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A posteriori error estimation for unilateral contact with matching and non-matching meshes

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Abstract

In this paper, we consider the unilateral contact problem between elastic bodies. We propose an error estimator based on the concept of error in the constitutive relation in order to evaluate the finite element approximation involving matching and non-matching meshes on the contact zone. The determination of the a posteriori error estimate is linked to the building of kinematically-admissible stress fields and statically-admissible stress fields. We then propose a finite element method for approximating the unilateral contact problem taking into account matching and non-matching meshes on the contact zone; then, we describe the construction of admissible fields. Lastly, we present optimized computations by using both the error estimates and a convenient mesh adaptivity procedure. © 2000 Elsevier Science S.A. All rights reserved.

1. Introduction

The numerical simulation of contact problems is more often carried out by finite element methods. For the user, one important aspect is obviously to evaluate the discretization errors due to the use of this type of approximation. From the point of view of mathematics, a unilateral contact problem corresponds to a variational inequality [1–3]. The convergence of the associated finite element methods has been studied by numerous authors on the basis of a priori error estimators. In particular, the case of two elastic bodies has been developed in [4] for matching meshes and in [5–8] for non-matching meshes. However, these a priori error estimations do not allow us to quantify the discretization errors. This quantification requires the definition of a posteriori error estimations. For linear problems, various research efforts have been performed: estimators based on the residual of the equilibrium equations [9], estimators using the smoothing of finite element stresses [10] and estimators based on the concept of error in the constitutive relation [11,12]. For non-linear problems, and especially for the non-linearity of contact, the work available is much less abundant. We can however cite Ref. [13] which, based on a penalty method, transforms the variational inequality into a variational equality that allows, within the classical framework, building an estimator based on the residuals. Nevertheless, this estimator explicitly uses the penalty parameter, which does represent a major drawback.

The aim of this paper is to propose, for unilateral contact problems without friction between elastic bodies under small perturbations, an error measure based on the concept of error in the constitutive relation. This error measure can be used for the classical numerical techniques for local treatment of the non-interpenetration condition, yet is particularly well-adapted to the global treatment of the non-interpenetration condition proposed in [8].

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The basis of the method is presented in Section 2. In particular, the development of an error measure in the constitutive relation relies on a classification of the kinematic conditions, equilibrium equations and constitutive relations. In the case of contact, in order to establish this classification, we consider as in [14] the contact zone as a mechanical entity with its own variables and its own constitutive relations.

The building of the error measure is the purpose of Section 3. The quality of an approximate admissible solution which, by definition, satisfies the kinematic conditions and equilibrium equations is evaluated by the manner in which the constitutive relations are satisfied. A link between the error measure in the constitutive relation and the classical errors in the solution is established. For these types of problems, such an approach can be considered as an extension of Prager–Synge's theorem [15] in elasticity.

The discretization of the problem by finite elements and the application of the proposed error measure are developed in Sections 4 and 5. In particular, the technique which allows building an approximate admissible solution from the finite element solution is detailed.

In Section 6 examples of the use of the error estimator as well as examples of adaptive computations are shown.

2. Problem set-up

2.1. General notations

We consider the bidimensional unilateral contact problem between two elastic bodies, denoted by Ω^1 and Ω^2 (Fig. 1), respectively. We assume that the boundary $\partial\Omega^\ell$ of Ω^ℓ, $\ell = 1,2$ is divided into three parts:

- On the first part, denoted by $\partial_1\Omega^\ell$, we suppose that the displacement field is given:

$$U^\ell_{|\partial_1\Omega^\ell} = U^\ell_{\mathrm{d}}, \quad \ell = 1,2. \tag{1}$$

For the sake of simplicity, we suppose in the following that: $U^\ell_{\mathrm{d}} = 0,\ \ell = 1,2$.

- On the second part, denoted by $\partial_2\Omega^\ell$, a surfacic density of forces F^ℓ_{d} is given.
- The complementary part, denoted by $\partial_C\Omega^\ell = \partial\Omega^\ell - (\partial_1\Omega^\ell \cup \partial_2\Omega^\ell)$, is the candidate contact zone.

We suppose that $\partial_C\Omega^1 = \partial_C\Omega^2$, which we denote by Γ_C. The body Ω^ℓ is submitted to a density of volumic forces f^ℓ_{d}. We assume that the strain tensor $\varepsilon(\cdot)$ is linearized and we denote K^ℓ by the elasticity operator associated with Ω^ℓ. The notation n^ℓ stands for the unit outward normal on the boundary of Ω^ℓ.

2.2. Formulation of the contact

In order to clearly express the error in the constitutive relation, we represent, as in [14], the contact zone $\Gamma_C = \partial_C\Omega^1 = \partial_C\Omega^2$ as a mechanical entity equipped with its constitutive relation. We choose the orientation of Γ_C by setting $n^C = n^1$. We then introduce on the interface Γ_C the functions W^1, W^2, R^1, R^2 and R^C,

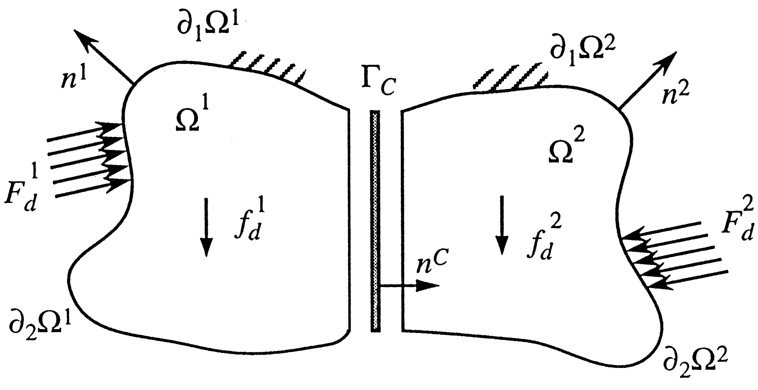

Fig. 1. The unilateral contact problem between two elastic bodies.

representing two displacement fields $W^1$and W^2 (on each side of the interface), two fields of surfacic densities of forces R^1 and R^2 (stresses transmitted to Ω^1 and Ω^2) and an "interior field" of a surfacic density of forces R^C.

The equilibrium of the interface is represented by:

$$R^C = R^1 \quad \text{and} \quad R^C = -R^2 \text{ on } \Gamma_C. \tag{2}$$

Let us define the jump in the displacement which, for the interface, plays a similar role as a strain:

$$W^C = W^1 - W^2. \tag{3}$$

For all of vector Z, set:

$$Z_n = Z^T n^C \quad \text{and} \quad Z_t = Z - Z_n n^C, \tag{4}$$

where the notation T represents transposition.

Coulomb's constitutive law, in the frictionless case, can be formulated as follows [1]:

$$W_n^C \leqslant 0, \tag{5}$$

$$R_n^C \leqslant 0, \tag{6}$$

$$R_n^C W_n^C = 0, \tag{7}$$

$$R_t^C = 0. \tag{8}$$

The inequality in (5) expresses the non_interpenetration of the two bodies; either contact or separation is allowed. The inequality in (6) states the sign condition on the normal constraint and (7) represents the complementary condition. Lastly, (8) states the nullity of the tangential component of the stress vector, which reflects the absence of friction.

Let us now introduce the conjugate convex potentials φ and φ^* [16]:

$$\varphi(V) = \begin{cases} 0 & \text{if } V_n \geqslant 0 \\ +\infty & \text{otherwise} \end{cases} \quad \text{and} \quad \varphi^*(Z) = \begin{cases} 0 & \text{if } Z_n \leqslant 0 \text{ and } Z_t = 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{9}$$

then, we have: $\varphi(V) + \varphi^*(Z) - Z^T V \geqslant 0 \ \forall V, \ \forall Z$.

The constitutive relation can be written in the three following equivalent forms:

$$\begin{aligned} &- W^C \in \partial\varphi^*(R^C), \\ &R^C \in \partial\varphi(-W^C), \\ &\varphi(-W^C) + \varphi^*(R^C) + R^{C^T} W^C = 0. \end{aligned} \tag{10}$$

2.3. Problem set-up

The problem of unilateral contact without friction can be formulated as follows:

Find (U^1, σ^1) defined in Ω^1, (U^2, σ^2), defined in Ω^2 and $[(W^1, W^2), (R^1, R^2, R^C)]$, defined on Γ_C such that:

- $(U^\ell, W^\ell), \ \ell = 1, 2$ satisfy the kinematic conditions:

$$U^\ell = 0 \text{ on } \partial_1\Omega^\ell \quad \text{and} \quad U^\ell = W^\ell \text{ on } \Gamma_C. \tag{11}$$

- $(\sigma^\ell, R^\ell, R^C), \ \ell = 1, 2$ satisfy the equilibrium equations:

$$\begin{aligned} &-\int_{\Omega^\ell} \mathrm{Tr}\big[\sigma^\ell \varepsilon(V^\ell)\big] \mathrm{d}\Omega + \int_{\Omega^\ell} f_d^{\ell T} V^\ell \, \mathrm{d}\Omega + \int_{\partial_2\Omega^\ell} F_d^{\ell T} V^\ell \, \mathrm{d}S + \int_{\Gamma_C} R^{\ell T} V^\ell \, \mathrm{d}S = 0 \quad \forall V^\ell \in U_0^\ell \\ &\quad = \big\{ U^\ell \text{ defined and enough regular on } \Omega^\ell \text{ such that } U^\ell = 0 \text{ on } \partial_1\Omega^\ell \big\} \end{aligned} \tag{12}$$

and: $R^C = R^1$ and $R^C = -R^2$ on Γ_C.

