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Abstract

We study the Nitsche-based finite element method for contact with Coulomb friction considering both static and dynamic situations. We provide existence and/or uniqueness results for the discretized problems under appropriate assumptions on physical and numerical parameters. In the dynamic case, existence and uniqueness of the space semi-discrete problem holds for every value of the friction coefficient and the Nitsche parameter. In the static case, if the Nitsche parameter is large enough, existence is ensured for any friction coefficient, and uniqueness can be obtained provided that the friction coefficient is below a bound that depends on the mesh size. These results are complemented by a numerical study.

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1. Introduction

Many problems in structural engineering involve frictional contact, and are approximated numerically using variational techniques, among them the Finite Element Method (FEM) [22, 23, 25, 32, 33, 49, 50]. Friction is generally taken into account using Coulomb's law, that is relevant for a broad range of applications.

The goal of this work is to present some first theoretical results for Coulomb friction discretized with the FEM and a Nitsche's formulation of frictional contact conditions (Nitsche-FEM). We present the method in the small strain framework, for the dynamic and the static settings. Our main results are, first, a well-posedness theorem in the dynamic setting, which guarantees the existence and uniqueness of a semi-discrete solution in space. Secondly, we obtain an existence result in the static setting, which ensures, if the Nitsche parameter is large enough, that there is at least one discrete solution, irrespectively of the values of the friction coefficient and of the mesh size. In this static setting, uniqueness of solutions is recovered with very restrictive conditions on the friction coefficient and the numerical parameters, which is something expected for this problem (see, *e.g.*, [24, 29]). These theoretical results are complemented by a numerical study. Particularly, this is the first time, to the best of our knowledge, that numerical results are obtained with the Nitsche-FEM for frictional elastodynamics. Part of the results presented here have been announced, without detailed proofs, in conference proceedings [10, 12].

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Let us put our work in perspective with the litterature. Contact with Coulomb friction in elastostatics remains a difficult problem, with still some open issues in its mathematical analysis, both for the continuous and the discrete problems. In the continuous case, there is indeed no complete characterization of existence and uniqueness when the friction coefficient is varied (see, *e.g.*, [20, 21, 40] for existence results when the friction coefficient is small). Moreover it can be proved that uniqueness is lost in some configurations and multiple solutions can be obtained, see, *e.g.*, [4, 27]. The existence and uniqueness of solutions to the discrete problem has been studied for instance in [28] and later on in [26, 34], for more general formulations, especially a variable friction coefficient. Also, comparatively to the frictionless case or to the Tresca friction case, more difficulties appear in the numerical analysis of the method: even in situations when uniqueness can be ensured at the continuous level (see, *e.g.*, [43]), the obtention of optimal error estimates in H^1–norm is still an open issue, see *e.g.*, [24, 29]. Moreover special care has to be taken in numerical solving when situations of non-uniqueness occur [28, 30, 36, 37]. In fact, it is well known that the finite element problem admits a solution and that the solution is unique if the friction coefficient is small enough, but the denomination small depends on the discretization parameter. Moreover, the improvement of numerical methods to solve contact with Coulomb friction represents still a very active research field (see for instance [3, 51]).

For Coulomb friction in elastodynamics, results are even more scattered (see [21] for a review). A pioneering work [38] addresses the problem of frictional contact with a normal compliance law, and existence and uniqueness of a solution is established. Another work [42] carries out the numerical analysis of a one dimensional contact problem with Coulomb friction. In [5], resp. [7], a well-posedness result is proven for a single particle, resp. a collection of particles, undergoing contact with Coulomb friction. A key assumption for the proof is the analyticity of the data. When the modified mass method of Khenous, Laborde and Renard [31] is used for the space semi-discretization with FEM of contact with Coulomb friction, well-posedness of the semi-discrete problem in space has been established in [35] and [19]. This latter reference provides also a convergence result.

In the last years, a Nitsche-FEM has been designed to handle contact conditions in elasticity, first for the frictionless contact problem of Signorini, in statics, and in the small strain framework [11, 15]. Nitsche's formulation [41] differs from standard penalization techniques which are generally non consistent. Besides, no Lagrange multiplier is needed and no discrete inf-sup condition must be fulfilled contrarily to mixed methods. A formulation for the Tresca friction problem has been made in [8]. Notably, it has been proven, without any assumption on the contact/friction set, the optimal convergence of the Nitsche-FEM in the $H^1(\Omega)$ norm, which is $O(h^{1/2+\nu})$ when the solution lies in $H^{3/2+\nu}(\Omega), 0 < \nu \leq 1/2$. The numerical analysis of this paper has been extended to the contact between two elastic bodies, with an unbiased formulation [16], and also for Hybrid High Order (HHO) discretization on polytope meshes [9].

Nitsche's formulation has also been extended to solve contact with Coulomb friction. This method has been first formulated, in the static case and in small strain, in [44]. This was accompanied by several numerical tests, in two and three dimensions, to assess the performance of a generalized Newton algorithm. Later on, an extension to contact in large deformations with Coulomb's friction has been performed and tested numerically in [39], for the quasi-static setting (see also [46]).

For contact in elastodynamics, a Nitsche-FEM has been devised and analyzed in [13, 14, 17], for frictionless contact and in small deformations. In this situation, the Nitsche-FEM leads to a well-posed semi-discrete problem in space, as for the penalty method and the modified mass method [31]. Several implicit or explicit time-marching schemes have been proposed, analysed and tested numerically, still in the frictionless case. The Nitsche-FEM has never been extended to Coulomb's friction in dynamics, though the method has been formulated in [10].

Let us introduce some useful notations. In what follows, bold letters like $\mathbf{u}, \mathbf{v}$, indicate vector or tensor valued quantities, while the capital ones (e.g., $\mathbf{V}, \mathbf{K} \ldots$) represent functional sets involving vector fields.

As usual, we denote by $(H^s(.))^d$, $s \in \mathbb{R}, d = 1, 2, 3$ the Sobolev spaces in one, two or three space dimensions (see [1]). The usual scalar product of $(H^s(D))^d$ is denoted by $(\cdot,\cdot)_{s,D}$, and $\|\cdot\|_{s,D} = (\cdot,\cdot)^{\frac{1}{2}}_{s,D}$ denotes the corresponding norm. We keep the same notation when $d = 1$ or $d > 1$. The letter C stands for a generic constant, independent of the discretization parameters.

2. Setting

Coulomb friction in small strain elasticity is first presented in the dynamic case, then in the static case.

2.1. The dynamic problem

We consider an elastic body Ω in $\mathbb{R}^d$ with $d = 2, 3$. Small strain assumptions are made (as well as plane strain when $d = 2$). The boundary $\partial\Omega$ of Ω is polygonal ($d = 2$) or polyhedral ($d = 3$). The outward unit normal vector on $\partial\Omega$ is denoted $\mathbf{n}$. We suppose that $\partial\Omega$ consists in three nonoverlapping parts Γ_D, Γ_N and the contact boundary Γ_C, with $\text{meas}(\Gamma_D) > 0$ and $\text{meas}(\Gamma_C) > 0$. The contact boundary is supposed to be a straight line segment when $d = 2$ or a polygon when $d = 3$ to simplify. In the reference configuration, the body is in frictional contact on Γ_C with a rigid foundation and we suppose that the unknown contact zone during deformation is included into Γ_C. The body is clamped on Γ_D for the sake of simplicity. It is subjected to volume forces $\mathbf{f}$ in Ω and to surface loads $\mathbf{g}$ on Γ_N.
We consider the unilateral contact problem with Coulomb friction in linear elastodynamics during a time interval $[0, T)$ where $T > 0$ is the final time. We denote by $\Omega_T := (0, T) \times \Omega$ the time-space domain, and similarly $\Gamma_{DT} := (0, T) \times \Gamma_D$, $\Gamma_{NT} := (0, T) \times \Gamma_N$ and $\Gamma_{CT} := (0, T) \times \Gamma_C$. The problem then consists in finding the displacement field $\mathbf{u} : [0, T) \times \Omega \to \mathbb{R}^d$ verifying the equations and conditions (1)–(5):

$$\rho\ddot{\mathbf{u}} - \mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u}) = \mathbf{f} \qquad \text{in } \Omega_T, \tag{1}$$

are the equations of motion for the body where the notation $\dot{\mathbf{x}}$ is used for the time-derivative of a vector field $\mathbf{x}$ so that $\dot{\mathbf{u}}$ is the velocity of the elastic body and $\ddot{\mathbf{u}}$ its acceleration. The notation $\boldsymbol{\sigma} = (\sigma_{ij})$, $1 \le i, j \le d$, stands for the stress tensor field and $\mathbf{div}$ denotes the divergence operator of tensor valued functions. The stress tensor is defined using the constitutive relation $\boldsymbol{\sigma}(\mathbf{u}) = \mathbf{A}\,\boldsymbol{\varepsilon}(\mathbf{u})$ where $\boldsymbol{\varepsilon}(\mathbf{v}) = (\boldsymbol{\nabla}\mathbf{v} + \boldsymbol{\nabla}\mathbf{v}^T)/2$ represents the linearized strain tensor field and $\mathbf{A}$ is the fourth order symmetric elasticity tensor having the usual uniform ellipticity and boundedness properties. The density of the elastic material denoted by ρ is supposed to be constant to simplify (this is not restrictive and the results can be extended straightforwardly for a variable density). The prescribed displacements and density of forces are expressed by the equations:

$$\begin{aligned} \mathbf{u} &= \mathbf{0} && \text{on } \Gamma_{DT}, \\ \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} &= \mathbf{g} && \text{on } \Gamma_{NT}. \end{aligned} \tag{2}$$

For any displacement field $\mathbf{v}$ and for any density of surface forces $\boldsymbol{\sigma}(\mathbf{v})\mathbf{n}$ defined on $\partial\Omega$ we adopt the following notation:

$$\mathbf{v} = v_n\mathbf{n} + \mathbf{v_t} \quad \text{and} \quad \boldsymbol{\sigma}(\mathbf{v})\mathbf{n} = \sigma_n(\mathbf{v})\mathbf{n} + \boldsymbol{\sigma}_\mathbf{t}(\mathbf{v}),$$

where $\mathbf{v_t}$ (resp. $\boldsymbol{\sigma}_\mathbf{t}(\mathbf{v})$) are the tangential components of $\mathbf{v}$ (resp. $\boldsymbol{\sigma}(\mathbf{v})\mathbf{n}$). The conditions describing unilateral contact on Γ_{CT} are:

$$u_n \le 0, \quad \sigma_n(\mathbf{u}) \le 0, \quad \sigma_n(\mathbf{u})\,u_n = 0 \tag{3}$$

and those modelling Coulomb friction on Γ_{CT} can be written as follows

$$\begin{aligned} \dot{\mathbf{u}}_{\mathbf{t}} = \mathbf{0} \implies & \quad |\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u})| \leq -\mathscr{F}\sigma_n(\mathbf{u}) \\ \dot{\mathbf{u}}_{\mathbf{t}} \neq \mathbf{0} \implies & \quad \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) = \mathscr{F}\sigma_n(\mathbf{u})\frac{\dot{\mathbf{u}}_{\mathbf{t}}}{|\dot{\mathbf{u}}_{\mathbf{t}}|} \end{aligned} \tag{4}$$

where $\mathscr{F} > 0$ stands for the friction coefficient ($\mathscr{F} = 0$ corresponds to the frictionless case). Finally we need to add the initial conditions

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0, \qquad \dot{\mathbf{u}}(0,\cdot) = \dot{\mathbf{u}}_0 \qquad \text{in } \Omega, \tag{5}$$

where $\mathbf{u}_0$ is the initial displacement and $\dot{\mathbf{u}}_0$ is the initial velocity. Note additionally that the initial displacement $\mathbf{u}_0$ should satisfy the compatibility condition $u_{0n} \leq 0$ on Γ_C.

Remark 2.1. *A quasi-static problem can be obtained if the inertial terms are neglected in the equations of motion. Then (1) simply becomes*

$$-\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u}) = \mathbf{f} \qquad \text{in } \Omega_T, \tag{6}$$

while the other equations (2)–(4) remain unchanged and (5) merely reduces to $\mathbf{u}(0,\cdot) = \mathbf{u}_0$ *in* Ω.

