}{2}(\lambda + G) & -(\lambda + G) & \lambda + G & \frac{1}{2}(-\lambda + G) & -(\lambda + 3G) \\ \frac{1}{2}(\lambda + G) & \lambda + 3G & \lambda + G & \frac{1}{2}(\lambda - G) & -(\lambda + G) & -(\lambda + 3G) \\ -(\lambda + G) & (\lambda + G) & 4\lambda + 12G & -(\lambda + 3G) & -(\lambda + 3G) & 0 \\ \lambda + G & \frac{1}{2}(\lambda - G) & -(\lambda + 3G) & \lambda + 3G & -\frac{1}{2}(\lambda + G) & -(\lambda + G) \\ \frac{1}{2}(-\lambda + G) & -(\lambda + G) & -(\lambda + 3G) & -\frac{1}{2}(\lambda + G) & \lambda + 3G & \lambda + G \\ -(\lambda + 3G) & -(\lambda + 3G) & 0 & -(\lambda + G) & \lambda + G & 4\lambda + 12G \end{pmatrix}.$$

The matrix of the eigenvalue problem in (3.6) is

$$(\tilde{K}_{nn})^{-1}(\tilde{K}_{nt}) = \frac{1}{(\lambda + 2G)(\lambda + 5G)} \begin{pmatrix} -(5G^2 + 5\lambda G + \lambda^2) & -G(5G + 2\lambda) \\ G(5G + 2\lambda) & 5G^2 + 5\lambda G + \lambda^2 \end{pmatrix}.$$

The two critical friction coefficients obtained from (3.6) are

$$\mu_{cr} = \pm \sqrt{\frac{(1 - \nu)(5 - 8\nu)}{\nu(3 - 4\nu)}} = \pm \sqrt{\frac{(\lambda + 2G)(\lambda + 5G)}{\lambda(\lambda + 3G)}}.$$

Note that these values are opposite since the mesh and the boundary conditions are symmetric. The behavior of the positive  $\mu_{cr}$  as function of  $\nu$  is shown in Figure 4. We observe that the positive eigenvalue tends to infinity when  $\nu \rightarrow 0$  and that it becomes 1 when  $\nu \rightarrow \frac{1}{2}$ .

## 5. Computational examples of non-uniqueness

This section shows two numerical experiments. In the first test we choose again the square geometry depicted in Figure 3 and we examine the convergence of the finite element procedure (3.6) with several meshes and types of finite elements. In the second test we show that the computed eigenvalues can be small (in fact as small as desired) on specific geometries. We conclude this section by explaining how an infinity of solutions located on a continuous branch can be always obtained when a positive critical friction coefficient is known.

### 5.1. First example

We consider the unit square introduced in Figure 3 and we solve the eigenvalue problem (3.6) with different meshes and various types of finite elements. We observe numerically that there always exist a positive eigenvalue that converges to a limiting value as the discretization parameter tends to zero, and this limit depends only on Poisson's ratio. Figure 5 represents the convergence of these critical friction coefficients obtained with various finite elements. The given Poisson ratio is 0.3 and the limit is approximately 1.945.

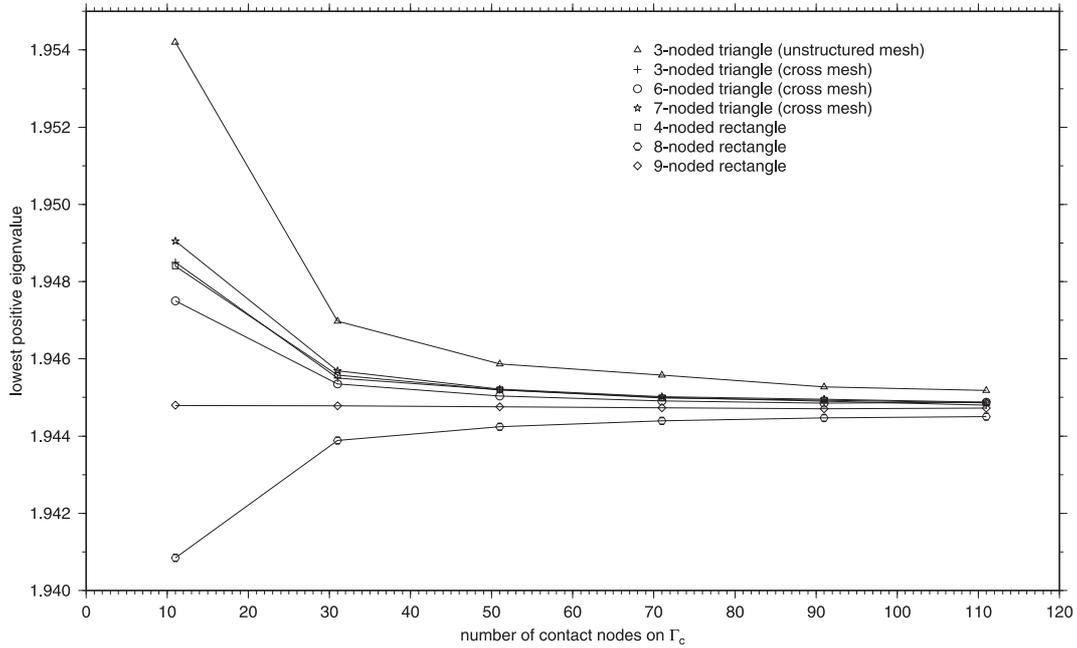


Figure 5: The convergence of the critical friction coefficient (lowest positive eigenvalue) with the mesh size for various finite elements ( $\nu = 0.3$ ) for the first computational example.