- $(U^\ell, W^\ell, \sigma^\ell, R^C)$, $\ell = 1, 2$ satisfy the constitutive relations:

$$\begin{aligned} &\sigma^\ell = K^\ell \varepsilon(U^\ell), \\ &\varphi(-W^C) + \varphi^*(R^C) + R^{C^{\mathrm{T}}} W^C = 0. \end{aligned} \tag{13}$$

3. Error in the constitutive relation

The concept of error in the constitutive relation is based on the classification of the equations in kinematic, equilibrium and constitutive relations.

3.1. Admissible fields

Definition. A pair $\hat{s} = (\hat{u}, \hat{c})$, $\hat{u} = (\hat{U}^1, \hat{U}^2, \hat{W}^1, \hat{W}^2)$, $\hat{c} = (\hat{\sigma}^1, \hat{\sigma}^2, \hat{R}^1, \hat{R}^2, \hat{R}^C)$ is said to be admissible if the kinematic relations (11) and the equilibrium equations (12) are satisfied.

3.2. Measure of the error in the constitutive relation

For all admissible $\hat{s}$, let's set:

$$e(\hat{s}) = \left[\sum_{\ell=1}^{2} \left\| \hat{\sigma}^\ell - K^\ell \varepsilon(\hat{U}^\ell) \right\|^2_{\sigma,\Omega^\ell} + 2 \int_{\Gamma_{\mathrm{C}}} \left(\varphi(-\hat{W}^C) + \varphi^*(\hat{R}^C) + \hat{R}^{C^{\mathrm{T}}} \hat{W}^C \right) \mathrm{d}S \right]^{1/2}, \tag{14}$$

where $\|\sigma\|^2_{\sigma,\Omega^\ell} = \int_{\Omega^\ell} \mathrm{Tr}[\sigma K^{-1} \sigma] \, \mathrm{d}\Omega$.

By definition, the quantity $e(\hat{s})$ is the measure of the error in the constitutive law corresponding to the admissible pair $\hat{s}$.

Property. For $\hat{s} = (\hat{u}, \hat{c})$ admissible, we have: $e(\hat{s})$ equal to zero if and only if $\hat{s}$ is the exact solution to the reference problem.

With the error in the constitutive law, we associate the relative error denoted ε and defined as follows:

$$\varepsilon = \left[\frac{\sum_{\ell=1}^{2} \left\| \hat{\sigma}^\ell - K^\ell \varepsilon(\hat{U}^\ell) \right\|^2_{\sigma,\Omega^\ell} + 2 \int_{\Gamma_{\mathrm{C}}} \left(\varphi(-\hat{W}^C) + \varphi^*(\hat{R}^C) + \hat{R}^{C^{\mathrm{T}}} \hat{W}^C \right) \mathrm{d}S}{\sum_{\ell=1}^{2} \| \hat{\sigma}^\ell + K^\ell \varepsilon(\hat{U}^\ell) \|^2_{\sigma,\Omega^\ell}} \right]^{1/2}. \tag{15}$$

Therefore, ε is a global error which allows evaluating the quality of the admissible pair $\hat{s}$. Let E be a part of Ω^ℓ. Then, we define the local contribution ε_E of E to the error (15) as follows:

$$\varepsilon_E = \left[\frac{\left\| \hat{\sigma}^\ell - K^\ell \varepsilon(\hat{U}^\ell) \right\|^2_{\sigma,E} + 2 \int_{\Gamma_{\mathrm{C}} \cap E} \left(\varphi(-\hat{W}^C) + \varphi^*(\hat{R}^C) + \hat{R}^{C^{\mathrm{T}}} \hat{W}^C \right) \mathrm{d}S}{\sum_{\ell=1}^{2} \| \hat{\sigma}^\ell + K^\ell \varepsilon(\hat{U}^\ell) \|^2_{\sigma,\Omega^\ell}} \right]^{1/2}. \tag{16}$$

where $\|\sigma\|^2_{\sigma,E} = \int_E \mathrm{Tr}[\sigma K^{-1} \sigma] \, \mathrm{d}\Omega$.

In practical situations, E is an element of the mesh's discretization. The local contributions enable obtaining information concerning the errors located on the structure. By construction, one has:

$$\varepsilon^2 = \sum_E \varepsilon_E^2. \tag{17}$$

When $\Gamma_C = \emptyset$, the definitions (14), (15) and (16) are already known and correspond to the case of elasticity [12].

3.3. Relation with the other errors

Proposition. *Let* $(U^1,\ U^2, \sigma^1,\ \sigma^2, W^1,\ W^2, R^1,\ R^2,\ R^C)$ *be a solution to the contact problem* (11)–(13). *For all* $\hat{s} = (\hat{U}^1,\ \hat{U}^2, \hat{\sigma}^1,\ \hat{\sigma}^2, \hat{W}^1,\ \hat{W}^2, \hat{R}^1,\ \hat{R}^2,\ \hat{R}^C)$ *admissible, one has:*

$$e^2(\hat{s}) - \sum_{\ell=1}^{2}\left(\left\|U^\ell - \hat{U}^\ell\right\|^2_{u,\Omega^\ell} + \left\|\sigma^\ell - \hat{\sigma}^\ell\right\|^2_{\sigma,\Omega^\ell}\right) \geqslant 0, \tag{18}$$

where $\|U\|^2_{u,\Omega^\ell} = \int_{\Omega^\ell} \mathrm{Tr}[\varepsilon(U)K\varepsilon(U)]\,\mathrm{d}\Omega$.

Therefore

$$\sqrt{\sum_{\ell=1}^{2}\left\|U^\ell - \hat{U}^\ell\right\|^2_{u,\Omega^\ell}} \leqslant e(\hat{s}) \quad \textit{and} \quad \sqrt{\sum_{\ell=1}^{2}\left\|\sigma^\ell - \hat{\sigma}^\ell\right\|^2_{\sigma,\Omega^\ell}} \leqslant e(\hat{s}). \tag{19}$$

This property is an extension of Prager–Synge's theorem in elasticity to the more general unilateral contact case.

Proof. For $\ell = 1, 2$, one has:

$$\begin{aligned}\left\|\hat{\sigma}^\ell - K^\ell\varepsilon(\hat{U}^\ell)\right\|^2_{\sigma,\Omega^\ell} &= \left\|\hat{\sigma}^\ell - \sigma^\ell + K^\ell\varepsilon(U^\ell - \hat{U}^\ell)\right\|^2_{\sigma,\Omega^\ell} \\ &= \left\|\hat{\sigma}^\ell - \sigma^\ell\right\|^2_{\sigma,\Omega^\ell} + \left\|U^\ell - \hat{U}^\ell\right\|^2_{u,\Omega^\ell} + 2\int_{\Omega^\ell} \mathrm{Tr}\left[(\hat{\sigma}^\ell - \sigma^\ell)\varepsilon(U^\ell - \hat{U}^\ell)\right]\mathrm{d}\Omega.\end{aligned} \tag{20}$$

Since σ^ℓ and $\hat{\sigma}^\ell$ satisfy (12) and U^ℓ and $\hat{U}^\ell$ satisfy (11), we can write:

$$\left\|\hat{\sigma}^\ell - K^\ell\varepsilon(\hat{U}^\ell)\right\|^2_{\sigma,\Omega^\ell} = \left\|\hat{\sigma}^\ell - \sigma^\ell\right\|^2_{\sigma,\Omega^\ell} + \left\|U^\ell - \hat{U}^\ell\right\|^2_{u,\Omega^\ell} + 2\int_{\Gamma_C} (\hat{R}^\ell - R^\ell)^{\mathrm{T}}(W^\ell - \hat{W}^\ell)\mathrm{d}S. \tag{21}$$

Then

$$\begin{aligned}\sum_{\ell=1}^{2}\left\|\hat{\sigma}^\ell - K^\ell\varepsilon(\hat{U}^\ell)\right\|^2_{\sigma,\Omega^\ell} = \sum_{\ell=1}^{2}\left\|\hat{\sigma}^\ell - \sigma^\ell\right\|^2_{\sigma,\Omega^\ell} + \sum_{\ell=1}^{2}\left\|U^\ell - \hat{U}^\ell\right\|^2_{u,\Omega^\ell} + 2\int_{\Gamma_C}\Big[(\hat{R}^1 - R^1)^{\mathrm{T}}(W^1 - \hat{W}^1) \\ + (\hat{R}^2 - R^2)^{\mathrm{T}}(W^2 - \hat{W}^2)\Big]\mathrm{d}S.\end{aligned} \tag{22}$$