2.2. The static problem

It consists of considering the quasi-static model and to approximate $\dot{\mathbf{u}}_{\mathbf{t}}(t,x)$ using a time increment: $\dot{\mathbf{u}}_{\mathbf{t}}(t,x) \approx (\mathbf{u}_{\mathbf{t}}(t,x) - \mathbf{u}_{\mathbf{t}}(t-\Delta t,x))/\Delta t$. Supposing to simplify that $\mathbf{u}_{\mathbf{t}}(t-\Delta t,x)$ equals zero yields the static friction model where (4) becomes

$$\begin{aligned} \mathbf{u}_{\mathbf{t}} = \mathbf{0} \implies & \quad |\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u})| \leq -\mathscr{F}\sigma_n(\mathbf{u}) \\ \mathbf{u}_{\mathbf{t}} \neq \mathbf{0} \implies & \quad \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) = \mathscr{F}\sigma_n(\mathbf{u})\frac{\mathbf{u}_{\mathbf{t}}}{|\mathbf{u}_{\mathbf{t}}|}. \end{aligned} \tag{7}$$

The static model consists then for a fixed t to find a displacement field $\mathbf{u} : \Omega \to \mathbb{R}^d$ satisfying (2), (3), (6) and (7).

3. Nitsche finite element discretizations

In this section we derive Nitsche-FEM for the dynamic and static settings presented above.

3.1. Preliminaries

We make use of the notation $[\cdot]_{\mathbb{R}^-}$, that stands for the projection onto $\mathbb{R}^-$ (i.e., $[x]_{\mathbb{R}^-} = \frac{1}{2}(x - |x|)$ for $x \in \mathbb{R}$). Moreover, for any $\alpha \in \mathbb{R}^+$, we introduce the notation $[\cdot]_\alpha$ for the orthogonal projection onto $\mathscr{B}(\mathbf{0},\alpha) \subset \mathbb{R}^{d-1}$, where $\mathscr{B}(\mathbf{0},\alpha)$ is the closed ball centered at the origin $\mathbf{0}$ and of radius α. This operation can be defined analytically, for $\mathbf{x} \in \mathbb{R}^{d-1}$ by:

$$[\mathbf{x}]_\alpha = \begin{cases} \mathbf{x} & \text{if } |\mathbf{x}| \leq \alpha, \\ \alpha\dfrac{\mathbf{x}}{|\mathbf{x}|} & \text{otherwise.} \end{cases}$$

It is easy to check that

$$|[\mathbf{x}]_\alpha - [\mathbf{y}]_\alpha| \leq |\mathbf{x} - \mathbf{y}|, \qquad ([\mathbf{x}]_\alpha - [\mathbf{y}]_\alpha)^2 \leq ([\mathbf{x}]_\alpha - [\mathbf{y}]_\alpha)(\mathbf{x} - \mathbf{y}), \tag{8}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{d-1}$. Let $\alpha, \beta \in \mathbb{R}_+$, it holds, for all $\mathbf{x} \in \mathbb{R}^{d-1}$:

$$|[\mathbf{x}]_\beta - [\mathbf{x}]_\alpha| \leq |\beta - \alpha|. \tag{9}$$

The derivation of a Nitsche-based method comes from the observation that the unilateral contact conditions (3), the static (resp. dynamic) Coulomb friction conditions (7) (resp. (4)) can be reformulated with only one equation as seen hereafter.

Proposition 3.1. *Let γ be a positive function defined on Γ_C.*
The unilateral contact conditions (3) can be reformulated as follows:

$$\sigma_n(\mathbf{u}) = [\sigma_n(\mathbf{u}) - \gamma\, u_n]_{\mathbb{R}^-}\ . \tag{10}$$

Suppose that (3) holds. Then the static (resp. dynamic) Coulomb friction conditions (7) (resp. (4)) can be reformulated as follows:

$$\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) = [\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) - \gamma\mathbf{u}_\mathbf{t}]_{(-\mathscr{F}\sigma_n(\mathbf{u}))} = [\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) - \gamma\mathbf{u}_\mathbf{t}]_{(-\mathscr{F}[\sigma_n(\mathbf{u})-\gamma u_n]_{\mathbb{R}^-})}\ , \tag{11}$$

respectively

$$\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) = [\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) - \gamma\dot{\mathbf{u}}_\mathbf{t}]_{(-\mathscr{F}\sigma_n(\mathbf{u}))} = [\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) - \gamma\dot{\mathbf{u}}_\mathbf{t}]_{(-\mathscr{F}[\sigma_n(\mathbf{u})-\gamma u_n]_{\mathbb{R}^-})}\ . \tag{12}$$

Proof: To establish equality (10) a direct proof can be found in [11, Proposition 2.1] (see also, e.g., [2, 44, 15]). The second identity (11) is a direct adaption of the proof made for the Tresca friction case in [8, Proposition 2.4] with some additional changes in the notations. Note that the second equality of (11) follows straightforwardly from (10). To render the paper more self contained, we next prove that the first equality in (11) is equivalent to (7).

• First we suppose that (7) holds.

Consider the case $\mathbf{u}_\mathbf{t} = \mathbf{0}$, we get $|\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u})| \leq -\mathscr{F}\sigma_n(\mathbf{u})$. Due to the property of the projection it results that $\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) = [\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u})]_{(-\mathscr{F}\sigma_n(\mathbf{u}))}$.

In the case $\mathbf{u}_\mathbf{t} \neq \mathbf{0}$, we have $|\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u})| = -\mathscr{F}\sigma_n(\mathbf{u})$ so $\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) = [\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u})]_{(-\mathscr{F}\sigma_n(\mathbf{u}))}$. In addition $\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u})$ either vanishes (obvious case) or $-\gamma\mathbf{u}_\mathbf{t} = \alpha\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u})$ with $\alpha > 0$, hence

$$\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) = [\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u})]_{(-\mathscr{F}\sigma_n(\mathbf{u}))} = [(1+\alpha)\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u})]_{(-\mathscr{F}\sigma_n(\mathbf{u}))} = [\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) - \gamma\mathbf{u}_\mathbf{t}]_{(-\mathscr{F}\sigma_n(\mathbf{u}))}$$

which proves (11).

• Suppose now that the condition (11) holds. Whatever the value of $\mathbf{u}_\mathbf{t} = \mathbf{0}$ is, we deduce immediately from (11) that $|\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u})| \leq -\mathscr{F}\sigma_n(\mathbf{u})$. Therefore we only have to consider the case $\mathbf{u}_\mathbf{t} \neq \mathbf{0}$ in (7). From $\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) = [\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) - \gamma\mathbf{u}_\mathbf{t}]_{(-\mathscr{F}\sigma_n(\mathbf{u}))}$ we see that:

- if $\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) = 0$ then necessarily $\sigma_n(\mathbf{u}) = 0$ (since $\mathscr{F} \neq 0$), so (7) holds,
- if $\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) \neq 0$ then necessarily $\sigma_n(\mathbf{u}) < 0$ and there exists $\beta \in (0,1)$ such that $\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) = \beta(\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) - \gamma\mathbf{u}_\mathbf{t})$, so

$$-\gamma\mathbf{u}_\mathbf{t} = \frac{1-\beta}{\beta}\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}), \tag{13}$$

with the quantity $(1-\beta)/\beta > 0$. Therefore (11) becomes $\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) = \left[\beta^{-1}\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u})\right]_{(-\mathscr{F}\sigma_n(\mathbf{u}))}$, so $|\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u})| = -\mathscr{F}\sigma_n(\mathbf{u})$ and finally (13) implies $\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) = \mathscr{F}\sigma_n(\mathbf{u})\mathbf{u}_\mathbf{t}/|\mathbf{u}_\mathbf{t}|$. Hence (7) is proved.

The equivalence between (12) and (4) is handled as the previous one by changing $\mathbf{u}_\mathbf{t}$ with $\dot{\mathbf{u}}_\mathbf{t}$. □

We introduce the following Hilbert space:

$$\mathbf{V} \quad := \quad \left\{ \mathbf{v} \in \left(H^1(\Omega)\right)^d \ : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D \right\}.$$

Suppose that $\mathbf{u}_0 \in \mathbf{V}$, with $u_{0n} \leq 0$ a.e. on Γ_C, and that $\dot{\mathbf{u}}_0 \in \left(L^2(\Omega)\right)^d$. Suppose also that $\mathbf{f} \in \mathscr{C}^0([0,T]; \left(L^2(\Omega)\right)^d)$ and $\mathbf{g} \in \mathscr{C}^0([0,T]; \left(L^2(\Gamma_N)\right)^d)$, which imply that they belong respectively to $\left(L^2(\Omega_T)\right)^d$ and $\left(L^2(\Gamma_{NT})\right)^d$.
Let us define now the following forms:

$$a(\mathbf{u}, \mathbf{v}) := \int_\Omega \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \, d\Omega, \qquad L(t)(\mathbf{v}) := \int_\Omega \mathbf{f}(t) \cdot \mathbf{v} \, d\Omega + \int_{\Gamma_N} \mathbf{g}(t) \cdot \mathbf{v} \, d\Gamma,$$

for any $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{V}$, for all $t \in [0, T)$.
Let $\mathbf{V}^h \subset \mathbf{V}$ be a family of finite dimensional vector spaces (see [18]) indexed by h coming from a family $\mathcal{T}^h$ of triangulations of the domain Ω ($h = \max_{K \in \mathcal{T}^h} h_K$ where h_K is the diameter of the triangle K). The family of triangulations is supposed:

- regular, i.e., there exists $\sigma > 0$ such that $\forall K \in \mathcal{T}^h, h_K/\rho_K \leq \sigma$ where ρ_K denotes the radius of the inscribed ball in K,
- conformal to the subdivision of the boundary into Γ_D, Γ_N and Γ_C, which means that a face of an element $K \in \mathcal{T}^h$ is not allowed to have simultaneous non-empty intersection with more than one part of the subdivision,
- quasi-uniform, i.e., there exists $c > 0$, such that, $\forall h > 0, \ \forall K \in \mathcal{T}^h, \ h_K \geq ch$.

To fix ideas, we choose a standard Lagrange finite element method of degree k with $k = 1$ or $k = 2$, i.e.:

$$\mathbf{V}^h = \left\{ \mathbf{v}^h \in (\mathscr{C}^0(\overline{\Omega}))^d : \mathbf{v}^h{}_{|K} \in (P_k(K))^d, \forall K \in \mathcal{T}^h, \mathbf{v}^h = \mathbf{0} \text{ on } \Gamma_D \right\}.$$

As usual for Nitsche's method (see e.g., [47, 6]), we introduce the following mesh- and parameter-dependent scalar product in $\mathbf{V}^h$:

$$(\mathbf{v}^h, \mathbf{w}^h)_\gamma := (\mathbf{v}^h, \mathbf{w}^h)_{1,\Omega} + (\gamma^{\frac{1}{2}} v^h_n, \gamma^{\frac{1}{2}} w^h_n)_{0,\Gamma_C} + (\gamma^{\frac{1}{2}} \mathbf{v}^h_{\mathbf{t}}, \gamma^{\frac{1}{2}} \mathbf{w}^h_{\mathbf{t}})_{0,\Gamma_C}.$$

We denote by $\| \cdot \|_\gamma := (\cdot, \cdot)^{\frac{1}{2}}_\gamma$ the associated norm.
We recall finally the discrete trace inequalities, proven in [8, Lemma 3.2] (see also [48, Lemma 2.1, p.24] for the scalar case):

Lemma 3.2. *There exists $C > 0$, independent of the parameter γ_0 and of the mesh size h, such that:*

$$\|\gamma^{-\frac{1}{2}} \sigma_n(\mathbf{v}^h)\|^2_{0,\Gamma_C} \leq C \gamma_0^{-1} \|\mathbf{v}^h\|^2_{1,\Omega}, \qquad \|\gamma^{-\frac{1}{2}} \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{v}^h)\|^2_{0,\Gamma_C} \leq C \gamma_0^{-1} \|\mathbf{v}^h\|^2_{1,\Omega} \tag{14}$$

for all $\mathbf{v}^h \in \mathbf{V}^h$.

