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Abstract

This study is concerned with the elastoplastic torsion problem and its standard finite element approximation using piecewise affine Lagrange finite elements. In the case of a polytopal convex domain in dimension $n \geq 2$ we obtain a H^1-error bound of order h for the solution. For a non convex domain we obtain an order $h^{3/4}$. This improves the existing bound of order $h^{1/2}$.

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1. Introduction

Problems written with weak formulations involving variational inequalities represent various nonlinear phenomena which occur in mechanics and physics [5, 12]. We focus on the elastoplastic torsion problem, as presented in, *e.g.*, [8] (see also [3, 9]). In the aforementioned reference, a direct piecewise linear Lagrange finite element approximation of the variational inequality is also presented, as well as a convergence result (Theorem 3.3), and two error estimates in the H^1-norm, in dimension one (Theorem 3.4) and in dimension two (Theorem 3.5). The error estimate in one dimension is optimal ($\mathcal{O}(h)$), whereas it remained suboptimal in dimension two, as it is of order $\mathcal{O}(h^{\frac{1}{2}-\frac{1}{p}})$ for a source term in L^p, $p > 2$. This bound has not been improved since then, up to our knowledge. Among the few existing results are weak and strong convergence results [13], and error estimates of $\mathcal{O}(h)$ for the L^2-norm of the gradient of the solution and under suitable restrictive assumptions, for mixed finite element approximations, using $\mathbb{P}_1/\mathbb{P}_0$ finite elements [7] or Raviart-Thomas finite elements [4].

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In this study we focus on a problem with a positive constant source term. In this case the variational inequality can be reformulated as an "obstacle" problem where the constraint involves the distance to the boundary, so the obstacle is nonsmooth and the usual techniques from the obstacle problem can not be directly applied. We present a new direct finite element approximation of the variational inequality, that makes use of piecewise linear, continuous, Lagrange finite elements, and in which the constraint involving the distance function is imposed at each node. When the domain is convex, the discretization is conforming and we prove error estimates in any dimension $n \geq 1$, with an optimal error bound of order $\mathcal{O}(h)$ for a regular enough continuous solution. In the case of a non convex domain, an extra term appear due to non conformity, that is challenging to bound. We manage to derive an error bound of $\mathcal{O}(h^{3/4})$ for a regular enough continuous solution.

As usual, we denote by $H^s(.)$, $s \in \mathbb{R}$, the Sobolev spaces. The usual norm of $H^s(D)$ is denoted by $\|\cdot\|_{s,D}$. The space $H^1_0(D)$ is the subspace of functions in $H^1(D)$ with vanishing trace on ∂D. The letter C stands for a generic constant, independent of the discretization parameters.

2. The elastoplastic torsion problem

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be an open bounded polytope, connected and with Lipschitz boundary. We consider the variational inequality modelling the torsion of an infinitely long elastoplastic cylinder of cross section Ω and plasticity yield $r > 0$. To simplify we assume that $r = 1$. The problem is to find the stress potential u such that

$$u \in K_1 \ : \qquad a(u, v-u) \geq L(v-u), \qquad \forall\, v \in K_1, \tag{1}$$

where $a : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ is the bilinear form given by:

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v, \ \ \forall\, u, v \in H^1_0(\Omega),$$

and $L(v) := \int_\Omega fv \ \ \forall\, v \in H^1_0(\Omega)$, $f \in L^2(\Omega)$. The notation K_1 represents the nonempty closed convex set of admissible stress potentials:

$$K_1 := \left\{ v \in H^1_0(\Omega) \ : \ |\nabla v| \leq 1 \text{ a.e. in } \Omega \right\},$$

where $|\cdot|$ denotes the euclidian norm in $\mathbb{R}^n$. From Stampacchia's theorem we deduce that Problem (1) admits a unique solution (see also, *e.g.*, [5, 8, 9, 12]).

Next we suppose that $f = \mathcal{C}$ is a constant function. In this case and according to [1] (see also [11]) the problem (1) can be rewritten as follows: find the stress potential u such that

$$u \in \overline{K} \ : \qquad a(u, v-u) \geq \mathcal{C} \int_\Omega (v-u), \qquad \forall\, v \in \overline{K}, \tag{2}$$

with

$$\overline{K} := \left\{ v \in H^1_0(\Omega) \ : \ |v| \leq d_{\partial\Omega} \text{ a.e. in } \Omega \right\},$$

and $d_{\partial\Omega}$ denotes the (interior) distance function with respect to the boundary $\partial\Omega$:

$$d_{\partial\Omega}(x) := \inf_{y \in \partial\Omega} |x-y|, \ \forall\, x \in \overline{\Omega}.$$

Note that (2) still admits a unique solution from Stampacchia's theorem. To lighten the discussion we can suppose without loss of generality that $\mathcal{C} > 0$, so problem (2) can be rewritten as follows: find the stress potential u such that

$$u \in K \ : \qquad a(u, v-u) \geq \mathcal{C} \int_\Omega (v-u), \qquad \forall\, v \in K, \tag{3}$$

with

$$K := \left\{ v \in H_0^1(\Omega) \ : \ 0 \leq v \leq d_{\partial\Omega} \text{ a.e. in } \Omega \right\}.$$

Again (3) admits a unique solution from Stampacchia's theorem. Now we consider problem (3) which can be seen as a kind of double obstacle problem.