By using (12) and $R^C_{\mathrm{t}} = 0$, we deduce:

$$e^2(\hat{s}) = \sum_{\ell=1}^{2}\left\|\hat{\sigma}^\ell - \sigma^\ell\right\|^2_{\sigma,\Omega^\ell} + \sum_{\ell=1}^{2}\left\|U^\ell - \hat{U}^\ell\right\|^2_{u,\Omega^\ell} + 2\int_{\Gamma_C} \hat{R}^{C^{\mathrm{T}}} W^C\,\mathrm{d}S + 2\int_{\Gamma_C} R^C_n \hat{W}^C_n\,\mathrm{d}S. \tag{23}$$

Definition (14) is only valid when $\hat{s}$ is such that $\varphi(-\hat{W}^C)$ and $\varphi^*(\hat{R}^C)$ are finite. Otherwise, the error $e(\hat{s})$ is equal to infinity and relation (18) is obviously satisfied. If $\varphi(-\hat{W}^C) = 0$ and $\varphi^*(\hat{R}^C) = 0$, property (18) is established by observing that:

$$\int_{\Gamma_C} \hat{R}^{C^{\mathrm{T}}} W^C\,\mathrm{d}S + \int_{\Gamma_C} R^{C^{\mathrm{T}}} \hat{W}^C\,\mathrm{d}S = \int_{\Gamma_C} \hat{R}^C_n W^C_n\,\mathrm{d}S + \int_{\Gamma_C} R^C_n \hat{W}^C_n\,\mathrm{d}S \geqslant 0. \tag{24}$$

This concludes the proof. □

When initially non-contact points can come into contact after deformation, we must take into account the initial gap between the bodies. The unilateral contact conditions (5)–(8) on Γ_{C} can then be written:

$$W_n^C \leqslant G_n, \tag{25}$$

$$R_n^C \leqslant 0, \tag{26}$$

$$R_n^C (W_n^C - G_n) = 0, \tag{27}$$

$$R_{\mathrm{t}}^C = 0, \tag{28}$$

where G_n is the non-negative function expressing the distance between the two bodies. It thus becomes possible to define the error in the constitutive relation in the case of an initial gap.

$$e(\hat{s}) = \left[\sum_{\ell=1}^{2} \left\| \hat{\sigma}^\ell - K^\ell \varepsilon(\hat{U}^\ell) \right\|^2_{\sigma,\Omega^\ell} + 2\int_{\Gamma_{\mathrm{C}}} \Big(\varphi(-\hat{W}^C + G) + \varphi^*(\hat{R}^C) + \hat{R}^{C^{\mathrm{T}}}(\hat{W}^C - G) \Big) \mathrm{d}S \right]^{1/2}, \tag{29}$$

where $G = G_n n^C$.

4. The continuous and discrete variational formulations for the unilateral contact problem

4.1. The continuous case: the variational inequality and the mixed formulation

Let $\mathbf{U}_0 = \mathbf{U}_0^1 \times \mathbf{U}_0^2$, where $\mathbf{U}_0^1$ and $\mathbf{U}_0^2$ have been introduced in (12). We set, for all $U = (U^1, U^2)$ and $V = (V^1, V^2)$ in $\mathbf{U}_0$

$$a(U,V) = \sum_{\ell=1}^{2} \int_{\Omega^\ell} \mathrm{Tr}\big[\varepsilon\big(U^\ell\big) K^\ell \varepsilon\big(V^\ell\big)\big] \mathrm{d}\Omega. \tag{30}$$

$a(.,.)$ is the bilinear symmetrical form in elasticity. We also set, for all V in $\mathbf{U}_0$

$$L(V) = \sum_{\ell=1}^{2} \left(\int_{\Omega^\ell} f_{\mathrm{d}}^{\ell^{\mathrm{T}}} V^\ell \,\mathrm{d}\Omega + \int_{\partial_2\Omega^\ell} F_{\mathrm{d}}^{\ell^{\mathrm{T}}} V^\ell \,\mathrm{d}S \right). \tag{31}$$

The linear form $L(.)$ takes into account the external loads f_{d}^ℓ and F_{d}^ℓ. We then define the convex of admissible displacements denoted $\mathbf{U}_{\mathrm{ad}}$ comprising the non-interpenetration condition between the bodies

$$\mathbf{U}_{\mathrm{ad}} = \left\{ V = (V^1, V^2) \in U_0,\ V^{1^{\mathrm{T}}} n^1 + V^{2^{\mathrm{T}}} n^2 \leqslant 0 \text{ on } \Gamma_{\mathrm{C}} \right\}. \tag{32}$$

The variational formulation associated with problem (11)–(13) is then [1,4,3]: find U such that:

$$U \in \mathbf{U}_{\mathrm{ad}},\ a(U, V-U) \geqslant L(V-U) \quad \forall V \in \mathbf{U}_{\mathrm{ad}}. \tag{33}$$

This problem is well-posed and admits a unique solution in the case where $\partial_1\Omega^1$ and $\partial_1\Omega^2$ are positive. If not, sufficient conditions that state the existence and uniqueness for well-oriented loads [4] are also available.

Next, we introduce the mixed variational formulation of the unilateral contact problem which consists of finding $(U, \lambda) \in \mathbf{U}_0 \times N(\Gamma_C)$ that satisfies:

$$
\begin{aligned}
&a(U,V) - \int_{\Gamma_C} \lambda\Big(V^{1^T}n^1 + V^{2^T}n^2\Big)\mathrm{d}S = L(V) \quad \forall V \in \mathbf{U}_0,\\
&\int_{\Gamma_C} (\mu-\lambda)\Big(U^{1^T}n^1 + U^{2^T}n^2\Big)\mathrm{d}S \geqslant 0 \qquad\qquad \forall \mu \in N(\Gamma_C),
\end{aligned}
\tag{34}
$$

where $N(\Gamma_C)$ is the convex cone of negative functions defined on Γ_C in some dual sense (for a detailed study, see [4]).

When $\partial_1\Omega^1$ and $\partial_1\Omega^2$ are positive, it is clear that problem (34) has a unique solution (U,λ) where U is the solution to (33) and $\lambda = R_n^C$ (7).

4.2. Finite element discretization for matching and non-matching meshes

We now consider the general case of non-matching meshes on the contact zone. A detailed study of contact problems with non-matching meshes on the contact zone can be found in [8]. In the present case, both polygonally-shaped bodies Ω^1 and Ω^2 are discretized independently, thereby leading to nodes which do not coincide on the contact zone.

We will denote the finite element translation of the space $\mathbf{U}_0^\ell$ by $\mathbf{U}_{0,h}^\ell$ and write $\mathbf{U}_{0,h} = \mathbf{U}_{0,h}^1 \times \mathbf{U}_{0,h}^2$. The functions $V_h^\ell \in \mathbf{U}_{0,h}^\ell$, both of whose components are continuous on Ω^ℓ and polynomial of degree one on each triangle, satisfy the embedding conditions on $\partial_1\Omega^\ell$. Let us denote the discretization parameter associated with Ω^ℓ by h_ℓ and set $h=(h_1,h_2)$ (see Fig. 2).

Next, we introduce the approximation convex cone, denoted by $\mathbf{U}_{\mathrm{ad},h}$ and defined as follows:

$$
\mathbf{U}_{\mathrm{ad},h} = \left\{ V_h = (V_h^1, V_h^2) \in \mathbf{U}_{0,h},\ \int_{\Gamma_C} \Big(V_h^{1^T}n^1 + V_h^{2^T}n^2\Big)\chi_h \,\mathrm{d}S \geqslant 0 \quad \forall \chi_h \in N_h^1(\Gamma_C) \right\},
\tag{35}
$$

where $N_h^1(\Gamma_C)$ is the closed convex cone of non-positive continuous functions, piecewise linear on the mesh of Ω^1 on Γ_C.

The discretized mixed variational formulation then becomes: find $U_h \in \mathbf{U}_{0,h}$ and $\lambda_h \in N_h^1(\Gamma_C)$ such that:

$$
\begin{aligned}
&a(U_h,V_h) - \int_{\Gamma_C} \lambda_h\Big(V_h^{1^T}n^1 + V_h^{2^T}n^2\Big)\mathrm{d}S = L(V_h) \quad \forall V_h \in \mathbf{U}_{0,h},\\
&\int_{\Gamma_C} (\mu_h-\lambda_h)\Big(U_h^{1^T}n^1 + U_h^{2^T}n^2\Big)\mathrm{d}S \geqslant 0 \qquad\qquad \forall \mu_h \in N_h^1(\Gamma_C).
\end{aligned}
\tag{36}
$$

It is simple to show that problem (36) admits a unique solution, denoted by (U_h, λ_h). It is also simply shown that U_h belongs to the convex defined in (35) and that the pair (U_h, λ_h) tends towards (U, R_n^C) if the discretization parameter $h=(h_1,h_2)$ tends towards 0. These convergence results and the a priori error estimates can be found in [5,8].