3.2. Nitsche discretizations

We provide the Nitsche-FEM formulations for the dynamic and static problems described in the previous sections, following the same path as in [10]. We consider in what follows γ, a positive piecewise constant function on the contact interface Γ_C which satisfies

$$\gamma_{|K\cap\Gamma_C} = \frac{\gamma_0}{h_K}, \tag{15}$$

for every K that has a non-empty intersection of dimension $d-1$ with Γ_C, and where γ_0 is a positive given constant (the Nitsche parameter). Note that the value of γ on element intersections has no influence. Given θ a fixed parameter, we introduce the discrete linear operators

$$\mathrm{P}^{\mathbf{n}}_{\theta,\gamma} : \begin{array}{ccc} \mathbf{V}^h & \to & L^2(\Gamma_C) \\ \mathbf{v}^h & \mapsto & \theta\sigma_n(\mathbf{v}^h) - \gamma v^h_n \end{array} \qquad \mathbf{P}^{\mathbf{t}}_{\theta,\gamma} : \begin{array}{ccc} \mathbf{V}^h & \to & (L^2(\Gamma_C))^{d-1} \\ \mathbf{v}^h & \mapsto & \theta\boldsymbol{\sigma}_t(\mathbf{v}^h) - \gamma \mathbf{v}^h_{\mathbf{t}} \end{array}$$

and

$$\mathbf{Q}^{\mathbf{t}}_{\gamma} : \begin{array}{ccc} \mathbf{V}^h \times \mathbf{V}^h & \to & L^2(\Gamma_C) \\ (\mathbf{v}^h, \dot{\mathbf{v}}^h) & \mapsto & \boldsymbol{\sigma}_t(\mathbf{v}^h) - \gamma \dot{\mathbf{v}}^h_{\mathbf{t}} \end{array}.$$

Define as well the bilinear form:

$$A_{\theta\gamma}(\mathbf{u}^h, \mathbf{v}^h) := a(\mathbf{u}^h, \mathbf{v}^h) - \int_{\Gamma_C} \frac{\theta}{\gamma}\, \boldsymbol{\sigma}(\mathbf{u}^h)\mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{v}^h)\mathbf{n}\, d\Gamma.$$

The Nitsche-FEM for the dynamic setting (1)–(5) reads:

$$\begin{cases} \text{Find } \mathbf{u}^h : [0,T] \to \mathbf{V}^h \text{ such that for } t \in [0,T] : \\ \langle \rho \ddot{\mathbf{u}}^h(t), \mathbf{v}^h \rangle + A_{\theta\gamma}(\mathbf{u}^h(t), \mathbf{v}^h) + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma} \left[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}^h(t))\right]_{\mathbb{R}^-} \mathrm{P}^{\mathbf{n}}_{\theta,\gamma}(\mathbf{v}^h)\, d\Gamma \\ + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma} \left[\mathbf{Q}^{\mathbf{t}}_{\gamma}(\mathbf{u}^h(t), \dot{\mathbf{u}}^h(t))\right]_{(-\mathcal{F}[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}^h(t))]_{\mathbb{R}^-})} \cdot \mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{v}^h)\, d\Gamma = L(t)(\mathbf{v}^h), \qquad \forall\, \mathbf{v}^h \in \mathbf{V}^h, \\ \mathbf{u}^h(0,\cdot) = \mathbf{u}^h_0, \qquad \dot{\mathbf{u}}^h(0,\cdot) = \dot{\mathbf{u}}^h_0, \end{cases} \tag{16}$$

where $\mathbf{u}^h_0$ (resp. $\dot{\mathbf{u}}^h_0$) is an approximation in $\mathbf{V}^h$ of the initial displacement $\mathbf{u}_0$ (resp. the initial velocity $\dot{\mathbf{u}}_0$), for instance the Lagrange interpolant or the $L^2(\Omega)$ projection of $\mathbf{u}_0$ (resp. $\dot{\mathbf{u}}_0$). The notation $\langle\cdot,\cdot\rangle$ stands for the $L^2(\Omega)$ inner product.

Remark 3.3. *For the quasi-static problem, the Nitsche-FEM reads:*

$$\begin{cases} \textit{Find } \mathbf{u}^h : [0,T] \to \mathbf{V}^h \textit{ such that for } t \in [0,T] : \\ A_{\theta\gamma}(\mathbf{u}^h(t), \mathbf{v}^h) + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma} \left[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}^h(t))\right]_{\mathbb{R}^-} \mathrm{P}^{\mathbf{n}}_{\theta,\gamma}(\mathbf{v}^h)\, d\Gamma \\ + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma} \left[\mathbf{Q}^{\mathbf{t}}_{\gamma}(\mathbf{u}^h(t), \dot{\mathbf{u}}^h(t))\right]_{(-\mathcal{F}[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}^h(t))]_{\mathbb{R}^-})} \cdot \mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{v}^h)\, d\Gamma = L(t)(\mathbf{v}^h), \qquad \forall\, \mathbf{v}^h \in \mathbf{V}^h, \\ \mathbf{u}^h(0,\cdot) = \mathbf{u}^h_0 \end{cases} \tag{17}$$

Finally, the Nitsche-FEM for the static setting (2), (3), (6) and (7) reads:

$$\begin{cases} \text{Find } \mathbf{u}^h \in \mathbf{V}^h \text{ such that:} \\ A_{\theta\gamma}(\mathbf{u}^h, \mathbf{v}^h) + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma} \left[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}^h)\right]_{\mathbb{R}^-} \mathrm{P}^{\mathbf{n}}_{\theta,\gamma}(\mathbf{v}^h)\, d\Gamma \\ + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma} \left[\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}^h)\right]_{\left(-\mathscr{F}\left[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}^h)\right]_{\mathbb{R}^-}\right)} \cdot \mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{v}^h)\, d\Gamma = L(\mathbf{v}^h), \qquad \forall\, \mathbf{v}^h \in \mathbf{V}^h. \end{cases} \tag{18}$$

4. Existence and uniqueness results

The aim of this section is to provide existence and uniqueness results, first in the dynamic setting (Problem (16)), and then in statics (Problem (18)).

4.1. The dynamic case

In contrast with the standard (mixed) finite element semi-discretization, Nitsche's formulation leads to a well-posed (Lipschitz) system of differential equations, as it will be shown below. In order to prove well-posedness we reformulate (16) as a system of (non-linear) second-order differential equations. To this purpose, using Riesz's representation theorem in $(\mathbf{V}^h, (\cdot,\cdot)_\gamma)$ we first introduce the mass operator $\mathbf{M}^h : \mathbf{V}^h \to \mathbf{V}^h$, which is defined for all $\mathbf{v}^h, \mathbf{w}^h \in \mathbf{V}^h$ by

$$(\mathbf{M}^h \mathbf{v}^h, \mathbf{w}^h)_\gamma = \langle \rho \mathbf{v}^h, \mathbf{w}^h \rangle.$$

Still using Riesz's representation theorem, we define the (non-linear) operator $\mathbf{B}^h : (\mathbf{V}^h)^2 \to \mathbf{V}^h$, by means of the formula

$$\begin{aligned}(\mathbf{B}^h(\mathbf{v}^h, \dot{\mathbf{v}}^h), \mathbf{w}^h)_\gamma \quad = \quad & A_{\theta\gamma}(\mathbf{v}^h, \mathbf{w}^h) + \int_{\Gamma_C} \frac{1}{\gamma} \left[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}^h)\right]_{\mathbb{R}^-} \mathrm{P}^{\mathbf{n}}_{\theta,\gamma}(\mathbf{w}^h) \, d\Gamma \\ & + \int_{\Gamma_C} \frac{1}{\gamma} \left[\mathbf{Q}^{\mathbf{t}}_\gamma(\mathbf{v}^h, \dot{\mathbf{v}}^h)\right]_{(-\mathcal{F}[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}^h)]_{\mathbb{R}^-})} \cdot \mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{w}^h) \, d\Gamma,\end{aligned}$$

for all $\mathbf{v}^h, \dot{\mathbf{v}}^h, \mathbf{w}^h \in \mathbf{V}^h$. Finally, we denote by $\mathbf{L}^h(t)$ the vector in $\mathbf{V}^h$ such that, for all $t \in [0,T]$ and for every $\mathbf{w}^h$ in $\mathbf{V}^h$:

$$(\mathbf{L}^h(t), \mathbf{w}^h)_\gamma = L(t)(\mathbf{w}^h).$$

Remark that, due to the assumptions on $\mathbf{f}$ and $\mathbf{g}$, $\mathbf{L}^h$ is continuous from $[0,T]$ onto $(\mathbf{V}_h, \|\cdot\|_\gamma)$.
With the above notation, Problem (16) reads:

$$\begin{cases} \text{Find } \mathbf{u}^h : [0,T] \to \mathbf{V}^h \text{ such that for } t \in [0,T] : \\ \mathbf{M}^h \ddot{\mathbf{u}}^h(t) + \mathbf{B}^h(\mathbf{u}^h(t), \dot{\mathbf{u}}^h(t)) = \mathbf{L}^h(t), \\ \mathbf{u}^h(0,\cdot) = \mathbf{u}^h_0, \qquad \dot{\mathbf{u}}^h(0,\cdot) = \dot{\mathbf{u}}^h_0. \end{cases} \tag{19}$$

We then show that Problem (16) (or equivalently Problem (19)) is well-posed.

Theorem 4.1. *The operator $\mathbf{B}^h$ is Lipschitz-continuous in the following sense: there exists a constant $C > 0$, independent of h, θ, γ_0 and $\mathscr{F}$ such that, for all $(\mathbf{v}^h_1, \dot{\mathbf{v}}^h_1), (\mathbf{v}^h_2, \dot{\mathbf{v}}^h_2) \in (\mathbf{V}^h)^2$:*

$$\begin{aligned}&\|\mathbf{B}^h(\mathbf{v}^h_1, \dot{\mathbf{v}}^h_1) - \mathbf{B}^h(\mathbf{v}^h_2, \dot{\mathbf{v}}^h_2)\|_\gamma \\ &\le C(1 + \gamma_0^{-\frac{1}{2}})(1 + |\theta|\gamma_0^{-\frac{1}{2}})(1 + \mathscr{F})\|\mathbf{v}^h_1 - \mathbf{v}^h_2\|_\gamma + C(1 + |\theta|\gamma_0^{-\frac{1}{2}})\|\dot{\mathbf{v}}^h_1 - \dot{\mathbf{v}}^h_2\|_\gamma.\end{aligned} \tag{20}$$

As a consequence, for every value of $\theta \in \mathbb{R}$ and $\gamma_0 > 0$, Problem (16) admits one unique solution $\mathbf{u}^h \in \mathscr{C}^2([0,T], \mathbf{V}^h)$.

Proof: Let us pick $\mathbf{v}_1^h, \mathbf{v}_2^h, \dot{\mathbf{v}}_1^h, \dot{\mathbf{v}}_2^h, \mathbf{w}^h \in \mathbf{V}^h$, then:

$$\begin{aligned}
& |(\mathbf{B}^h(\mathbf{v}_1^h, \dot{\mathbf{v}}_1^h) - \mathbf{B}^h(\mathbf{v}_2^h, \dot{\mathbf{v}}_2^h), \mathbf{w}^h)_\gamma| \\
\leq \; & \left|A_{\theta\gamma}(\mathbf{v}_1^h - \mathbf{v}_2^h, \mathbf{w}^h)\right| + \left|\int_{\Gamma_C} \frac{1}{\gamma} \left([\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}_1^h)]_{\mathbb{R}^-} - [\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}_2^h)]_{\mathbb{R}^-}\right) \mathrm{P}^{\mathbf{n}}_{\theta,\gamma}(\mathbf{w}^h)\, d\Gamma\right| \\
& + \left|\int_{\Gamma_C} \frac{1}{\gamma} \left([\mathbf{Q}^{\mathbf{t}}_\gamma(\mathbf{v}_1^h, \dot{\mathbf{v}}_1^h)]_{-\mathscr{F}[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}_1^h)]_{\mathbb{R}^-}} - [\mathbf{Q}^{\mathbf{t}}_\gamma(\mathbf{v}_2^h, \dot{\mathbf{v}}_2^h)]_{-\mathscr{F}[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}_2^h)]_{\mathbb{R}^-}}\right) \mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{w}^h)\, d\Gamma\right| \\
\leq \; & C(1 + |\theta|\gamma_0^{-1})\|\mathbf{v}_1^h - \mathbf{v}_2^h\|_{1,\Omega}\|\mathbf{w}^h\|_{1,\Omega} + \int_{\Gamma_C} \frac{1}{\gamma} \left|[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}_1^h)]_{\mathbb{R}^-} - [\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}_2^h)]_{\mathbb{R}^-}\right| |\mathrm{P}^{\mathbf{n}}_{\theta,\gamma}(\mathbf{w}^h)|\, d\Gamma \\
& + \int_{\Gamma_C} \frac{1}{\gamma} \left|[\mathbf{Q}^{\mathbf{t}}_\gamma(\mathbf{v}_1^h, \dot{\mathbf{v}}_1^h)]_{-\mathscr{F}[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}_1^h)]_{\mathbb{R}^-}} - [\mathbf{Q}^{\mathbf{t}}_\gamma(\mathbf{v}_2^h, \dot{\mathbf{v}}_2^h)]_{-\mathscr{F}[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}_2^h)]_{\mathbb{R}^-}}\right| |\mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{w}^h)|\, d\Gamma,
\end{aligned}$$