Remark 2.1. *In the case where $f = \mathcal{C}$ is a positive constant, note that we could write the same variational inequality as in (1),(2) and (3) but with the convex set*

$$\widetilde{K} := \left\{ v \in H_0^1(\Omega) \ : \ v \leq d_{\partial\Omega} \textit{ a.e. in } \Omega \right\}.$$

So the torsion problem can be simply seen as an obstacle problem but with a nonsmooth obstacle which is the distance function (roughly speaking the distance function does not lie in $H^2(\Omega)$). This implies that the classical finite element error analysis for the obstacle problem can not be directly applied in the forthcoming analysis.

3. Finite element discretization

Let V_h be a family of Lagrange finite element spaces of degree one indexed by h, and coming from a family T_h of simplicial meshes of the domain Ω ($h := \max_{T \in T_h} h_T$ where h_T is the diameter of T). The family of meshes is supposed regular and quasi-uniform. We define:

$$V_h := \{ v_h \in \mathscr{C}(\overline{\Omega}) \cap H_0^1(\Omega) : \ v_{h|T} \in \mathbb{P}_1(T), \ \forall\, T \in T_h \}.$$

Let $\mathcal{N}_h$ be the set of the nodes of the mesh and set

$$K_h := \{ v_h \in V_h : 0 \leq v_h(a) \leq d_{\partial\Omega}(a), \ \forall a \in \mathcal{N}_h \}.$$

The discrete problem is as follows (recall that $\mathcal{C} > 0$):

$$u_h \in K_h \ : \qquad a(u_h, v_h - u_h) \geq \mathcal{C} \int_\Omega (v_h - u_h) \qquad \forall\, v_h \in K_h, \tag{4}$$

and it admits a unique solution.

Remark 3.1. *If Ω contains a reentrant corner (take for instance a L-shaped domain when $n = 2$), it is easy to check that generally $K_h \not\subset K$. Note however that, when Ω is a convex set in $\mathbb{R}^n$ there holds $K_h \subset K$, since the hypograph of $d_{\partial\Omega}$ is convex [10, Chapter B, Section 1.3].*

4. *A priori* error estimate

Our main result is:

Theorem 4.1. *Let $\Omega \in \mathbb{R}^n$ be an open bounded polytope, connected and with Lipschitz boundary.*

1. Let Ω be convex, $n \le 3$, $u \in K \cap H^2(\Omega)$ and $u_h \in K_h$ be the solutions to problems (3) and (4), respectively. There holds

$$\|u - u_h\|_{1,\Omega} \le Ch\|u\|_{2,\Omega}. \tag{5}$$

2. Let Ω be non convex, $u \in K \cap H^\alpha(\Omega)$ ($\alpha > \max(1, n/2)$), $\Delta u \in L^2(\Omega)$ and $u_h \in K_h$ be the solutions to problems (3) and (4), respectively. There holds

$$\|u - u_h\|_{1,\Omega} \le Ch^{\min(3/4, \alpha-1)}\|u\|_{\alpha,\Omega}. \tag{6}$$

Remark 4.1. *1. In the one dimensional case, since Ω is connected, it is necessarily convex, so we recover the well known optimal result of order $\mathcal{O}(h)$ (see,* e.g.*, [9]).*

2. In the convex case, the solution is known to be in $W^{2,p}(\Omega)$ for any $1 < p < \infty$ [2].

3. In the non convex case we have to add the assumption $\Delta u \in L^2(\Omega)$ which is necessary to write Falk's lemma in its standard form. Otherwise this would lead to additional technicalities (and changes for the convergence rate of course) which are beyond the scope of this paper. Note that reference [3] investigates some regularity properties of the solution to the torsion problem near reentrant corners.

Proof: From standard Falk's Lemma (see, *e.g.*, [9]), since $\Delta u \in L^2(\Omega)$ and since $\mathcal{I}_h u \in K_h$, where $\mathcal{I}_h$ is the Lagrange interpolation operator mapping onto V_h, we get

$$\begin{aligned}\|u - u_h\|_{1,\Omega}^2 &\le C\Big[\inf_{v_h \in K_h}(\|u - v_h\|_{1,\Omega}^2 + \|u - v_h\|_{0,\Omega}) + \inf_{v \in K}\|v - u_h\|_{0,\Omega}\Big] \\ &\le C\Big[\|u - \mathcal{I}_h u\|_{1,\Omega}^2 + \|u - \mathcal{I}_h u\|_{0,\Omega} + \inf_{v \in K}\|v - u_h\|_{0,\Omega}\Big],\end{aligned} \tag{7}$$

where the constant C depends on $\|\Delta u\|_{L^2(\Omega)}$.