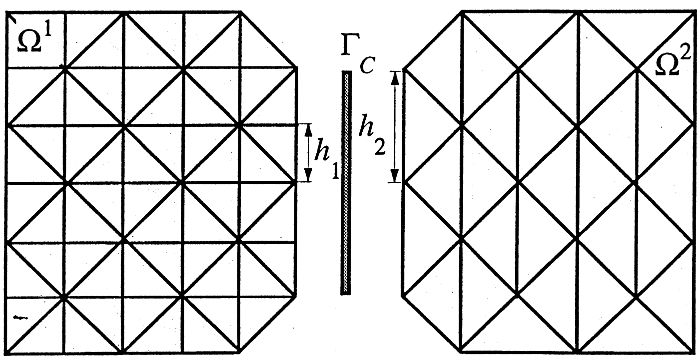

Fig. 2. Non-matching meshes.

Remark. *The mixed problem* (36) *is equivalent to finding the saddle point on* $\mathbf{U}_{0,h} \times N_h^1(\Gamma_C)$ *associated with the following Lagragian*:

$$\mathscr{L}(V_h, \mu_h) = \frac{1}{2} a(V_h, V_h) - \int_{\Gamma_C} \mu_h \left(V_h^{1^T} n^1 + V_h^{2^T} n^2 \right) \mathrm{d}S - L(V_h). \tag{37}$$

In the general case of nonmatching meshes, we choose the mesh of Ω^1 *when defining the Lagrangian multipliers in* $N_h^1(\Gamma_C)$. *In the case of matching meshes, there is obviously no choice in defining the set of the multipliers.*

4.3. The matrix formulation

In order to provide the matrix formulation of the previous mixed problem, we hold fixed h. Thus, we consider a discretization comprising N_ℓ nodes belonging to Ω^ℓ, $\ell = 1, 2$; let $N = N_1 + N_2$. For the sake of simplicity, we assume the absence of embedding conditions. Let m be the number of nodes on Γ_C belonging to the mesh of Ω^1, and let numbers 1 through m correspond to these nodes. We denote the number of nodes on Γ_C belonging to the mesh of Ω^2 by n, and let the numbers between $N_1 + 1$ and $N_1 + 1 + n$ correspond to these nodes. Let $\psi_k, 1 \leqslant k \leqslant m$ and $\varphi_k, 1 \leqslant k \leqslant n$ be the scalar basis functions on the mesh of Ω^1 and Ω^2, respectively, on Γ_C.

The matrix formulation of Eq. (36) then becomes:

$$\begin{pmatrix} \mathbf{K}^1 & 0 \\ 0 & \mathbf{K}^2 \end{pmatrix} \begin{pmatrix} q^1 \\ q^2 \end{pmatrix} - \begin{pmatrix} \mathbf{M}^1 \\ 0 \\ \mathbf{C}^{2,1} \\ 0 \end{pmatrix} \Lambda = \begin{pmatrix} F^1 \\ F^2 \end{pmatrix} \tag{38}$$

or, with obvious notations:

$$\mathbf{K}q - \mathbf{A}\Lambda = F \quad \text{in } \mathbb{R}^N, \tag{39}$$

where q^1 is the vector of components $U_h^1(i),\ 1 \leqslant i \leqslant N_1$; q^2 the vector of components $U_h^2(i),\ N_1 + 1 \leqslant i \leqslant N$; Λ the vector of components $\lambda_h(i),\ 1 \leqslant i \leqslant m$; $\mathbf{K}^1$ (resp. $\mathbf{K}^2$) the N_1-by-N_1 (resp. N_2-by-N_2) stiffness matrix corresponding to Ω^1 (resp. Ω^2); $\mathbf{M}^1$ the m-by-m matrix of coefficients $m_{j,k}^1 = \int_{\Gamma_C} \psi_k \psi_j \,\mathrm{d}S,\ 1 \leqslant j,k \leqslant m$; $\mathbf{C}^{2,1}$ the n-by-m rectangular matrix of coefficients $c_{j,k} = \int_{\Gamma_C} \psi_k \varphi_j \,\mathrm{d}S,\ N_1 + 1 \leqslant j \leqslant N_1 + n + 1,\ 1 \leqslant k \leqslant m$, and $F^1$and F^2 the vectors representing the external loads.

The inequality in (36) yields the remaining conditions. Denoting the vector corresponding to the normal displacements of the nodes of Ω^ℓ on Γ_C by q_N^ℓ, we deduce the matrix formulation corresponding to (36) as follows:

Find $q \in \mathbb{R}^N$ and $\Lambda \in \mathbb{R}^m$ satisfying:

$$\begin{aligned} \mathbf{K}q - \mathbf{A}\Lambda &= F && \text{in } \mathbb{R}^N, \\ \mathbf{M}^1 q_N^1 +^t \mathbf{C}^{2,1} q_N^2 &\leqslant 0 && \text{in } \mathbb{R}^m, \\ \Lambda &\leqslant 0 && \text{in } \mathbb{R}^m, \\ (\mathbf{M}^1 q_N^1 +^t \mathbf{C}^{2,1} q_N^2)^{\mathrm{T}} \Lambda &= 0 && \text{in } \mathbb{R}. \end{aligned} \tag{40}$$

In the current case, the integral condition in (35) can be written: $\mathbf{M}^1 q_N^1 + {}^t\mathbf{C}^{2,1} q_N^2 \leqslant 0$. Let us note that $\mathbf{M}^1$ represents the symmetrical mass matrix on the mesh of Ω^1 on Γ_C. We denote the coupling matrix between the two meshes on the contact zone as $\mathbf{C}^{2,1}$.

Remark.
1. *When the nodes of both bodies fit together on the contact zone*, $\mathbf{M}^1 = \mathbf{C}^{2,1}$ *and the previous condition becomes* $\mathbf{M}^1(q_N^1 + q_N^2) \leqslant 0$; *this does not represent the node-on-node condition which is* $q_N^1 + q_N^2 \leqslant 0$. *The condition* $\mathbf{M}^1(q_N^1 + q_N^2) \leqslant 0$ *allows some slight interpenetration of the bodies.*
2. *The multiplier* Λ *expressing the contact pressure on the contact zone satisfies the non-positiveness condition on* Γ_C.

Let $V_h = (V_h^1, V_h^2) \in \mathbf{U}_{0,h}$ and $\mu_h \in N_h^1(\Gamma_C)$. Denoting the vector of components $V_h^1(i), 1 \leqslant i \leqslant N_1$ by $\tilde{q}^1$, the vector of components $V_h^2(i), N_1+1 \leqslant i \leqslant N$ by $\tilde{q}^2$ and the vector of components $\mu_h(i), 1 \leqslant i \leqslant m$ by $\tilde{\Lambda}$ and in applying the Remark from Section 4.2, it becomes straightforward that the matrix formulation in (40) can also be written as:

$$\max_{\tilde{\Lambda} \leqslant 0} \left(\min_{\tilde{q}} \frac{1}{2} \tilde{q}^{\mathrm{T}} \mathbf{K} \tilde{q} - \tilde{q}^{\mathrm{T}} F - \left(\mathbf{A} \mathrm{T} \tilde{q} \right)^{\mathrm{T}} \tilde{\Lambda} \right), \quad \text{where } \tilde{q}^{\mathrm{T}} = \begin{pmatrix} \tilde{q}^1 \\ \tilde{q}^2 \end{pmatrix}. \tag{41}$$

Using the property $\mathbf{K}q - \mathbf{A}\Lambda = F$ (see (39)) implies that (41) becomes a minimization problem of a quadratic function with convex constraints:

$$\min_{\tilde{\Lambda} \leqslant 0} \left(\frac{1}{2} \tilde{\Lambda}^{\mathrm{T}} \mathbf{A}^{\mathrm{T}} \mathbf{K}^{-1} \mathbf{A} \tilde{\Lambda} + \tilde{\Lambda}^{\mathrm{T}} \mathbf{A}^{\mathrm{T}} \mathbf{K}^{-1} F + \frac{1}{2} F^{\mathrm{T}} \mathbf{K}^{-1} F \right). \tag{42}$$

The problem in (42) is a classical minimization problem and is solved by using an iterative Frank and Wolfe algorithm. With the values of Λ now available, we can obtain q by simply computing $\mathbf{K}^{-1}(F - \mathbf{A}\Lambda)$.

5. Construction of admissible fields

The aim of this section is to describe the construction of the admissible fields by applying the properties of the finite element solution [12].