as the estimate (14) yields $\|A_{\theta\gamma}\| \leq C(1 + |\theta|\gamma_0^{-1})$.
With the inequality $|[x]_{\mathbb{R}^-} - [y]_{\mathbb{R}^-}| \leq |x - y|$, for all $x, y \in \mathbb{R}$, and using the linearity of $\mathrm{P}^{\mathbf{n}}_{1,\gamma}$, we note that:

$$\begin{aligned}
& \int_{\Gamma_C} \frac{1}{\gamma} \left|[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}_1^h)]_{\mathbb{R}^-} - [\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}_2^h)]_{\mathbb{R}^-}\right| |\mathrm{P}^{\mathbf{n}}_{\theta,\gamma}(\mathbf{w}^h)|\, d\Gamma \\
\leq \; & \int_{\Gamma_C} \frac{1}{\gamma} |\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}_1^h - \mathbf{v}_2^h)||\mathrm{P}^{\mathbf{n}}_{\theta,\gamma}(\mathbf{w}^h)|\, d\Gamma \\
\leq \; & \|\gamma^{-\frac{1}{2}}\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}_1^h - \mathbf{v}_2^h)\|_{0,\Gamma_C}\|\gamma^{-\frac{1}{2}}\mathrm{P}^{\mathbf{n}}_{\theta,\gamma}(\mathbf{w}^h)\|_{0,\Gamma_C} \\
\leq \; & \left(\|\gamma^{\frac{1}{2}}(v_{1,n}^h - v_{2,n}^h)\|_{0,\Gamma_C} + \|\gamma^{-\frac{1}{2}}\sigma_n(\mathbf{v}_1^h - \mathbf{v}_2^h)\|_{0,\Gamma_C}\right)\left(\|\gamma^{\frac{1}{2}}w_n^h\|_{0,\Gamma_C} + |\theta|\|\gamma^{-\frac{1}{2}}\sigma_n(\mathbf{w}^h)\|_{0,\Gamma_C}\right) \\
\leq \; & \left(\|\gamma^{\frac{1}{2}}(v_{1,n}^h - v_{2,n}^h)\|_{0,\Gamma_C} + C\gamma_0^{-\frac{1}{2}}\|\mathbf{v}_1^h - \mathbf{v}_2^h\|_{1,\Omega}\right)\left(\|\gamma^{\frac{1}{2}}w_n^h\|_{0,\Gamma_C} + C|\theta|\gamma_0^{-\frac{1}{2}}\|\mathbf{w}^h\|_{1,\Omega}\right) \\
\leq \; & C(1 + \gamma_0^{-\frac{1}{2}})(1 + |\theta|\gamma_0^{-\frac{1}{2}})\|\mathbf{v}_1^h - \mathbf{v}_2^h\|_\gamma\|\mathbf{w}^h\|_\gamma. \qquad (21)
\end{aligned}$$

In the last lines, we used the Cauchy-Schwarz and triangular inequalities, the estimate (14) and the continuity of the trace operator from $H^1(\Omega)$ to $L^2(\Gamma_C)$.
Besides, using a triangular inequality, together with properties (8) and (9):

$$\begin{aligned}
& \int_{\Gamma_C} \frac{1}{\gamma} \left|[\mathbf{Q}^{\mathbf{t}}_\gamma(\mathbf{v}_1^h, \dot{\mathbf{v}}_1^h)]_{-\mathscr{F}[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}_1^h)]_{\mathbb{R}^-}} - [\mathbf{Q}^{\mathbf{t}}_\gamma(\mathbf{v}_2^h, \dot{\mathbf{v}}_2^h)]_{-\mathscr{F}[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}_2^h)]_{\mathbb{R}^-}}\right| |\mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{w}^h)|\, d\Gamma \\
\leq \; & \int_{\Gamma_C} \frac{1}{\gamma} \left|[\mathbf{Q}^{\mathbf{t}}_\gamma(\mathbf{v}_1^h, \dot{\mathbf{v}}_1^h)]_{-\mathscr{F}[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}_1^h)]_{\mathbb{R}^-}} - [\mathbf{Q}^{\mathbf{t}}_\gamma(\mathbf{v}_2^h, \dot{\mathbf{v}}_2^h)]_{-\mathscr{F}[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}_1^h)]_{\mathbb{R}^-}}\right| |\mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{w}^h)|\, d\Gamma \\
& + \int_{\Gamma_C} \frac{1}{\gamma} \left|[\mathbf{Q}^{\mathbf{t}}_\gamma(\mathbf{v}_2^h, \dot{\mathbf{v}}_2^h)]_{-\mathscr{F}[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}_1^h)]_{\mathbb{R}^-}} - [\mathbf{Q}^{\mathbf{t}}_\gamma(\mathbf{v}_2^h, \dot{\mathbf{v}}_2^h)]_{-\mathscr{F}[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}_2^h)]_{\mathbb{R}^-}}\right| |\mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{w}^h)|\, d\Gamma \\
\leq \; & \int_{\Gamma_C} \frac{1}{\gamma} \left|\mathbf{Q}^{\mathbf{t}}_\gamma(\mathbf{v}_1^h, \dot{\mathbf{v}}_1^h) - \mathbf{Q}^{\mathbf{t}}_\gamma(\mathbf{v}_2^h, \dot{\mathbf{v}}_2^h)\right| |\mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{w}^h)|\, d\Gamma \\
& + \int_{\Gamma_C} \frac{\mathscr{F}}{\gamma} \left|[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}_1^h)]_{\mathbb{R}^-} - [\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}_2^h)]_{\mathbb{R}^-}\right| |\mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{w}^h)|\, d\Gamma \\
\leq \; & \int_{\Gamma_C} \frac{1}{\gamma} \left(\left|\mathbf{Q}^{\mathbf{t}}_\gamma(\mathbf{v}_1^h - \mathbf{v}_2^h, \dot{\mathbf{v}}_1^h - \dot{\mathbf{v}}_2^h)\right| + \mathscr{F}\left|\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}_1^h - \mathbf{v}_2^h)\right|\right) |\mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{w}^h)|\, d\Gamma. \qquad (22)
\end{aligned}$$

Then, we apply Cauchy-Schwarz inequality, use the discrete trace inequalities(14) and get:

$$
\begin{aligned}
&\int_{\Gamma_C} \frac{1}{\gamma} \left| [\mathbf{Q}^{\mathbf{t}}_{\gamma}(\mathbf{v}_1^h, \dot{\mathbf{v}}_1^h)]_{-\mathscr{F}[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}_1^h)]_{\mathbb{R}^-}} - [\mathbf{Q}^{\mathbf{t}}_{\gamma}(\mathbf{v}_2^h, \dot{\mathbf{v}}_2^h)]_{-\mathscr{F}[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}_2^h)]_{\mathbb{R}^-}} \right| |\mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{w}^h)| \, d\Gamma \\
\leq\ & \Big(\|\gamma^{\frac{1}{2}}(\dot{\mathbf{v}}^h_{1,\mathbf{t}} - \dot{\mathbf{v}}^h_{2,\mathbf{t}}) - \gamma^{-\frac{1}{2}} \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{v}_1^h - \mathbf{v}_2^h)\|_{0,\Gamma_C} + \mathscr{F} \|\gamma^{\frac{1}{2}}(v^h_{1,n} - v^h_{2,n}) \\
&-\gamma^{-\frac{1}{2}} \sigma_n(\mathbf{v}_1^h - \mathbf{v}_2^h)\|_{0,\Gamma_C} \Big) \|\gamma^{-\frac{1}{2}} \mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{w}^h)\|_{0,\Gamma_C} \\
\leq\ & C \left(\|\dot{\mathbf{v}}_1^h - \dot{\mathbf{v}}_2^h\|_{\gamma} + (1 + \gamma_0^{-\frac{1}{2}})(1 + \mathscr{F}) \|\mathbf{v}_1^h - \mathbf{v}_2^h\|_{\gamma} \right) (1 + |\theta| \gamma_0^{-\frac{1}{2}}) \|\mathbf{w}^h\|_{\gamma}.
\end{aligned}
$$

Taking this bound into account, we now combine the above estimations to obtain:

$$
\begin{aligned}
& |(\mathbf{B}^h(\mathbf{v}_1^h, \dot{\mathbf{v}}_1^h) - \mathbf{B}^h(\mathbf{v}_2^h, \dot{\mathbf{v}}_2^h), \mathbf{w}^h)_{\gamma}| \\
\leq\ & C(1 + \gamma_0^{-\frac{1}{2}})(1 + |\theta| \gamma_0^{-\frac{1}{2}})(1 + \mathscr{F}) \|\mathbf{v}_1^h - \mathbf{v}_2^h\|_{\gamma} \|\mathbf{w}^h\|_{\gamma} + C(1 + |\theta| \gamma_0^{-\frac{1}{2}}) \|\dot{\mathbf{v}}_1^h - \dot{\mathbf{v}}_2^h\|_{\gamma} \|\mathbf{w}^h\|_{\gamma}.
\end{aligned}
$$

It results that

$$
\begin{aligned}
& \|\mathbf{B}^h(\mathbf{v}_1^h, \dot{\mathbf{v}}_1^h) - \mathbf{B}^h(\mathbf{v}_2^h, \dot{\mathbf{v}}_2^h)\|_{\gamma} \\
=\ & \sup_{\mathbf{w}^h \in \mathbf{V}^h} \frac{(\mathbf{B}^h(\mathbf{v}_1^h, \dot{\mathbf{v}}_1^h) - \mathbf{B}^h(\mathbf{v}_2^h, \dot{\mathbf{v}}_2^h), \mathbf{w}^h)_{\gamma}}{\|\mathbf{w}^h\|_{\gamma}} \\
\leq\ & C(1 + \gamma_0^{-\frac{1}{2}})(1 + |\theta| \gamma_0^{-\frac{1}{2}})(1 + \mathscr{F}) \|\mathbf{v}_1^h - \mathbf{v}_2^h\|_{\gamma} + C(1 + |\theta| \gamma_0^{-\frac{1}{2}}) \|\dot{\mathbf{v}}_1^h - \dot{\mathbf{v}}_2^h\|_{\gamma}.
\end{aligned} \tag{23}
$$

This proves the first assertion of the theorem.

Then we recast (19) in the canonical form of a first-order system:

$$
\frac{d}{dt} \mathbf{x}^h(t) = \mathbf{F}^h(t, \mathbf{x}^h(t)), \quad \mathbf{x}^h(0) = \mathbf{x}_0^h,
$$

where:

$$
\mathbf{x}^h(t) := \begin{bmatrix} \dot{\mathbf{u}}^h \\ \mathbf{u}^h \end{bmatrix}(t), \quad \mathbf{x}_0^h := \begin{bmatrix} \dot{\mathbf{u}}_0^h \\ \mathbf{u}_0^h, \end{bmatrix}, \qquad \mathbf{F}^h(t, \mathbf{x}^h(t)) := \begin{bmatrix} (\mathbf{M}^h)^{-1}(\mathbf{L}^h(t) - \mathbf{B}^h(\mathbf{u}^h(t), \dot{\mathbf{u}}^h(t))) \\ \dot{\mathbf{u}}^h(t) \end{bmatrix}.
$$

It holds for arbitrary $t \in [0, T]$ and $\mathbf{x}_1^h, \mathbf{x}_2^h \in (\mathbf{V}^h)^2$:

$$
\|\mathbf{F}^h(t, \mathbf{x}_1^h) - \mathbf{F}^h(t, \mathbf{x}_2^h)\|^2_{\gamma \times \gamma} = \|(\mathbf{M}^h)^{-1}(\mathbf{B}^h(\mathbf{u}_2^h, \dot{\mathbf{u}}_2^h) - \mathbf{B}^h(\mathbf{u}_1^h, \dot{\mathbf{u}}_1^h)\|^2_{\gamma} + \|\dot{\mathbf{u}}_1^h - \dot{\mathbf{u}}_2^h\|^2_{\gamma},
$$

where $\| \cdot \|_{\gamma \times \gamma}$ denotes the product norm on $(\mathbf{V}^h)^2$.