## 5.2. Second example

Next, we consider the inclined body represented in Figure 6. The geometrical properties of  $\Omega$  are  $H/L = H/L' = 3$ . The computations are performed on a fixed mesh comprising 28084 linear triangles, 14251 nodes and 51 nodes on  $\Gamma_C$ .

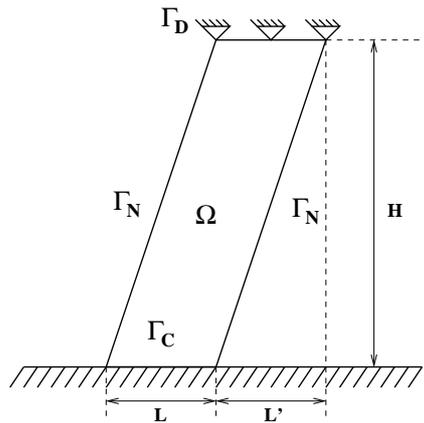


Figure 6: Setting of the problem for the second computational example.

Figure 7 shows the behavior of the lowest positive eigenvalue as a function of Poisson ratio. Let us notice that the computed eigenvalues range between 0.55 and 0.61. Such values are commonly observed friction coefficients. Of course these values depend also on  $H, L, L'$  and we notice numerically that the eigenvalues tend to zero when the ratios  $H/L = H/L'$  tend to infinity.

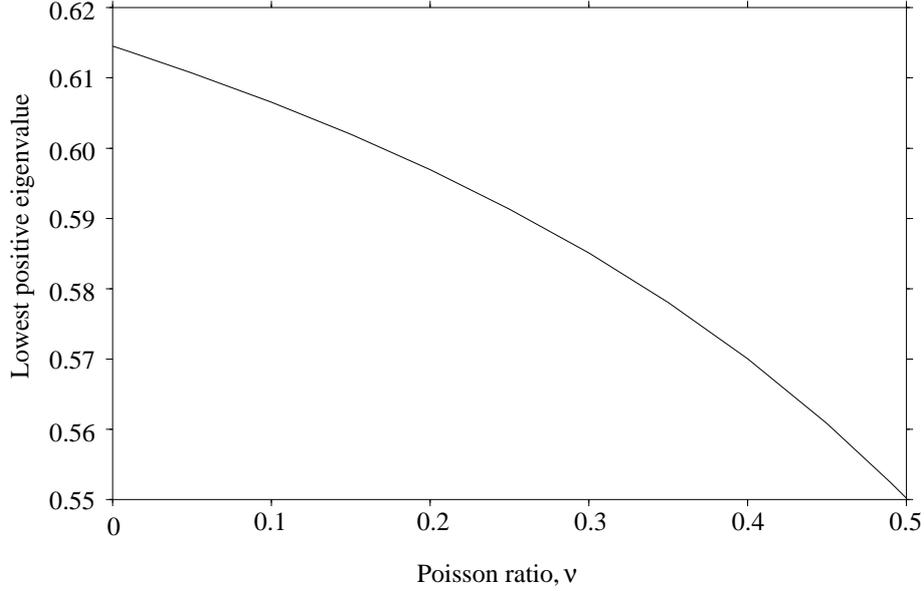


Figure 7: The behavior of the critical friction coefficient (lowest positive eigenvalue) as a function of Poisson ratio  $\nu$  for the second computational example.

Finally the eigenfunction  $\Phi$  corresponding to  $\nu = 0.3$  is computed from (3.5) and depicted in Figure 8. Using the constitutive relation (4.1) allows the computation of the Von-Mises stress field shown in Figure 9.