5.1. Building of the kinematically-admissible displacement field

If both domains are discretized with matching or non-matching meshes on the contact zone, the only difficulty lies in building the displacement fields $\hat{U}^1$ and $\hat{U}^2$ which satisfy the non-interpenetration condition in the case where the integral conditions in (35) have been adopted. The field $U_h = (U_h^1, U_h^2)$ given by the algorithm does not satisfy the non-interpenetration condition (see Remark 1, Section 4.3). We thus set:

$$\begin{aligned} \hat{U}^{1^{\mathrm{T}}} n^1 &= \hat{W}^{1^{\mathrm{T}}} n^1 = U_h^{1^{\mathrm{T}}} n^1 - \frac{E_2}{E_1 + E_2} \max \left(\left(U_h^{1^{\mathrm{T}}} n^1 + U_h^{2^{\mathrm{T}}} n^2 \right), 0 \right), \\ \hat{U}^{2^{\mathrm{T}}} n^2 &= \hat{W}^{2^{\mathrm{T}}} n^2 = U_h^{2^{\mathrm{T}}} n^2 - \frac{E_1}{E_1 + E_2} \max \left(\left(U_h^{1^{\mathrm{T}}} n^1 + U_h^{2^{\mathrm{T}}} n^2 \right), 0 \right), \end{aligned} \tag{43}$$

where E_1 (resp. E_2) denotes the Young's modulus of Ω^1 (resp. Ω^2). With respect to the tangential displacements, we can write:

$$\hat{U}_{\mathrm{t}}^1 = \hat{W}_{\mathrm{t}}^1 = U_{\mathrm{ht}}^1 \quad \text{and} \quad \hat{U}_{\mathrm{t}}^2 = \hat{W}_{\mathrm{t}}^2 = U_{\mathrm{ht}}^2. \tag{44}$$

For the nodes i which are not located on Γ_C, we set:

$$\hat{U}^1(i) = U_h^1(i) \quad \text{and} \quad \hat{U}^2(i) = U_h^2(i) \tag{45}$$

and $\hat{U}^1$ (resp. $\hat{U}^2$) is built on Ω^1 (resp. Ω^2) in a piecewise linear fashion on each element by using the previous values.

5.2. Building of the statically-admissible stress field

The algorithm used to solve problem (42) yields a normal contact pressure Λ which is the vector introduced in (39) corresponding to the nodal values of λ_h (36). We recall that λ_h is continuous, non-positive and piecewise linear on the mesh of Ω^1 on Γ_C.

We set $\hat{R}^1 = \lambda_h n^1$, $\hat{R}^2 = \lambda_h n^2$ and $\hat{R}^C = \hat{R}^1$ so that the equilibrium of the interface (2) is satisfied. One then obtain: $\hat{R}_n^C = \hat{R}^{C^{\mathrm{T}}} n^C = \lambda_h \leqslant 0$ and $\hat{R}_{\mathrm{t}}^C = \hat{R}^C - \hat{R}_n^C n^C = 0$; conditions (6) and (8) are thus fulfilled. The first equation in (36) becomes:

$$\begin{aligned}
&\int_{\Omega^1} \mathrm{Tr}\left[\varepsilon(U_h^1)K^1\varepsilon(V_h^1)\right]\mathrm{d}\Omega = \int_{\Omega^1} f_\mathrm{d}^{1^\mathrm{T}} V_h^1 \,\mathrm{d}\Omega + \int_{\partial_2\Omega^1} F_\mathrm{d}^{1^\mathrm{T}} V_h^1 \,\mathrm{d}S + \int_{\Gamma_\mathrm{C}} \hat{R}^{C^\mathrm{T}} V_h^1 \,\mathrm{d}S \quad \forall V_h^1 \in U_{0,h}^1,\\
&\int_{\Omega^2} \mathrm{Tr}\left[\varepsilon(U_h^2)K^2\varepsilon(V_h^2)\right]\mathrm{d}\Omega = \int_{\Omega^2} f_\mathrm{d}^{2^\mathrm{T}} V_h^2 \,\mathrm{d}\Omega + \int_{\partial_2\Omega^2} F_\mathrm{d}^{2^\mathrm{T}} V_h^2 \,\mathrm{d}S + \int_{\Gamma_\mathrm{C}} \hat{R}^{C^\mathrm{T}} V_h^2 \,\mathrm{d}S \quad \forall V_h^2 \in U_{0,h}^2.
\end{aligned} \tag{46}$$

Denoting $\sigma_h^1 = K^1\varepsilon(U_h^1)$, it becomes obvious that (U_h^1, σ_h^1) is the finite element solution to a linear elasticity problem on Ω^1 without unilateral contact conditions and with given loads f_d^1, F_d^1 and $\hat{R}^C$. Supposing that the volume loads f_d^1 are constant (on each triangle) and that the surface loads F_d^1 are piecewise linear on the mesh of $\partial_2\Omega^1$, the construction of the statically-admissible stress field $\hat{\sigma}^1$ on Ω^1 is then classical [12].

Denoting $\sigma_h^2 = K^2\varepsilon(U_h^2)$, it is also clear that (U_h^2, σ_h^2) is the finite element solution to a linear elasticity problem on Ω^2 with given loads f_d^2, F_d^2 and $\hat{R}^C$.

In the case of matching meshes on the contact zone, the construction of the statically-admissible stress field $\hat{\sigma}^2$ on Ω^2 is also classical.

When considering the construction of the statically-admissible stress field $\hat{\sigma}^2$ in the general case of non-matching meshes, the situation is more complicated than for the construction of $\hat{\sigma}^1$. As a matter of fact, $\hat{R}_n^C$ is piecewise linear on Γ_C on the mesh of Ω^1 but, in the general case, not on the mesh of Ω^2.

Remark.
1. *If the mesh of Ω^2 on Γ_C is strictly finer than the mesh of Ω^1 on Γ_C, then the construction is also classical.*
2. *If the mesh of Ω^1 is strictly finer than the mesh of Ω^2 on Γ_C, it is possible to provide a symmetrical definition of the closed convex cone $N_h^1(\Gamma_\mathrm{C})$ of the Lagrangian multipliers in* (35) *and then to apply multipliers, which are non-positive and piecewise linear on the mesh of Ω^2 on Γ_C.*

If we wish to reconstruct the admissible field $\hat{\sigma}^2$ in the most general case of non-matching meshes, then we must take into account that $\hat{R}_n^C$ is only continuous and piecewise linear on each edge of E^2, a boundary element of Ω^2 (see Fig. 3).

The theoretical construction of $\hat{\sigma}^2$ by using the classical method then becomes possible by dividing the triangles into subtriangles. In practical cases, this is only feasible when the number of subtriangles is small and the triangles are not very flat (Fig. 4 represents the kinds of subtriangles that lead to difficulties).

5.3. Practical constructions

We have observed in the previous sections that the theoretical construction of an admissible $\hat{s} = (\hat{U}^1, \hat{U}^2, \hat{\sigma}^1, \hat{\sigma}^2, \hat{W}^1, \hat{W}^2, \hat{R}^1, \hat{R}^2, \hat{R}^C)$ satisfying $\varphi^*(\hat{R}^C) = 0$ and $\varphi(\hat{W}^2 - \hat{W}^1) = 0$ is always possible by applying the appropriate properties of the finite element solution (36). In order to "simplify" the numerical implementation, we now propose two reasonable alternatives which will be tested and then compared to the analytical construction. The latter is considered first.

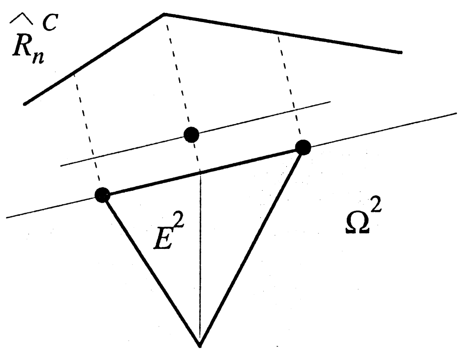

Fig. 3. The given contact pressure on the triangle E^2.

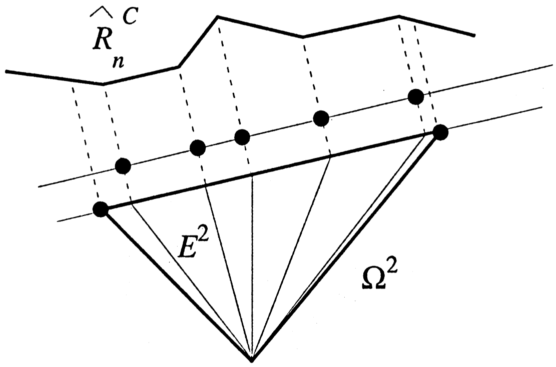

Fig. 4. Flat subtriangles.

5.3.1. Analytical construction of a strictly-admissible stress field

This building process is classically carried out in two steps:

- during the first step, densities of forces $\hat{F}$ are constructed on the edge of each element; these densities are in equilibrium with the body forces f_d^ℓ, and
- during the second step, the strictly statically-admissible field $\hat{\sigma}^\ell$ is constructed element-by-element, using the densities as boundary conditions.

This construction is easy to accomplish when there are matching meshes on the contact zone or if one mesh is a submesh of the other. Otherwise, the theoretical construction is practically possible when the subdivision of the triangles does not lead to overly-flat triangles.

The other case (leading to practical difficulties) appears when some nodes of Ω^1 and Ω^2 are very close in comparison with the discretization parameters h_1 and h_2; in this case, very flat triangles cannot be avoided.

Next, we propose two alternatives for treating the latter case.