From (20) and re-arranging the terms we get (for a more precise estimate of the subordinated matrix norm term $\|(\mathbf{M}^h)^{-1}\|_{\gamma}$, see [13]):

$$
\begin{aligned}
\|(\mathbf{M}^h)^{-1}(\mathbf{B}^h(\mathbf{u}_2^h, \dot{\mathbf{u}}_2^h) - \mathbf{B}^h(\mathbf{u}_1^h, \dot{\mathbf{u}}_1^h))\|_{\gamma} &\leq \|(\mathbf{M}^h)^{-1}\|_{\gamma} \|\mathbf{B}^h(\mathbf{u}_2^h, \dot{\mathbf{u}}_2^h) - \mathbf{B}^h(\mathbf{u}_1^h, \dot{\mathbf{u}}_1^h)\|_{\gamma} \\
&\leq C(\rho, \gamma_0, h, |\theta|, \mathscr{F})(\|\mathbf{u}_2^h - \mathbf{u}_1^h\|_{\gamma} + \|\dot{\mathbf{u}}_2^h - \dot{\mathbf{u}}_1^h\|_{\gamma}). \\
&\leq C(\rho, \gamma_0, h, |\theta|, \mathscr{F}) \|\mathbf{x}_2^h - \mathbf{x}_1^h\|_{\gamma \times \gamma}.
\end{aligned}
$$

Hence

$$
\|\mathbf{F}^h(t, \mathbf{x}_1^h) - \mathbf{F}^h(t, \mathbf{x}_2^h)\|_{\gamma \times \gamma} \leq C(\rho, \gamma_0, h, |\theta|, \mathscr{F}) \|\mathbf{x}_2^h - \mathbf{x}_1^h\|_{\gamma \times \gamma}.
$$

So the second assertion of the theorem results of the Lipschitz-continuity of $\mathbf{F}^h$ and of the Cauchy-Lipschitz (Picard-Lindelöf) theorem. □

Remark 4.2. *Note that there is no condition on γ_0 for the space (semi-)discretization, which remains well-posed even if γ_0 is small. The same remark applies for the friction coefficient.*

4.2. The static case

In this section we prove that the discrete problem (18) admits solutions when γ_0 is large (here the denomination "large" depends on θ) and that the solution is unique under an additional smallness assumption on $\mathscr{F}\gamma_0 h^{-1}$.

The proof of the unique solution uses the Banach-Picard fixed point theorem for contractive functions in a metric space. Surprisingly the calculus in the existence theorem proved thanks to the Brouwer theorem is slightly more complicated since we have to use distances instead of norm terms. The main result of this section is stated below:

Theorem 4.3. *For any value of $\theta \in \mathbb{R}$, Problem* (18) *admits at least a solution when γ_0 is large enough. Moreover, if the quantity $\mathscr{F}^2\gamma_0 h^{-1}$ is small enough, this solution is unique.*

Proof: Let us first introduce the problem of (Tresca) friction with a fixed threshold $g \in L^2(\Gamma_C)$, which admits a unique solution according to [8]:

$$\mathcal{P}(g) \begin{cases} \text{Find } \mathbf{u}^h \in \mathbf{V}^h \text{ such that:} \\ A_{\theta\gamma}(\mathbf{u}^h, \mathbf{v}^h) + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma} \left[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}^h)\right]_{\mathbb{R}^-} \mathrm{P}^{\mathbf{n}}_{\theta,\gamma}(\mathbf{v}^h)\, d\Gamma \\ + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma} \left[\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}^h)\right]_g \cdot \mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{v}^h)\, d\Gamma = L(\mathbf{v}^h), \qquad \forall\, \mathbf{v}^h \in \mathbf{V}^h. \end{cases}$$

Remark now that the solutions to Coulomb discrete problem (16) are the fixed points of the application $\phi^h : \mathbf{V}^h \to \mathbf{V}^h$ defined as follows: $\phi^h(\mathbf{w}^h)$ is the solution to $\mathcal{P}(-\mathscr{F}[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{w}^h)]_{\mathbb{R}^-})$.

Step 1. To apply the Banach fixed point theorem in a metric space, we have to prove that the mapping ϕ^h is contractive on $\mathbf{V}^h$. Set for $\mathbf{v}^h$ and $\mathbf{w}^h$ in $\mathbf{V}^h$:

$$d(\mathbf{v}^h, \mathbf{w}^h) = a(\mathbf{v}^h - \mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h)^{1/2} + \|\gamma^{-\frac{1}{2}}([\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{v}^h)]_{\mathbb{R}^-} - [\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{w}^h)]_{\mathbb{R}^-})\|_{0,\Gamma_C}.$$

It is easy to check that $d(\cdot,\cdot)$ is a distance on $\mathbf{V}^h$.

Let $\mathbf{w}_1^h, \mathbf{w}_2^h \in \mathbf{V}^h$, and denote by

$$\mathbf{u}_1^h := \phi^h(\mathbf{w}_1^h), \qquad \mathbf{u}_2^h := \phi^h(\mathbf{w}_2^h)$$

the respective solutions to $\mathcal{P}(-\mathscr{F}[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{w}_1^h)]_{\mathbb{R}^-})$ and $\mathcal{P}(-\mathscr{F}[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{w}_2^h)]_{\mathbb{R}^-})$. To lighten the notations in the proof, we write x_1 (resp. x_2) instead of $-\mathscr{F}[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{w}_1^h)]_{\mathbb{R}^-}$ (resp. $-\mathscr{F}[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{w}_2^h)]_{\mathbb{R}^-}$).

We can write for all $\mathbf{v}^h \in \mathbf{V}^h$

$$A_{\theta\gamma}(\mathbf{u}_1^h, \mathbf{v}^h) + \int_{\Gamma_C} \frac{1}{\gamma} [\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}_1^h)]_{\mathbb{R}^-} \mathrm{P}^{\mathbf{n}}_{\theta,\gamma}(\mathbf{v}^h)\, d\Gamma + \int_{\Gamma_C} \frac{1}{\gamma} [\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_1^h)]_{x_1} \mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{v}^h)\, d\Gamma = L(\mathbf{v}^h), \qquad (24)$$

and

$$A_{\theta\gamma}(\mathbf{u}_2^h, \mathbf{v}^h) + \int_{\Gamma_C} \frac{1}{\gamma} [\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}_2^h))]_{\mathbb{R}^-} \mathrm{P}^{\mathbf{n}}_{\theta,\gamma}(\mathbf{v}^h)\, d\Gamma + \int_{\Gamma_C} \frac{1}{\gamma} [\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_2^h)]_{x_2} \mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{v}^h)\, d\Gamma = L(\mathbf{v}^h). \qquad (25)$$

So by taking $\mathbf{v}^h = \mathbf{u}_1^h - \mathbf{u}_2^h$ in (24) and $\mathbf{v}^h = \mathbf{u}_2^h - \mathbf{u}_1^h$ in (25), and after summation of the two equalities, we obtain:

$$A_{\theta\gamma}(\mathbf{u}_1^h - \mathbf{u}_2^h, \mathbf{u}_1^h - \mathbf{u}_2^h) + \int_{\Gamma_C} \frac{1}{\gamma} \left([\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}_1^h)]_{\mathbb{R}^-} - [\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}_2^h)]_{\mathbb{R}^-}\right) \mathrm{P}^{\mathbf{n}}_{\theta,\gamma}(\mathbf{u}_1^h - \mathbf{u}_2^h)\, d\Gamma$$
$$+ \int_{\Gamma_C} \frac{1}{\gamma} \left([\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_1^h)]_{x_1} - [\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_2^h)]_{x_2}\right) \mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{u}_1^h - \mathbf{u}_2^h)\, d\Gamma = 0.$$

Then we obtain after splitting the term associated to friction:

$$
\begin{aligned}
&A_{\theta\gamma}(\mathbf{u}_1^h-\mathbf{u}_2^h,\mathbf{u}_1^h-\mathbf{u}_2^h)+\int_{\Gamma_C}\frac{1}{\gamma}\left([\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}_1^h)]_{\mathbb{R}^-}-[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}_2^h)]_{\mathbb{R}^-}\right)\mathrm{P}^{\mathbf{n}}_{\theta,\gamma}(\mathbf{u}_1^h-\mathbf{u}_2^h)\,d\Gamma\\
&+\int_{\Gamma_C}\frac{1}{\gamma}\left([\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_1^h)]_{x_1}-[\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_2^h)]_{x_1}\right)\mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{u}_1^h-\mathbf{u}_2^h)\,d\Gamma\\
&+\int_{\Gamma_C}\frac{1}{\gamma}\left([\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_2^h)]_{x_1}-[\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_2^h)]_{x_2}\right)\mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{u}_1^h-\mathbf{u}_2^h)\,d\Gamma=0. \qquad (26)
\end{aligned}
$$

We now use the splitting $\mathrm{P}^{\mathbf{n}}_{\theta,\gamma}(\cdot)=\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\cdot)+(\theta-1)\sigma_n(\cdot)$ (and the same for the tangential counterpart):

$$
\begin{aligned}
&A_{\theta\gamma}(\mathbf{u}_1^h-\mathbf{u}_2^h,\mathbf{u}_1^h-\mathbf{u}_2^h)+\int_{\Gamma_C}\frac{1}{\gamma}\left([\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}_1^h)]_{\mathbb{R}^-}-[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}_2^h)]_{\mathbb{R}^-}\right)\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}_1^h-\mathbf{u}_2^h)\,d\Gamma\\
&+(\theta-1)\int_{\Gamma_C}\frac{1}{\gamma}\left([\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}_1^h)]_{\mathbb{R}^-}-[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}_2^h)]_{\mathbb{R}^-}\right)\sigma_n(\mathbf{u}_1^h-\mathbf{u}_2^h)\,d\Gamma\\
&+\int_{\Gamma_C}\frac{1}{\gamma}\left([\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_1^h)]_{x_1}-[\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_2^h)]_{x_1}\right)\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_1^h-\mathbf{u}_2^h)\,d\Gamma\\
&+(\theta-1)\int_{\Gamma_C}\frac{1}{\gamma}\left([\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_1^h)]_{x_1}-[\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_2^h)]_{x_1}\right)\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}_1^h-\mathbf{u}_2^h)\,d\Gamma\\
&+\int_{\Gamma_C}\frac{1}{\gamma}\left([\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_2^h)]_{x_1}-[\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_2^h)]_{x_2}\right)\mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{u}_1^h-\mathbf{u}_2^h)\,d\Gamma=0. \qquad (27)
\end{aligned}
$$

By using the second property in (8) and the inequality $(x+y)^2\le 2(x^2+y^2)$ we get

$$
\begin{aligned}
&\frac{1}{2}d^2(\mathbf{u}_1^h,\mathbf{u}_2^h)-\theta\|\gamma^{-\frac{1}{2}}\boldsymbol{\sigma}(\mathbf{u}_1^h-\mathbf{u}_2^h)\mathbf{n}\|^2_{0,\Gamma_C}+\|\gamma^{-\frac{1}{2}}([\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_1^h)]_{x_1}-[\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_2^h)]_{x_1})\|^2_{0,\Gamma_C}\\
\le\ & A_{\theta\gamma}(\mathbf{u}_1^h-\mathbf{u}_2^h,\mathbf{u}_1^h-\mathbf{u}_2^h)+\|\gamma^{-\frac{1}{2}}([\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}_1^h)]_{\mathbb{R}^-}-[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}_2^h)]_{\mathbb{R}^-})\|^2_{0,\Gamma_C}\\
&+\|\gamma^{-\frac{1}{2}}([\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_1^h)]_{x_1}-[\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_2^h)]_{x_1})\|^2_{0,\Gamma_C}\\
\le\ & (1-\theta)\int_{\Gamma_C}\frac{1}{\gamma}\left([\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}_1^h)]_{\mathbb{R}^-}-[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}_2^h)]_{\mathbb{R}^-}\right)\sigma_n(\mathbf{u}_1^h-\mathbf{u}_2^h)\,d\Gamma\\
&+(1-\theta)\int_{\Gamma_C}\frac{1}{\gamma}\left([\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_1^h)]_{x_1}-[\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_2^h)]_{x_1}\right)\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}_1^h-\mathbf{u}_2^h)\,d\Gamma\\
&-\int_{\Gamma_C}\frac{1}{\gamma}\left([\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_2^h)]_{x_1}-[\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_2^h)]_{x_2}\right)\mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{u}_1^h-\mathbf{u}_2^h)\,d\Gamma. \qquad (28)
\end{aligned}
$$