When problem (3.6) admits a real eigenvalue  $\mu$  then the pair geometry-material is open to the non-uniqueness for the Coulomb friction problem. As a matter of fact, one can think of a distribution of loads  $\mathbf{F}, \mathbf{f}$  and a displacement field  $\mathbf{U}_h$  such that a solution  $(\mathbf{u}_h, \lambda_{hn}, \lambda_{ht})$  of (2.8) for this particular friction coefficient  $\mu$  satisfies (3.8). We consider as in the previous examples a geometry  $\Omega$  in which  $\Gamma_C$  is a straight line segment located on the  $0x_1$ -axis with  $\mathbf{n} = (0, -1)$  and  $\mathbf{t} = (1, 0)$ . We choose as example

$$\mathbf{U}_h(\mathbf{x}) = \left( \alpha + 2\mu \frac{1-\nu}{1-2\nu} x_2, -x_2 \right), \quad \mathbf{F}(\mathbf{x}) = \sigma \mathbf{n}(\mathbf{x}), \quad \mathbf{f} = \mathbf{0}$$

with  $\alpha > 0$  and  $\sigma_{11} = -(E\nu)/((1-2\nu)(1+\nu))$ ,  $\sigma_{22} = -(E(1-\nu))/((1-2\nu)(1+\nu))$ ,  $\sigma_{12} = \sigma_{21} = -\mu\sigma_{22}$ . Taking  $\mathbf{u}_h(\mathbf{x}) = \mathbf{U}_h(\mathbf{x})$ , for all  $\mathbf{x} \in \Omega$ ,  $\lambda_{hn}(\mathbf{x}) = \sigma_{22}$ ,  $\lambda_{ht} = -\sigma_{12}$ , for all  $\mathbf{x} \in \Gamma_C$ , one can easily check that  $(\mathbf{u}_h, \lambda_{hn}, \lambda_{ht})$  is a solution of (2.8). Since  $I_f = I_s = \emptyset$ ,  $\lambda_{hn}(\mathbf{x}) = \sigma_{22} < 0$  and  $U_t(\mathbf{x}) = \alpha > 0$  we deduce that the sufficient conditions of Theorem 3.3 hold.

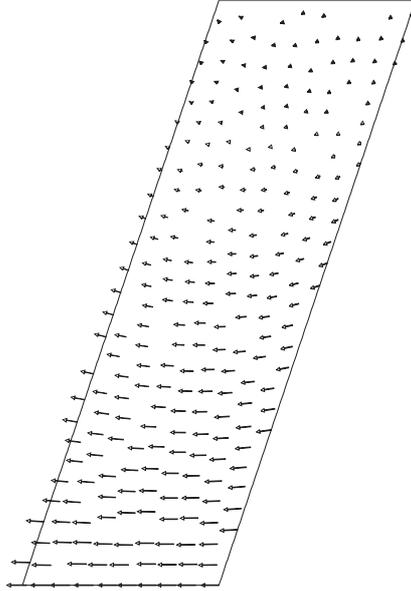


Figure 8: The eigenfunction  $\Phi$  corresponding to  $\nu = 0.3$  for the second computational example.

## 6. Conclusion

The problem of uniqueness of the static (or quasi-static) Coulomb friction problem in linear elasticity is studied using a specific eigenvalue problem involving mixed finite elements with two multipliers. If this problem admits a positive eigenvalue called critical friction coefficient, then the Coulomb friction problem is open to non-uniqueness. More precisely if the friction coefficient is equal with this critical value then the problem exhibits an infinity of solutions located on a continuous branch. This critical coefficient depends exclusively on the geometry (the shape of the domain and the distribution of different types of boundaries) and on the Poisson ratio.

When the mesh size tends to zero the sequence of the “discrete” first eigenvalues is convergent to a critical friction coefficient. The mixed finite element procedure with two multipliers used in this paper is very efficient in detecting the eigenvalues. The numerical experiments obtained with this method clearly show that the computed critical friction coefficient is independent on the mesh type and on the degree of the elements.

The loss of uniqueness which exhibits an infinity of non-isolated solutions (continuous branch) can be associated with a loss of validity of the static or quasi-static approximations and the presence of dynamic instabilities. This loss of stability for a specific friction coefficient has to be analyzed in the context of state-dependent friction coefficients. Indeed for the slip weakening or slip-rate weakening friction models the friction coefficient is continuously decreasing (from the static value down to a dynamic value) during the quasi-static slip and the loss of stability (or uniqueness) occurs for a specific (critical) friction coefficient. These questions are actually under investigation in [12].

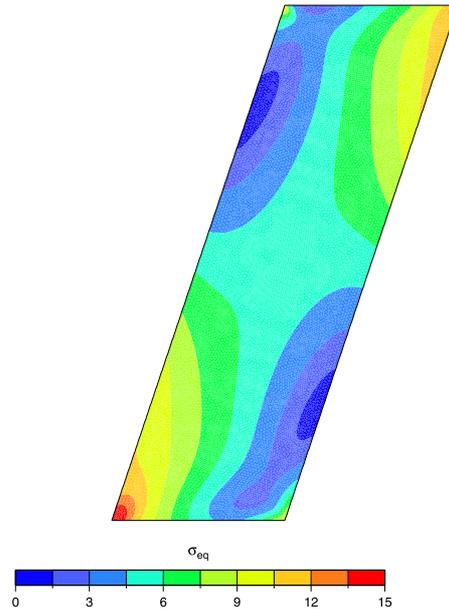


Figure 9: The Von-Mises stress field corresponding to the eigenfunction  $\Phi$  for the second computational example.

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