5.3.2. Numerical construction by use of higher-degree polynomials

The chosen technique consists of searching the stress field $\hat{\sigma}^\ell$ on the element E of Ω^ℓ in the following form [17]:

$$\hat{\sigma}^\ell_{|E} = K^\ell \varepsilon(V_E), \tag{47}$$

where V_E is a displacement field defined on E, a polynomial of degree $p + k$ with p being the degree of the element used to carry out the finite element analysis and k being a strictly positive integer.

More precisely, V_E is the solution to the following finite element problem on a single element E: find V_E, a polynomial of degree $p + k$ on E, such that:

$$\forall V^* \text{ displacement field of degree } p + k \text{ on } E,$$
$$\int_E \mathrm{Tr}\left[K^\ell \varepsilon(V_E)\varepsilon(V^*)\right]\mathrm{d}E = \int_E f_d^\ell V^* \,\mathrm{d}E + \int_{\partial E} \hat{F}_{|\partial E} V^* \,\mathrm{d}S. \tag{48}$$

This problem can be solved, apart from one displacement field of a solid, on account of the equilibrium of the densities $\hat{F}$ with the body forces f_d^ℓ.

On each element E of the mesh, a small linear system must be solved:

$$\mathbf{K}_{p+k} U = \mathbf{F}_{p+k}, \tag{49}$$

where $\mathbf{K}_{p+k}$ denotes the classical rigidity matrix constructed with the interpolation polynomials φ_i^{p+k} of degree $p + k$ and $\mathbf{F}_{p+k}$ denotes the vector of generalized forces.

Thus, we obtain an approximate field which is no longer strictly admissible, yet which leads to efficient error estimators as shown in [17].

5.3.3. Approximation of the construction by applying the specific properties of the finite element solution in (36)

This technique consists of building $\hat{s} = (\hat{U}^1, \hat{U}^2, \hat{\sigma}^1, \hat{\sigma}^2, \hat{W}^1, \hat{W}^2, \hat{R}^1, \hat{R}^2, \hat{R}^C)$ that satisfies (11), $\varphi^*(\hat{R}^C) = 0$, $\varphi(\hat{W}^2 - \hat{W}^1) = 0$, (12) with a weak interpretation of condition (2) which will be specified later.

The construction of $\hat{U}^1, \hat{U}^2, \hat{W}^1, \hat{W}^2$ represents the theoretical construction proposed in (43)–(45). Next, we set $\hat{R}^1 = \lambda_h n^1$ (with λ_h corresponding to the solution of (42)), $\hat{R}^C = \hat{R}^1$ and $\hat{R}^2 = \lambda_h^2 n^2$, where λ_h^2 is the continuous, piecewise linear function on the mesh of Ω^2 on Γ_C satisfying:

$$\int_{\Gamma_C} \left(\lambda_h - \lambda_h^2\right) \chi_h^2 \, dS = 0 \tag{50}$$

for all χ_h^2 which is continuous and piecewise linear on the mesh of Ω^2 on Γ_C.

The first equation of (36) then becomes:

$$\begin{aligned} &\int_{\Omega^1} \mathrm{Tr}\left[\varepsilon(U_h^1) K^1 \varepsilon(V_h^1)\right] d\Omega = \int_{\Omega^1} f_d^{1^T} V_h^1 \, d\Omega + \int_{\partial_2 \Omega^1} F_d^{1^T} V_h^1 \, dS + \int_{\Gamma_C} \hat{R}^{C^T} V_h^1 \, dS \quad \forall V_h^1 \in U_{0,h}^1, \\ &\int_{\Omega^2} \mathrm{Tr}\left[\varepsilon(U_h^2) K^2 \varepsilon(V_h^2)\right] d\Omega = \int_{\Omega^2} f_d^{2^T} V_h^2 \, d\Omega + \int_{\partial_2 \Omega^2} F_d^{2^T} V_h^2 \, dS + \int_{\Gamma_C} \hat{R}^{2^T} V_h^2 \, dS \quad \forall V_h^2 \in U_{0,h}^2. \end{aligned} \tag{51}$$

The last equality results from the construction of $\hat{R}^2$, whose components are piecewise linear on the mesh of Ω^2 on Γ_C. The construction of $\hat{\sigma}^1$ and $\hat{\sigma}^2$ is then classical. This technique consists of approximating the theoretical construction with fields $\hat{R}^1, \hat{R}^2$ and $\hat{R}^C$, which satisfy, albeit weakly (see (50)), Eq. (2).

6. Numerical studies

6.1. Example

In the first example, we consider the problem of two elastic bodies initially in contact. The upper body is submitted to a uniform load (see Fig. 5). In this problem, we have adopted symmetry conditions in order to avoid a greater number of singularities. The material characteristics are: $E = 200$ GPa, $\upsilon = 0.3$.

Due to the lack of an analytical solution for such a problem, we use a reference solution, denoted U_{ref}, which is a finite element solution associated with a very refined mesh. Then, the exact error ε_{ex} can be defined as follows:

$$\varepsilon_{\text{ex}} = \left[\frac{\sum_{\ell=1}^{2} \|U_{\text{ref}} - U_h\|_{u,\Omega^\ell}^2}{\sum_{\ell=1}^{2} \|U_{\text{ref}} + U_h\|_{u,\Omega^\ell}^2} \right]^{1/2}. \tag{52}$$

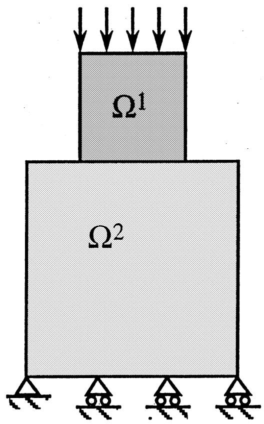

Fig. 5. Problem set-up.

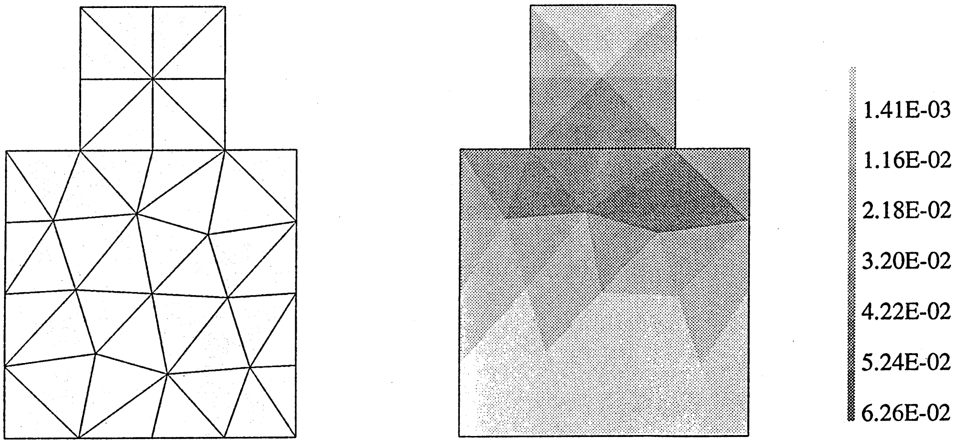

Fig. 6. Initial matching mesh and map of contributions ε_E to the error.

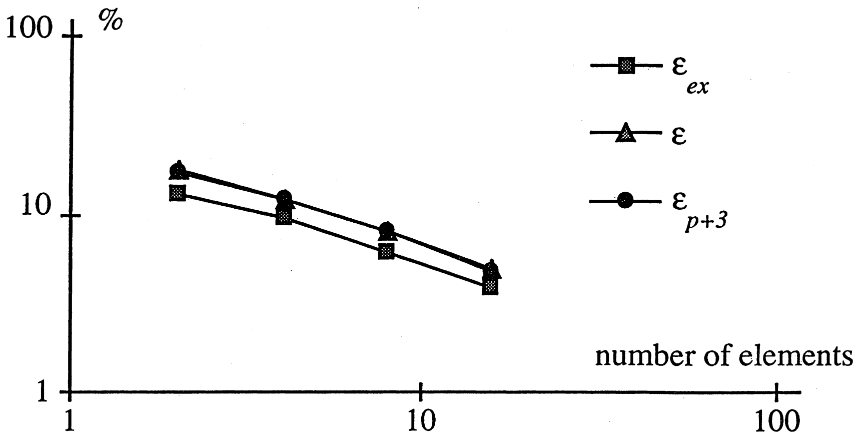

Fig. 7. Convergence of the error estimators as a function of the number of elements on the contact zone with matching meshes.

By subcutting the initial mesh (see Fig. 6), the trend in the errors can be studied. In Fig. 7, the convergence rates of the exact error $\varepsilon_{\mathrm{ex}}$ and the computed error ε defined in (15) are compared as a function of the number of elements on the contact zone. We compare the two errors ε estimated with strictly-admissible stress fields (see Section 5.3.1) and ε_{p+3} with numerical constructions (see Section 5.3.2).

We can observe that the convergence rate of the exact error is the same as the convergence rate of the errors in the constitutive relation in the case of matching meshes.