With the notation $\mathcal{T}_1$ for the right part of the previous inequality, we deduce by using Cauchy-Schwarz inequality:

$$
\begin{aligned}
\mathcal{T}_1\ \le\ & |1-\theta|\|\gamma^{-\frac{1}{2}}([\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}_1^h)]_{\mathbb{R}^-}-[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}_2^h)]_{\mathbb{R}^-})\|_{0,\Gamma_C}\|\gamma^{-\frac{1}{2}}\sigma_n(\mathbf{u}_1^h-\mathbf{u}_2^h)\|_{0,\Gamma_C}\\
&+|1-\theta|\|\gamma^{-\frac{1}{2}}([\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_1^h)]_{x_1}-[\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_2^h)]_{x_1})\|_{0,\Gamma_C}\|\gamma^{-\frac{1}{2}}\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}_1^h-\mathbf{u}_2^h)\|_{0,\Gamma_C}\\
&+\|\gamma^{-\frac{1}{2}}([\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_2^h)]_{x_1}-[\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_2^h)]_{x_2})\|_{0,\Gamma_C}\|\gamma^{-\frac{1}{2}}\mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{u}_1^h-\mathbf{u}_2^h)\|_{0,\Gamma_C}. \qquad (29)
\end{aligned}
$$

Applying Young's inequality for $\beta_1, \beta_2 > 0$ and (9) together with the definitions of x_1, x_2, we obtain:

$$\begin{aligned}
\mathcal{T}_1 \quad \leq \quad & \frac{|1-\theta|\beta_1}{2}\|\gamma^{-\frac{1}{2}}\sigma_n(\mathbf{u}_1^h-\mathbf{u}_2^h)\|^2_{0,\Gamma_C} + \frac{|1-\theta|}{2\beta_1}\|\gamma^{-\frac{1}{2}}([\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}_1^h)]_{\mathbb{R}^-} - [\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}_2^h)]_{\mathbb{R}^-})\|^2_{0,\Gamma_C} \\
& + \frac{|1-\theta|\beta_2}{2}\|\gamma^{-\frac{1}{2}}\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}_1^h-\mathbf{u}_2^h)\|^2_{0,\Gamma_C} + \frac{|1-\theta|}{2\beta_2}\|\gamma^{-\frac{1}{2}}([\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_1^h)]_{x_1} - [\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_2^h)]_{x_1})\|^2_{0,\Gamma_C} \\
& + \frac{1}{4}\|\gamma^{-\frac{1}{2}}([\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{w}_1^h)]_{\mathbb{R}^-} - [\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{w}_2^h)]_{\mathbb{R}^-})\|^2_{0,\Gamma_C} + \mathscr{F}^2\|\gamma^{-\frac{1}{2}}\mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{u}_1^h-\mathbf{u}_2^h)\|^2_{0,\Gamma_C}. \qquad (30)
\end{aligned}$$

Applying the triangular inequality, and then the continuity of the trace from $H^1(\Omega)$ into $L^2(\Gamma_C)$ as well as the assumption of quasi-uniformity of the mesh $\mathcal{T}^h$ yields:

$$\begin{aligned}
\|\gamma^{-\frac{1}{2}}\mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{u}_1^h-\mathbf{u}_2^h)\|_{0,\Gamma_C} &\leq \|\gamma^{\frac{1}{2}}(\mathbf{u}_{1\mathbf{t}}^h-\mathbf{u}_{2\mathbf{t}}^h)\|_{0,\Gamma_C} + |\theta|\|\gamma^{-\frac{1}{2}}\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}_1^h-\mathbf{u}_2^h)\|_{0,\Gamma_C} \\
&\leq C_2\gamma_0^{\frac{1}{2}}h^{-\frac{1}{2}}\|\mathbf{u}_1^h-\mathbf{u}_2^h\|_{1,\Omega} + |\theta|\|\gamma^{-\frac{1}{2}}\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}_1^h-\mathbf{u}_2^h)\|_{0,\Gamma_C}, \qquad (31)
\end{aligned}$$

with $C_2 > 0$.
Let us now combine the previous results:

$$\begin{aligned}
& \frac{1}{2}d^2(\mathbf{u}_1^h,\mathbf{u}_2^h) - \frac{|1-\theta|}{2\beta_1}\|\gamma^{-\frac{1}{2}}([\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}_1^h)]_{\mathbb{R}^-} - [\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}_2^h)]_{\mathbb{R}^-})\|^2_{0,\Gamma_C} \\
& \left(1-\frac{|1-\theta|}{2\beta_2}\right)\|\gamma^{-\frac{1}{2}}([\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_1^h)]_{x_1} - [\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_2^h)]_{x_1})\|^2_{0,\Gamma_C} \\
\leq \quad & \left(\frac{|1-\theta|\beta_1}{2}+\theta\right)\|\gamma^{-\frac{1}{2}}\sigma_n(\mathbf{u}_1^h-\mathbf{u}_2^h)\|^2_{0,\Gamma_C} + \left(\frac{|1-\theta|\beta_2}{2}+2\mathscr{F}^2\theta^2+\theta\right)\|\gamma^{-\frac{1}{2}}\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}_1^h-\mathbf{u}_2^h)\|^2_{0,\Gamma_C} \\
& +2\mathscr{F}^2C_2\gamma_0h^{-1}\|\mathbf{u}_1^h-\mathbf{u}_2^h\|^2_{1,\Omega} + \frac{1}{4}\|\gamma^{-\frac{1}{2}}\left([\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{w}_1^h)]_{\mathbb{R}^-} - [\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{w}_2^h)]_{\mathbb{R}^-}\right)\|^2_{0,\Gamma_C} \qquad (32)
\end{aligned}$$

If $\theta = 1$, then

$$\begin{aligned}
& \frac{1}{2}d^2(\mathbf{u}_1^h,\mathbf{u}_2^h) + \|\gamma^{-\frac{1}{2}}([\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_1^h)]_{x_1} - [\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_2^h)]_{x_1})\|^2_{0,\Gamma_C} \\
\leq \quad & \|\gamma^{-\frac{1}{2}}\sigma_n(\mathbf{u}_1^h-\mathbf{u}_2^h)\|^2_{0,\Gamma_C} + (1+2\mathscr{F}^2)\|\gamma^{-\frac{1}{2}}\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}_1^h-\mathbf{u}_2^h)\|^2_{0,\Gamma_C} \\
& +2\mathscr{F}^2C_2\gamma_0h^{-1}\|\mathbf{u}_1^h-\mathbf{u}_2^h\|^2_{1,\Omega} + \frac{1}{4}\|\gamma^{-\frac{1}{2}}\left([\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{w}_1^h)]_{\mathbb{R}^-} - [\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{w}_2^h)]_{\mathbb{R}^-}\right)\|^2_{0,\Gamma_C} \\
\leq \quad & C\left(\gamma_0^{-1}+\mathscr{F}^2(\gamma_0^{-1}+\gamma_0h^{-1})\right)\|\mathbf{u}_1^h-\mathbf{u}_2^h\|^2_{1,\Omega} + \frac{1}{4}d^2(\mathbf{w}_1^h,\mathbf{w}_2^h).
\end{aligned}$$

So ϕ^h is contractive if γ_0^{-1} and $\mathscr{F}^2\gamma_0h^{-1}$ are small enough.
Suppose now that $\theta \neq 1$. We choose in (32) $\beta_1 = \beta_2 = 4|1-\theta|$, so

$$\begin{aligned}
& \frac{1}{2}d^2(\mathbf{u}_1^h,\mathbf{u}_2^h) - \frac{1}{8}\|\gamma^{-\frac{1}{2}}([\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}_1^h)]_{\mathbb{R}^-} - [\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}_2^h)]_{\mathbb{R}^-})\|^2_{0,\Gamma_C} \\
& +\frac{7}{8}\|\gamma^{-\frac{1}{2}}([\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_1^h)]_{x_1} - [\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}_2^h)]_{x_1})\|^2_{0,\Gamma_C} \\
\leq \quad & \left(2(1-\theta)^2+\theta\right)\|\gamma^{-\frac{1}{2}}\sigma_n(\mathbf{u}_1^h-\mathbf{u}_2^h)\|^2_{0,\Gamma_C} + \left(2(1-\theta)^2+\theta+2\mathscr{F}^2\theta^2\right)\|\gamma^{-\frac{1}{2}}\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}_1^h-\mathbf{u}_2^h)\|^2_{0,\Gamma_C} \\
& +2\mathscr{F}^2C_2\gamma_0h^{-1}\|\mathbf{u}_1^h-\mathbf{u}_2^h\|^2_{1,\Omega} + \frac{1}{4}d^2(\mathbf{w}_1^h,\mathbf{w}_2^h).
\end{aligned}$$

So

$$\frac{3}{8}d^2(\mathbf{u}_1^h,\mathbf{u}_2^h) \le C\left(\gamma_0^{-1}\left(2(1-\theta)^2+\theta+2\mathscr{F}^2\theta^2\right)+\mathscr{F}^2\gamma_0 h^{-1}\right)\|\mathbf{u}_1^h-\mathbf{u}_2^h\|_{1,\Omega}^2+\frac{1}{4}d^2(\mathbf{w}_1^h,\mathbf{w}_2^h).$$

If γ_0^{-1} and $\mathscr{F}^2\gamma_0 h^{-1}$ are small enough such that e.g.,

$$C\left(\gamma_0^{-1}\left(2(1-\theta)^2+\theta+2\mathscr{F}^2\theta^2\right)+\mathscr{F}^2\gamma_0 h^{-1}\right)\|\mathbf{u}_1^h-\mathbf{u}_2^h\|_{1,\Omega}^2 \le \frac{1}{16}d^2(\mathbf{u}_1^h,\mathbf{u}_2^h)$$

then ϕ^h is contractive and (16) admits a unique solution according to the Banach-Picard fixed point theorem.

Step 2. Now we consider Brouwer fixed point theorem to establish existence in a more general case without conditions on the friction coefficient and on the mesh size.

• We first prove continuity of ϕ^h. We consider again the last inequality in (28) where the right part of the inequality, denoted $\mathcal{T}_1$ as before, is bounded as in (30) excepted for the last term of (29) which remains unchanged. Choosing $\beta_1=\beta_2=|1-\theta|/2$ yields

$$\begin{aligned} A_{\theta\gamma}(\mathbf{u}_1^h-\mathbf{u}_2^h,\mathbf{u}_1^h-\mathbf{u}_2^h) \quad \le \quad & \frac{|1-\theta|^2}{4}\|\gamma^{-\frac{1}{2}}\sigma_n(\mathbf{u}_1^h-\mathbf{u}_2^h)\|_{0,\Gamma_C}^2+\frac{|1-\theta|^2}{4}\|\gamma^{-\frac{1}{2}}\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}_1^h-\mathbf{u}_2^h)\|_{0,\Gamma_C}^2 \\ & +\|\gamma^{-\frac{1}{2}}([\mathbf{P}_{1,\gamma}^{\mathbf{t}}(\mathbf{u}_2^h)]_{x_1}-[\mathbf{P}_{1,\gamma}^{\mathbf{t}}(\mathbf{u}_2^h)]_{x_2})\|_{0,\Gamma_C}\|\gamma^{-\frac{1}{2}}\mathbf{P}_{\theta,\gamma}^{\mathbf{t}}(\mathbf{u}_1^h-\mathbf{u}_2^h)\|_{0,\Gamma_C}. \end{aligned}$$