1. In the first case (*i.e.*, Ω is convex and $u \in K \cap H^2(\Omega)$) the first two terms in (7) are bounded by Ch^2 and the second infimum disappears according to Remark 3.1. So bound (5) holds.

2. Let Ω be non convex, $u \in K \cap H^\alpha(\Omega)$ with $\max(1, n/2) < \alpha \le 2$ and $\Delta u \in L^2(\Omega)$. From standard approximation bounds the first two terms in (7) are bounded by $Ch^{2(\alpha-1)}$. To bound the infimum on K, we set $v := \min(u_h, d_{\partial\Omega})$. Clearly $v \in H^1(\Omega)$. Indeed for Ω a bounded polytope there holds $d_{\partial\Omega} \in H^1(\Omega)$, and the minimum of two functions in $H^1(\Omega)$ remains in $H^1(\Omega)$ [14, Lemma 1.1]. Moreover we have $v = 0$ on $\partial\Omega$ and $0 \le v \le d_{\partial\Omega}$, which guarantees that $v \in K$.

Now set $1_h := \{x \in \Omega,\ d_{\partial\Omega}(x) < u_h(x)\}$. This set is generally nonempty since $K_h \not\subset K$. If $x \notin 1_h$, then $v(x) = u_h(x)$ by definition. So

$$\|v - u_h\|_{0,\Omega}^2 = \int_\Omega (v - u_h)^2 = \int_{1_h} (d_{\partial\Omega} - u_h)^2.$$

Since $u_h \in K_h$ we have $u_h(a) \leq d_{\partial\Omega}(a) = \mathcal{I}_h d_{\partial\Omega}(a), \forall a \in \mathcal{N}_h$. So $\mathcal{I}_h d_{\partial\Omega} - u_h \geq 0$ in Ω. Let $x \in 1_h$, then we bound:

$$\begin{aligned} 0 < |(u_h - d_{\partial\Omega})(x)| &= (u_h - d_{\partial\Omega})(x) \\ &= (u_h - \mathcal{I}_h d_{\partial\Omega})(x) + (\mathcal{I}_h d_{\partial\Omega} - d_{\partial\Omega})(x) \\ &\leq (\mathcal{I}_h d_{\partial\Omega} - d_{\partial\Omega})(x). \end{aligned}$$

Therefore

$$\|v - u_h\|_{0,\Omega}^2 = \int_{1_h} (d_{\partial\Omega} - u_h)^2 \leq \int_{1_h} (I_h d_{\partial\Omega} - d_{\partial\Omega})^2 \leq \|d_{\partial\Omega} - \mathcal{I}_h d_{\partial\Omega}\|_{0,\Omega}^2.$$

Note that since Ω is a polytope, then $d_{\partial\Omega} - \mathcal{I}_h d_{\partial\Omega} = 0$ on all the simplices T of the mesh excepted those which intersect the regions where $d_{\partial\Omega}$ is not differentiable (since $d_{\partial\Omega}$ is affine on these simplices). These regions where $d_{\partial\Omega}$ is not differentiable are straight line segments when $n = 2$ and bounded plane polygons when $n = 3$. The measure (in $\mathbb{R}^{n-1}$) of these regions is finite and only depends on the geometry of Ω. Since the mesh is regular and quasi-uniform, there are at most C/h^{n-1} simplices T where $d_{\partial\Omega} - \mathcal{I}_h d_{\partial\Omega} \neq 0$. We consider such a simplex T, and using the interpolation estimate $\|d_{\partial\Omega} - \mathcal{I}_h d_{\partial\Omega}\|_{L^\infty(T)} \leq C h_T \|\nabla d_{\partial\Omega}\|_{L^\infty(T)}$ [6, Theorem 1.103], we bound as follows:

$$\|d_{\partial\Omega} - \mathcal{I}_h d_{\partial\Omega}\|_{0,T}^2 \leq h_T^n \|d_{\partial\Omega} - \mathcal{I}_h d_{\partial\Omega}\|_{L^\infty(T)}^2 \leq C h_T^{n+2}.$$

Since there are at most C/h^{n-1} simplices T concerned by the above estimate, we get $\|d_{\partial\Omega} - \mathcal{I}_h d_{\partial\Omega}\|_{0,\Omega}^2 \leq Ch^3$. As a result we obtain $\inf_{v \in K} \|v - u_h\|_{0,\Omega} \leq Ch^{3/2}$ and the final bound

$$\|u - u_h\|_{1,\Omega} \leq Ch^{\min(3/4, \alpha-1)}$$

follows. □

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