Moreover, the lower bound and the upper bound of the effectivity index $\gamma = \varepsilon/\varepsilon_{\mathrm{ex}}$ are 1.28 and 1.40, respectively (see Fig. 8), whereas for numerical constructions, the bounds of $\gamma_{p+3} = \varepsilon_{p+3}/\varepsilon_{ex}$ are 1.26 and 1.36, respectively.

In Fig. 10, the convergence rates are compared in the case of non-matching meshes (see Fig. 9). The mesh of Ω^2 is a submesh of Ω^1 on the contact zone that allows us to build strictly-admissible fields.

Moreover, the lower and the upper bound of the effectivity index γ are 1.17 and 1.47, respectively (see Fig. 11), whereas the bounds of γ_{p+3} are 1.16 and 1.42, respectively.

6.2. Mesh adaptivity

6.2.1. Algorithm

The goal of a mesh adaptation procedure is to guarantee to the finite element user a certain level of precision by minimizing the computation costs. We will use the h-generation, which is the most

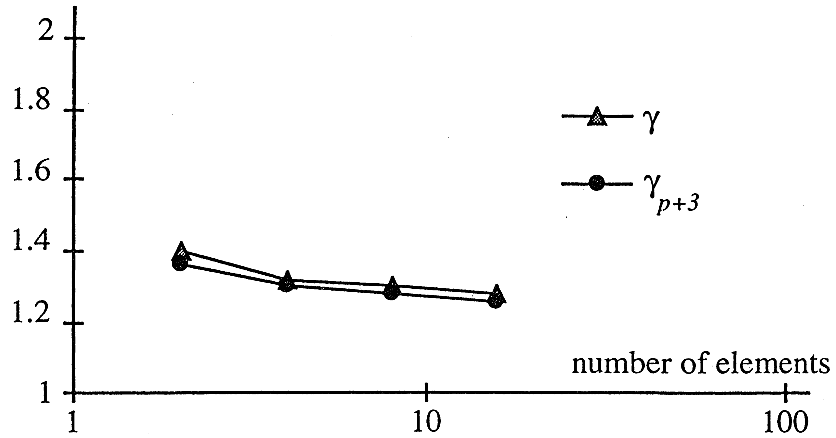

Fig. 8. Effectivity of the error estimators as a function of the number of elements on the contact zone with matching meshes.

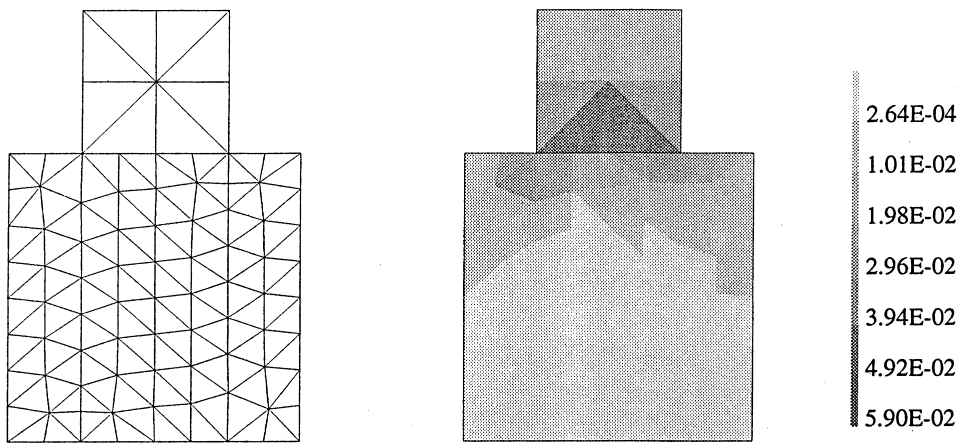

Fig. 9. Initial non-matching mesh and map of contributions ε_E to the error.

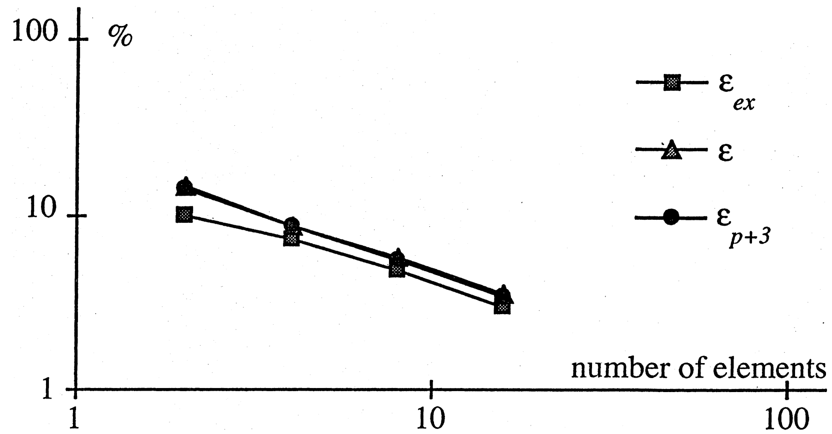

Fig. 10. Convergence of the error estimators as a function of the number of elements of Ω^1 on the contact zone with non-matching meshes.

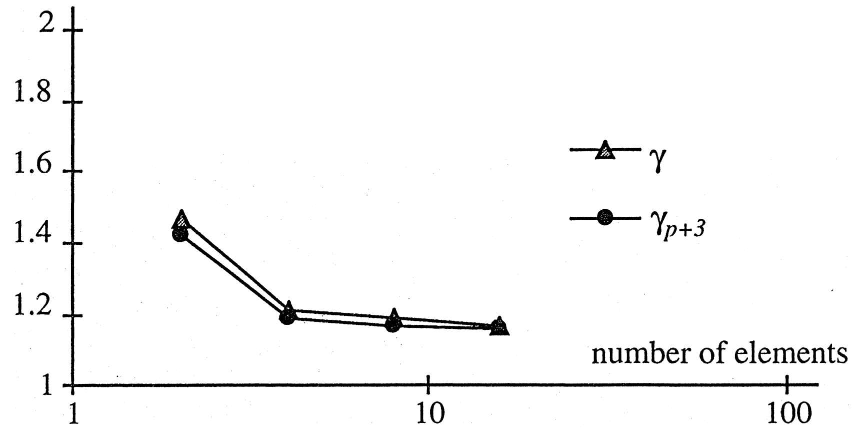

Fig. 11. Effectivity of the error estimators as a function of the number of elements of Ω^1 on the contact zone with non-matching meshes.

frequently-employed procedure: the size and topology of the elements are changed while the type of finite element functions on the different meshes remains the same. A mesh $\mathbf{T}^*$ is said to be optimal [11] for an error measure ε if:

$$\begin{aligned} &\varepsilon = \varepsilon_0, \ \text{the accuracy prescribed by the user,} \\ &N^* \ \text{is minimized (element number of mesh } \mathbf{T}^*). \end{aligned} \tag{53}$$

In order to solve the problem in (53), we adopt the following technique [18]:

- computation on a coarse mesh $\mathbf{T}$;
- computation of both the global relative error ε and the local contributions ε_E;
- determination of the optimized mesh $\mathbf{T}^*$ and
- second computation on the new mesh $\mathbf{T}^*$.

6.2.2. Examples

For the first example, we once again take the problem of two elastic bodies initially in contact (see Fig. 5). The problem is optimized for a desired error of 5%.

The initial mesh comprises 640 three-node triangles and this yields a computed error ε of 7.64% (see Fig. 12). The deflection is shown in Fig. 13. The optimized mesh comprises 514 triangles for a computed error of 4.42% (see Fig. 14).

For the second example, we take the case of a cork (see Fig. 15). The initial mesh comprises 112 triangles and this yields an error of 32.39% (see Fig. 16). The desired error is set at 10%. The map of contributions ε_E to the error is shown in Fig. 17.

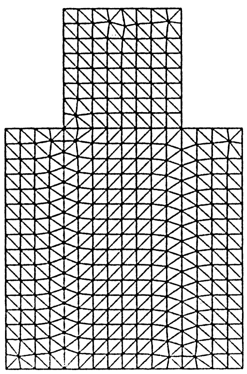

Fig. 12. Initial mesh: 640 3-node elements, 370 nodes, $\varepsilon = 7.64\%$, $\varepsilon_{ex} = 6.12\%$, $\varepsilon_0 = 5\%$.

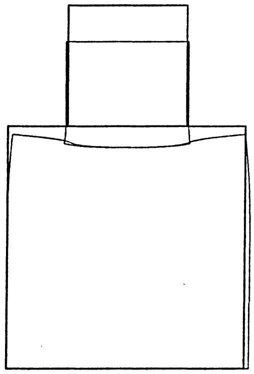

Fig. 13. Deflection.

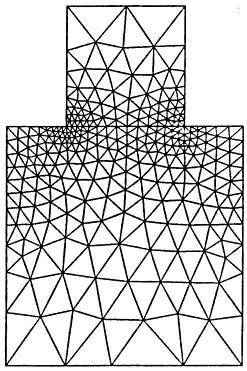

Fig. 14. Optimized mesh: 514 3-node elements, 303 nodes, $\varepsilon = 4.42\%$, $\varepsilon_{ex} = 4.08\%$.