Bounding $\|\gamma^{-\frac{1}{2}}([\mathbf{P}_{1,\gamma}^{\mathbf{t}}(\mathbf{u}_2^h)]_{x_1}-[\mathbf{P}_{1,\gamma}^{\mathbf{t}}(\mathbf{u}_2^h)]_{x_2})\|_{0,\Gamma_C}$ using (9) together with the definitions of x_1,x_2, bounding $\|\gamma^{-\frac{1}{2}}\mathbf{P}_{\theta,\gamma}^{\mathbf{t}}(\mathbf{u}_1^h-\mathbf{u}_2^h)\|_{0,\Gamma_C}$ as in (31), and applying two Young inequalities with $\beta_3>0$ and $\beta_4>0$ gives:

$$\begin{aligned} & \|\gamma^{-\frac{1}{2}}([\mathbf{P}_{1,\gamma}^{\mathbf{t}}(\mathbf{u}_2^h)]_{x_1}-[\mathbf{P}_{1,\gamma}^{\mathbf{t}}(\mathbf{u}_2^h)]_{x_2})\|_{0,\Gamma_C}\|\gamma^{-\frac{1}{2}}\mathbf{P}_{\theta,\gamma}^{\mathbf{t}}(\mathbf{u}_1^h-\mathbf{u}_2^h)\|_{0,\Gamma_C} \\ \le \quad & \mathscr{F}\|\gamma^{-\frac{1}{2}}\left([\mathrm{P}_{1,\gamma}^{\mathbf{n}}(\mathbf{w}_1^h)]_{\mathbb{R}^-}-[\mathrm{P}_{1,\gamma}^{\mathbf{n}}(\mathbf{w}_2^h)]_{\mathbb{R}^-}\right)\|_{0,\Gamma_C}\left(C_2\gamma_0^{\frac{1}{2}}h^{-\frac{1}{2}}\|\mathbf{u}_1^h-\mathbf{u}_2^h\|_{1,\Omega}+|\theta|\|\gamma^{-\frac{1}{2}}\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}_1^h-\mathbf{u}_2^h)\|_{0,\Gamma_C}\right) \\ \le \quad & \beta_3\|\mathbf{u}_1^h-\mathbf{u}_2^h\|_{1,\Omega}^2+\left(\frac{C_2^2\gamma_0 h^{-1}}{4\beta_3}+\frac{1}{4\beta_4}\right)\mathscr{F}^2\|\gamma^{-\frac{1}{2}}\left([\mathrm{P}_{1,\gamma}^{\mathbf{n}}(\mathbf{w}_1^h)]_{\mathbb{R}^-}-[\mathrm{P}_{1,\gamma}^{\mathbf{n}}(\mathbf{w}_2^h)]_{\mathbb{R}^-}\right)\|_{0,\Gamma_C}^2 \\ & +\beta_4|\theta|^2\|\gamma^{-\frac{1}{2}}\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}_1^h-\mathbf{u}_2^h)\|_{0,\Gamma_C}^2. \end{aligned}$$

Putting together both previous estimates, choosing β_3 and β_4 small enough, using four times (14), the definition of $A_{\theta\gamma}$ and the **V**-ellipticity of a gives the following estimate (here $C(\theta)$ and $C(\gamma_0,h,\mathscr{F},\theta)$ denote positive constants depending on θ and $\gamma_0,h,\mathscr{F},\theta$, respectively):

$$\begin{aligned} (C-C(\theta)\gamma_0^{-1})\|\mathbf{u}_1^h-\mathbf{u}_2^h\|_{1,\Omega}^2 \quad & \le \quad C\left(\gamma_0 h^{-1}+1\right)\mathscr{F}^2\|\gamma^{-\frac{1}{2}}\left([\mathrm{P}_{1,\gamma}^{\mathbf{n}}(\mathbf{w}_1^h)]_{\mathbb{R}^-}-[\mathrm{P}_{1,\gamma}^{\mathbf{n}}(\mathbf{w}_2^h)]_{\mathbb{R}^-}\right)\|_{0,\Gamma_C}^2 \\ & \le \quad C(\gamma_0,h,\mathscr{F},\theta)\|\mathbf{w}_1^h-\mathbf{w}_2^h\|_{1,\Omega}^2 \end{aligned}$$

where the last bound is obtained by using $|[x]_{\mathbb{R}^-}-[y]_{\mathbb{R}^-}|\le|x-y|$, estimate (14) and the continuity of the trace operator (as in (31)). As a result ϕ^h is continuous when γ_0^{-1} is small enough.

• We next prove boundedness of ϕ^h. Let $\mathbf{w}^h\in\mathbf{V}^h$, denote by $\mathbf{u}^h:=\phi^h(\mathbf{w}^h)$ and as before we lighten the notations by writing x instead of $-\mathscr{F}[\mathrm{P}_{1,\gamma}^{\mathbf{n}}(\mathbf{w}^h)]_{\mathbb{R}^-}$. Choosing $\mathbf{v}^h=\mathbf{u}^h$ in $\mathcal{P}(-\mathscr{F}[\mathrm{P}_{1,\gamma}^{\mathbf{n}}(\mathbf{w}^h)]_{\mathbb{R}^-})$ gives

$$A_{\theta\gamma}(\mathbf{u}^h,\mathbf{u}^h)+\int_{\Gamma_C}\frac{1}{\gamma}\,[\mathrm{P}_{1,\gamma}^{\mathbf{n}}(\mathbf{u}^h)]_{\mathbb{R}^-}\,\mathrm{P}_{\theta,\gamma}^{\mathbf{n}}(\mathbf{u}^h)\,d\Gamma+\int_{\Gamma_C}\frac{1}{\gamma}\,[\mathbf{P}_{1,\gamma}^{\mathbf{t}}(\mathbf{u}^h)]_x\mathbf{P}_{\theta,\gamma}^{\mathbf{t}}(\mathbf{u}^h)\,d\Gamma=L(\mathbf{u}^h).$$

Then we obtain after splitting as in (26) and (27):

$$A_{\theta\gamma}(\mathbf{u}^h,\mathbf{u}^h) + \int_{\Gamma_C} \frac{1}{\gamma}\left[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}^h)\right]_{\mathbb{R}^-} \mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}^h)\, d\Gamma + (\theta-1)\int_{\Gamma_C} \frac{1}{\gamma}\left[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}^h)\right]_{\mathbb{R}^-} \sigma_n(\mathbf{u}^h)\, d\Gamma$$
$$+\int_{\Gamma_C} \frac{1}{\gamma}\left[\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}^h)\right]_x \mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}^h)\, d\Gamma + (\theta-1)\int_{\Gamma_C} \frac{1}{\gamma}\left[\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}^h)\right]_x \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}^h)\, d\Gamma \;=\; L(\mathbf{u}^h).$$

Using the property in (8) with $y = 0$, Cauchy-Schwarz and Young inequalities as in (29), (30), we get

$$\begin{aligned}
&A_{\theta\gamma}(\mathbf{u}^h,\mathbf{u}^h) + \|\gamma^{-\frac{1}{2}}\left[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}^h)\right]_{\mathbb{R}^-}\|^2_{0,\Gamma_C} + \|\gamma^{-\frac{1}{2}}\left[\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}^h)\right]_x\|^2_{0,\Gamma_C} \\
&-\frac{|1-\theta|\beta_1}{2}\|\gamma^{-\frac{1}{2}}\sigma_n(\mathbf{u}^h)\|^2_{0,\Gamma_C} - \frac{|1-\theta|}{2\beta_1}\|\gamma^{-\frac{1}{2}}\left[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}^h)\right]_{\mathbb{R}^-}\|^2_{0,\Gamma_C} \\
&-\frac{|1-\theta|\beta_2}{2}\|\gamma^{-\frac{1}{2}}\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}^h)\|^2_{0,\Gamma_C} - \frac{|1-\theta|}{2\beta_2}\|\gamma^{-\frac{1}{2}}\left[\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}^h)\right]_x\|^2_{0,\Gamma_C} \le L(\mathbf{u}^h).
\end{aligned}$$

If $\theta \neq 1$, we choose $\beta_1 = \beta_2 = |1-\theta|/2$ and use three times (14) to conclude that if γ_0^{-1} is small enough (here the denomination small depends on θ) there is a positive constant C s.t. $Ca(\mathbf{u}^h,\mathbf{u}^h) \le L(\mathbf{u}^h)$, so $\|\mathbf{u}^h\|_{1,\Omega}$ is bounded and the conclusion follows. The same conclusion holds for the simpler case $\theta = 1$. □

5. Numerical experiments

We achieve the numerical implementation with the open source finite element library GetFEM++ [45]. We study, in two dimensions, the impact of a disc on a rigid support in the dynamic setting. The physical parameters are the following: the diameter of the disc is $D = 40$, the Lamé coefficients are $\lambda = 30$ and $\mu = 30$, the material density is $\rho = 1$, the volume load in the vertical direction is set to $\|f\| = 0.05$ (gravity, oriented towards the support). On the upper part of the boundary we apply a homogeneous Neumann condition $g = 0$ and the lower part of the boundary is the contact with Coulomb's friction region. We have chosen an initial vertical displacement ($\mathbf{u}_0 = 1$) and no initial velocity ($\dot{\mathbf{u}}_0 = 0$). There is an initial gap between the disc and the support. For space semi-discretization, Lagrange isoparametric finite elements of order $k = 2$ have been used. The mesh size is $h = 4$. Integrals of the non-linear term on Γ_{CT} are computed with standard quadrature formulas of order 4. The Nitsche parameters are $\theta = 1, \gamma_0 = 1000$. We limit ourselves to the symmetric variant $\theta = 1$ that has attractive properties of energy conservation in the dynamic, frictionless, setting [13, 14, 17]. We first present the results obtained by combination of Nitsche-FEM and Verlet scheme which is a second order, explicit, consistent scheme. We denote by $\tau > 0$ the time step. We consider a uniform discretization of the time interval $[0,T] : (t^0,...,t^N)$, with $t^n = n\tau, n = 0,..,N$. Let $\alpha \in [0,1]$, we use the notation $\mathbf{x}^{h,n+\alpha} = (1-\alpha)\mathbf{x}^{h,n} + \alpha\mathbf{x}^{h,n+1}$ and we denote by $\mathbf{u}^{h,n}, \dot{\mathbf{u}}^{h,n}, \ddot{\mathbf{u}}^{h,n}$ the discretized displacement, velocity and acceleration at time step t^n. The time discretization of the space semi-discrete problem (16), with the velocity-Verlet scheme, reads:

$$\begin{cases}
\text{Find } \mathbf{u}^{h,n+1}, \dot{\mathbf{u}}^{h,n+1}, \ddot{\mathbf{u}}^{h,n+1} \in \mathbf{V}^h \text{ such that :} \\
\mathbf{M}^h \ddot{\mathbf{u}}^{h,n+1} + \mathbf{B}^h(\mathbf{u}^{h,n+1}, \dot{\mathbf{u}}^{h,n+1}) = \mathbf{L}^{h,n+1}, \\
\mathbf{u}^{h,n+1} = \mathbf{u}^{h,n} + \tau\dot{\mathbf{u}}^{h,n} + \dfrac{\tau^2}{2}\ddot{\mathbf{u}}^{h,n}, \\
\dot{\mathbf{u}}^{h,n+1} = \dot{\mathbf{u}}^{h,n} + \tau\ddot{\mathbf{u}}^{h,n+\frac{1}{2}}
\end{cases}$$

with initial conditions $\mathbf{u}^{h,0} = \mathbf{u}^h_0, \dot{\mathbf{u}}^{h,0} = \dot{\mathbf{u}}^h_0, \ddot{\mathbf{u}}^{h,0} = \ddot{\mathbf{u}}^h_0$, and the notation $\mathbf{L}^{h,n+1} = \mathbf{L}^h(t^{n+1})$, the initial value $\ddot{\mathbf{u}}^h_0$ being obtained through $\mathbf{M}^h\ddot{\mathbf{u}}^{h,0} = \mathbf{L}^{h,0} - \mathbf{B}^h(\mathbf{u}^h_0, \dot{\mathbf{u}}^h_0)$. The value of the time-step has been fixed

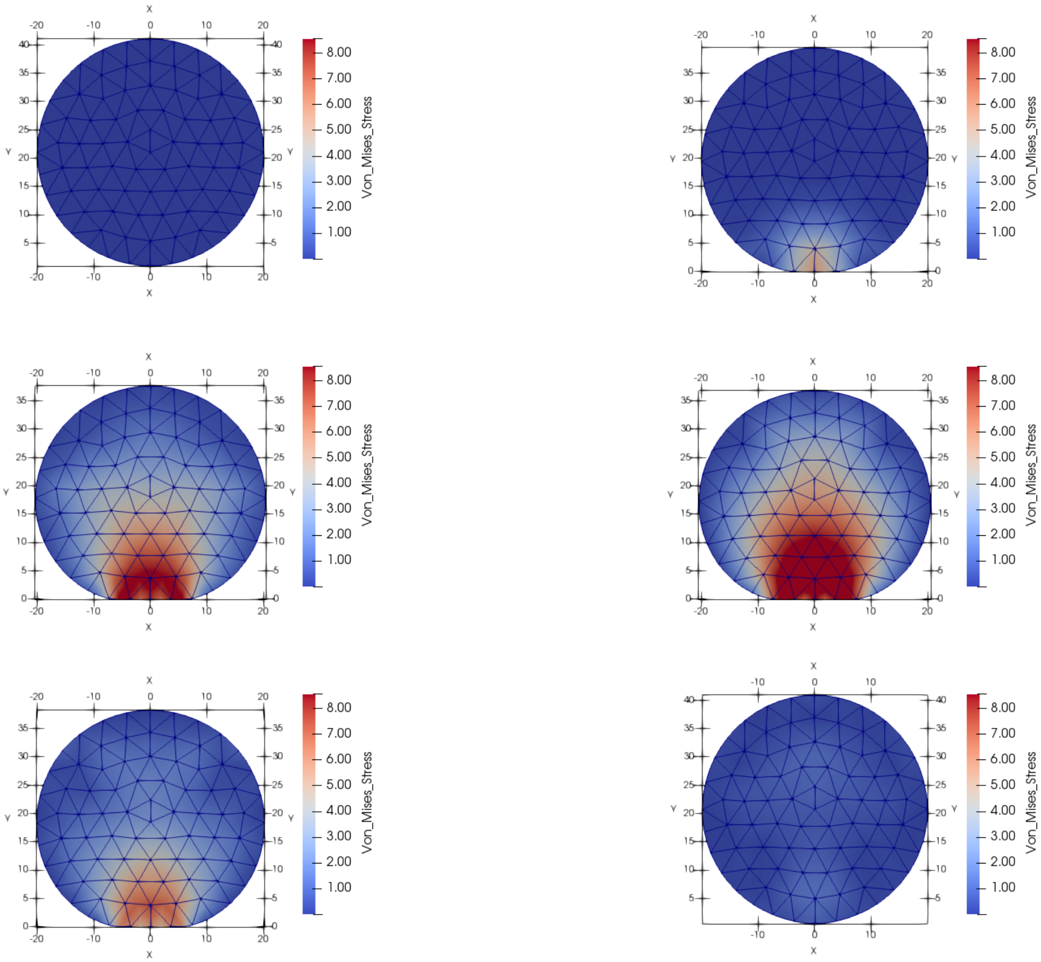

Figure 1: Impact of a disc, with Coulomb's friction ($\mathcal{F} = 0.7$) and symmetric Nitsche-FEM ($\theta = 1$) with Verlet's scheme. Deformed configuration and Von Mises stress at t=0, 8, 12, 15, 20, 23.