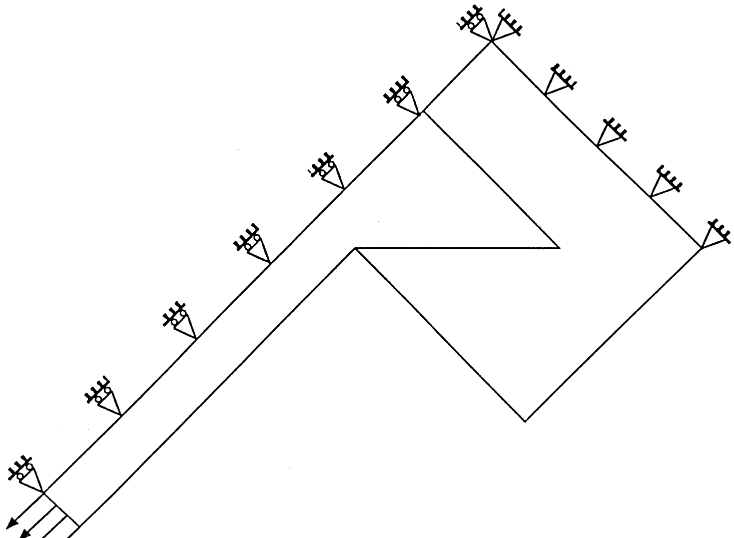

Fig. 15. Problem set-up.

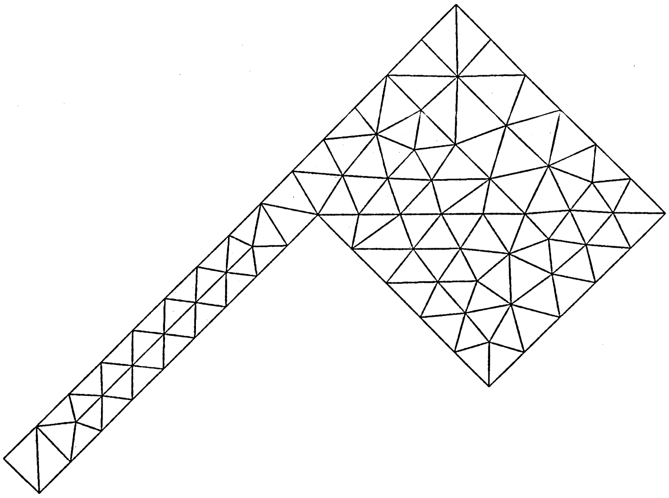

Fig. 16. Initial mesh: 112 3-node elements, 87 nodes, $\varepsilon = 32.39\%$, $\varepsilon_{ex} = 22.3\%$, $\varepsilon_0 = 10\%$.

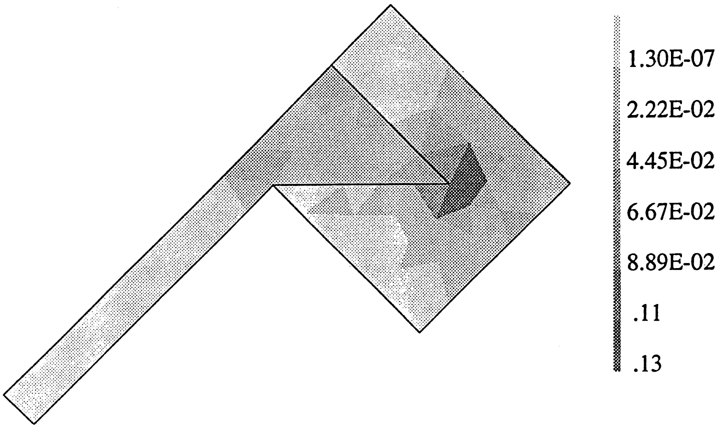

Fig. 17. Map of contributions ε_E to the error.

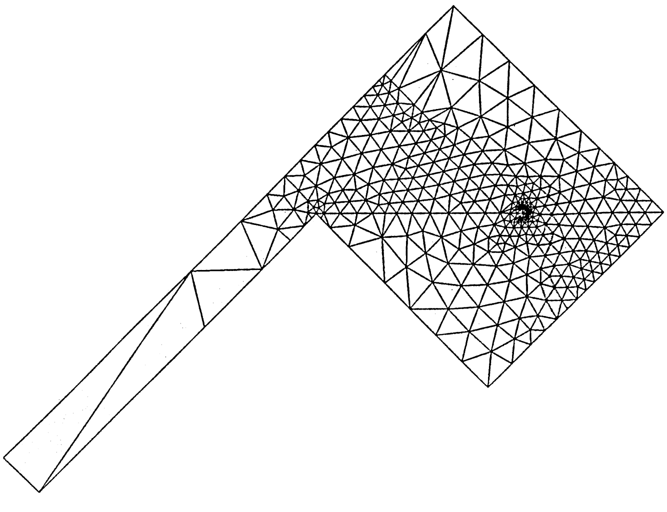

Fig. 18. Optimized mesh with matching mesh: 696 3-node elements, 411 nodes, $\varepsilon = 11.78\%$, $\varepsilon_{ex} = 9.27\%$.

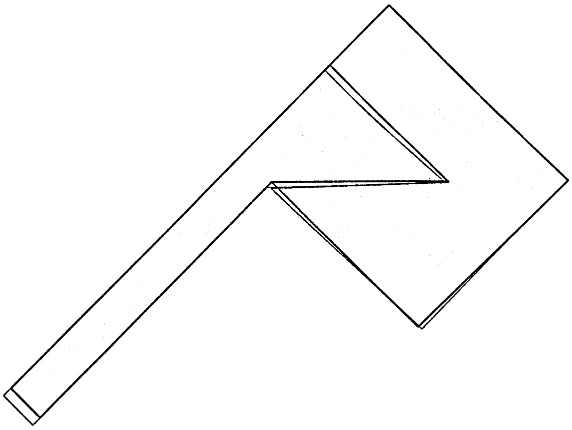

Fig. 19. Deflection.

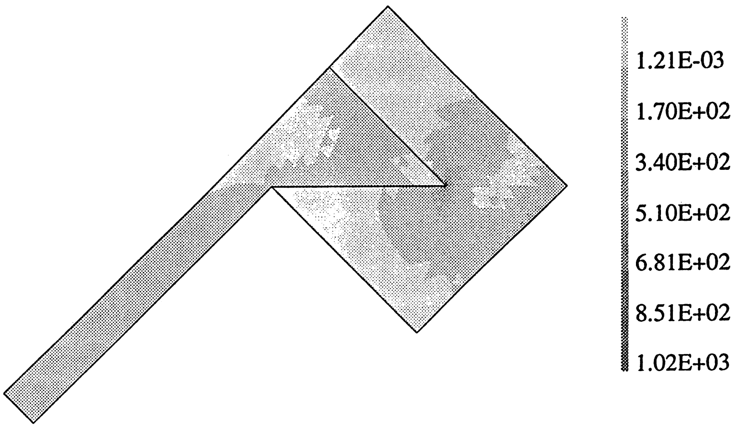

Fig. 20. Map of Von Mises stresses.

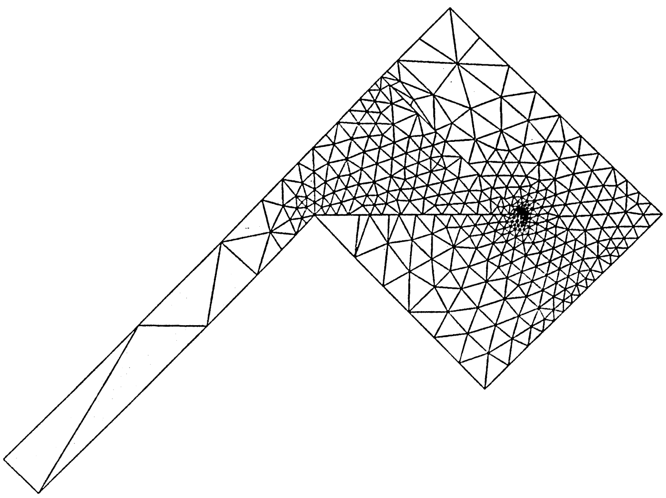

Fig. 21. Optimized mesh with non-matching mesh: 709 3-node elements, 414 nodes, $\varepsilon_{ex} = 8.96\%$.

By using a technique of mesh automation [19], in three steps we obtain an optimized mesh with 696 elements, a computed error of 11.78% and an exact error of 9.27% (see Fig. 18) in the case of matching meshes. The deflection and the map of Von Mises stresses are shown in Figs. 19 and 20.

In two steps, in the case of non-matching meshes, we obtain an optimized mesh with 709 elements and an exact error of 8.96% (see Fig. 21). The computed error is 11.22% using the method presented in Section 5.3.2 and 11.42% with the method in Section 5.3.3.

These various examples show that optimized meshes for the level of accuracy prescribed by the user, as well as efficient effectivity indexes, can be obtained.

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