to $\tau = 0.01$. A snapshot of the evolution of the disc during the first bounce can be seen Figure 1. The Von Mises stress as well as the deformed configuration are depicted.
We compare our results with the penalty method, combined with the velocity-Verlet scheme:

$$\begin{cases} \text{Find } \mathbf{u}^{h,n+1}, \dot{\mathbf{u}}^{h,n+1}, \ddot{\mathbf{u}}^{h,n+1} \in \mathbf{V}^h \text{ such that } : \\ \mathbf{M}^h \ddot{\mathbf{u}}^{h,n+1} + \mathbf{B}_p^h(\mathbf{u}^{h,n+1}, \dot{\mathbf{u}}^{h,n+1}) = \mathbf{L}^{h,n+1}, \\ \mathbf{u}^{h,n+1} = \mathbf{u}^{h,n} + \tau \dot{\mathbf{u}}^{h,n} + \dfrac{\tau^2}{2} \ddot{\mathbf{u}}^{h,n}, \\ \dot{\mathbf{u}}^{h,n+1} = \dot{\mathbf{u}}^{h,n} + \tau \ddot{\mathbf{u}}^{h,n+\frac{1}{2}} \end{cases}$$

with the non linear operator $\mathbf{B}_p^h : (\mathbf{V}^h)^2 \to \mathbf{V}^h$, defined by

$$(\mathbf{B}_p^h(\mathbf{v}^h, \dot{\mathbf{v}}^h), \mathbf{w}^h)_\gamma \quad = \quad a(\mathbf{v}^h, \mathbf{w}^h) + \int_{\Gamma_C} \gamma\, [v_n^h]_{\mathbb{R}^+}\, w_n^h \, d\Gamma + \int_{\Gamma_C} \gamma\, [\dot{\mathbf{v}}_t^h]_{(\mathcal{F}[v_n^h]_{\mathbb{R}^+})} \cdot \mathbf{w}_t^h \, d\Gamma.$$

Note that we still use the notation γ for the penalty parameter.
For each method and two different friction coefficients ($\mathscr{F} = 0.1$ and $\mathscr{F} = 0.7$), we depict, for the lowest point on Γ_C, the normal and tangential displacement, and the normal and tangential stress. The comparison between the penalty method and the Nitsche method can be seen Figures 2 and 3 for a friction coefficient equal to 0.1, and in Figures 4 and 5 for a friction coefficient equal to 0.7, for a total duration $T = 150$, which allows five impacts. It can be seen that, as expected, the non penetration condition is better respected with the Nitsche method. Moreover we can see that the approximation of the stress is polluted by spurious oscillations on the friction zone which are more important for the last two rebounds in the case of the penalty method.

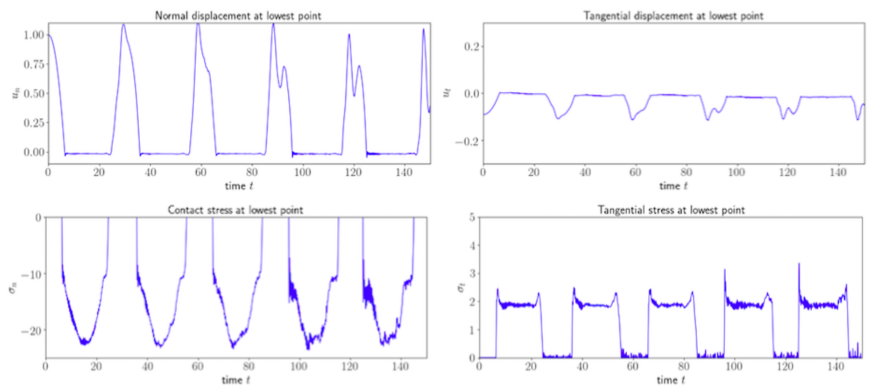

Figure 2: Nitsche's method with Verlet scheme for $\tau = 0.01, \gamma_0 = 1000, \mathscr{F} = 0.1$.

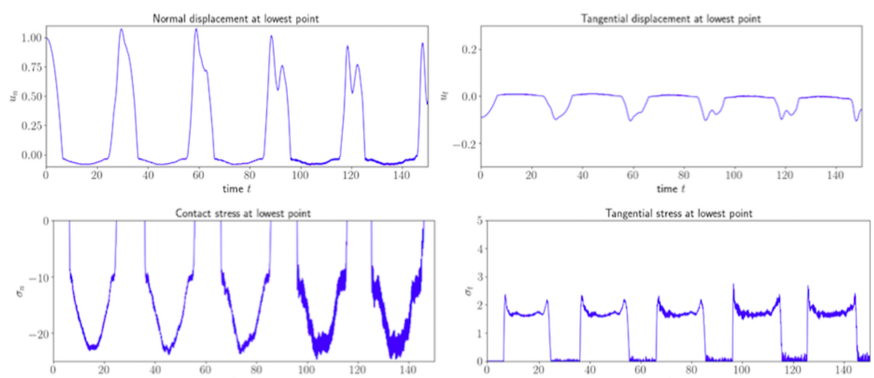

Figure 3: Penalty method with Verlet scheme for $\tau = 0.01, \gamma_0 = 1000, \mathscr{F} = 0.1$.

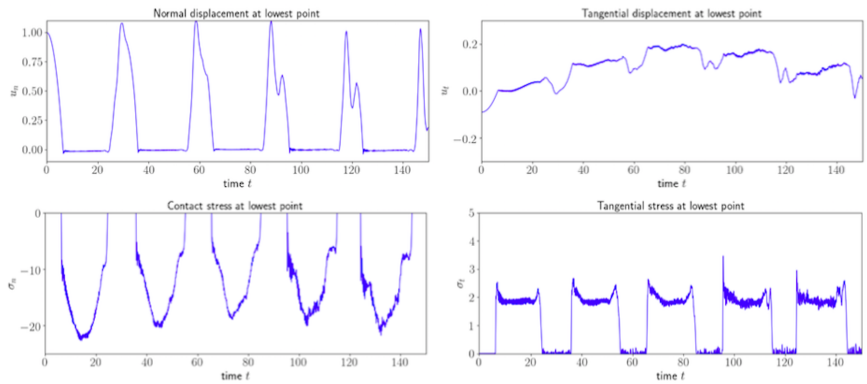

Figure 4: Nitsche's method with Verlet scheme for $\tau = 0.01, \gamma_0 = 1000, \mathscr{F} = 0.7$.

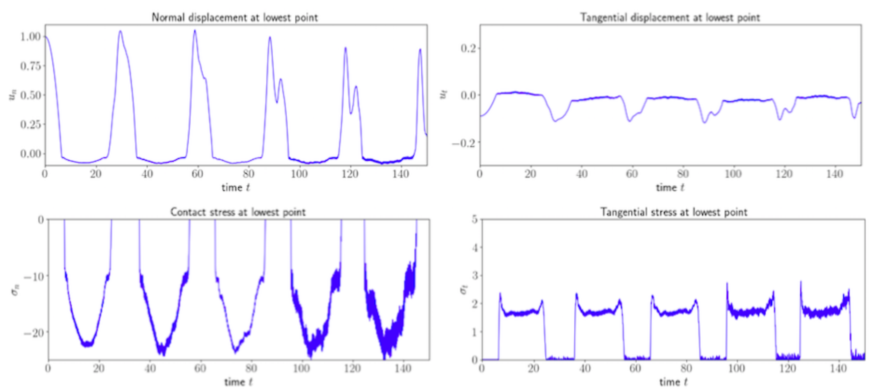

Figure 5: Penalty method with Verlet scheme for $\tau = 0.01, \gamma_0 = 1000, \mathscr{F} = 0.7$.

We also depict the discrete energies, defined, for Nitsche's method as follows $\forall t \in [0, T]$:

$$E^{h,n}(t) = \frac{1}{2}\rho\|\dot{\mathbf{u}}^{h,n}(t)\|_{0,\Omega}^2 + \frac{1}{2}a(\mathbf{u}^{h,n}(t), \mathbf{u}^{h,n}(t)) - \frac{1}{2}\left(\|\gamma^{-\frac{1}{2}}\sigma_n(\mathbf{u}^{h,n})\|_{0,\Gamma_C}^2 - \|\gamma^{-\frac{1}{2}}[\mathrm{P}_{1,\gamma}^{\mathbf{n}}(\mathbf{u}^{h,n})]_{\mathbb{R}^-}\|_{0,\Gamma_C}^2\right).$$

For the penalty method, the discrete energy is:

$$E_p^{h,n}(t) = \frac{1}{2}\rho\|\dot{\mathbf{u}}^{h,n}(t)\|_{0,\Omega}^2 + \frac{1}{2}a(\mathbf{u}^{h,n}(t), \mathbf{u}^{h,n}(t)) + \frac{1}{2}\int_{\Gamma_C} \gamma\, [u_n^h]_{\mathbb{R}^+}^2 \, d\Gamma.$$

Figures 6, 7 depict the evolution of discrete energies. They allow to assess the effect of dissipation caused by friction, which depends on the magnitude of the friction coefficient. The evolution of the discrete energy is comparable for the two methods.

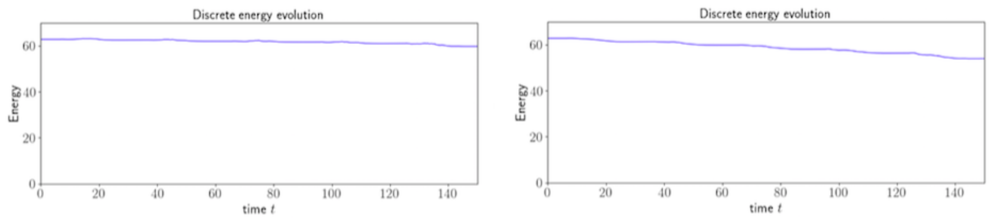

Figure 6: Discrete energy evolution for Nitsche's method. Left: $\mathscr{F} = 0.1$, Right: $\mathscr{F} = 0.7$.

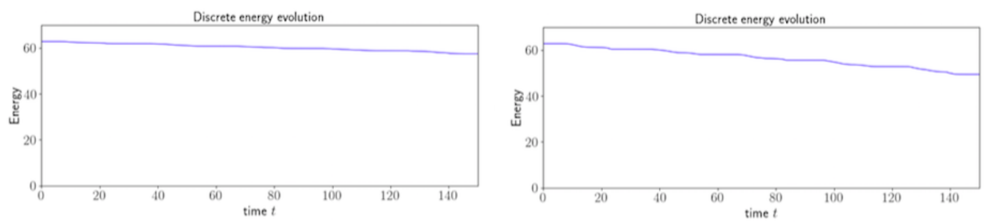

Figure 7: Discrete energy evolution for penalty method. Left: $\mathscr{F} = 0.1$, Right: $\mathscr{F} = 0.7$.

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