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Abstract

This study is concerned with the elastoplastic torsion problem, in dimension $n \geq 1$, and in a polytopal, convex or not, domain. In the physically relevant case where the source term is a constant, this problem can be reformulated using the distance function to the boundary. We combine the aforementioned reformulation with a Nitsche-type discretization as in [Burman, Erik, et al. Computer Methods in Applied Mechanics and Engineering 313 (2017): 362-374]. This has two advantages: 1) it leads to optimal error bounds in the natural norm, even for nonconvex domains; 2) it is easy to implement within most of finite element libraries. We establish the well-posedness and convergence properties of the method, and illustrate its behavior with numerical experiments.

Keywords: variational inequalities; elastoplastic torsion problem; finite elements; Nitsche; error estimates.
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1. Introduction

Problems written with weak formulations involving variational inequalities represent various nonlinear phenomena which occur in mechanics and physics [12, 22]. We focus on the elastoplastic torsion problem, as presented in, *e.g.*, [16] (see also [6, 17]). In the aforementioned reference, a direct piecewise affine Lagrange finite element approximation of the variational inequality is also presented, as well as a convergence result (Theorem 3.3), and two error estimates in the H^1-norm, in dimension one (Theorem 3.4) and in dimension two (Theorem 3.5). The error estimate in one dimension is optimal ($\mathcal{O}(h)$), whereas it remained suboptimal in dimension two, as it is of order $\mathcal{O}(h^{\frac{1}{2}-\frac{1}{p}})$ for a source term in L^p, $p > 2$. Among the first and few existing results are weak and strong convergence results [25], and error estimates of $\mathcal{O}(h)$ for the L^2-norm of the gradient of the solution and under suitable restrictive assumptions, for mixed finite element approximations, using $\mathbb{P}_1/\mathbb{P}_0$ finite elements [14] or Raviart-Thomas finite elements [7].

In this paper, we focus on the torsion problem with a positive constant source term, corresponding to constant shear modulus and angle of twist. In this case the variational inequality can be reformulated as an "obstacle" problem where the constraint involves the distance to the boundary, so the obstacle is nonsmooth and the usual techniques from the obstacle problem cannot be

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directly applied: for instance in [11, Theorem 5.1.2], the obstacle is supposed of Sobolev regularity H^2. In a previous paper [9], a direct finite element approximation of the variational inequality has been proposed, that makes use of piecewise affine, continuous, Lagrange finite elements, and in which the constraint involving the distance function is imposed at each node. When the domain is convex, error estimates have been established in any dimension $n = 1, 2, 3$, with an optimal error bound of $\mathcal{O}(h)$, for a regular enough continuous solution. In the case of a nonconvex domain, an error bound of $\mathcal{O}(h^{\frac{3}{4}})$ has been proven for a solution of Sobolev regularity H^α, $\alpha \geq 7/4$.

In the present paper, we propose a new method that combines both the reformulation with the distance function, as in [9], and a Nitsche term that allows to incorporate weakly the inequality constraint, following [5] and related works on Nitsche's method for variational inequalities, see, *e.g.*, [8] and references therein. For this discretization, we manage to derive optimal error estimates, for linear and quadratic finite elements, and even in the nonconvex situation, which improves the result of [9]. Moreover, this method is easy to implement into modern finite element librairies, and we provide also some numerical experiments, that allow to confirm the expected theoretical convergence rates.

As usual, we denote by $H^s(\cdot)$, $s \in \mathbb{R}$, the Sobolev spaces. The usual norm of $H^s(D)$ is denoted by $\|\cdot\|_{s,D}$, and the corresponding semi-norm is denoted by $|\cdot|_{s,D}$. The space $H^1_0(D)$ is the subspace of functions in $H^1(D)$ with vanishing trace on ∂D. The letter C stands for a generic constant, independent of the mesh size, which value can changes at different occurences.

2. The elastoplastic torsion problem

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be an open bounded polytope, connected and with Lipschitz boundary. We consider the variational inequality which, for $n = 2$, models the torsion of an infinitely long elastoplastic cylinder of cross section Ω and plasticity yield $r > 0$. To simplify we assume that $r = 1$. The problem is to find the stress potential u such that

$$u \in K_1 \ : \qquad a(u, v-u) \geq L(v-u) \qquad \forall\, v \in K_1, \tag{1}$$

where $a : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ is the bilinear form given by:

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v, \ \ \forall\, u, v \in H^1_0(\Omega),$$

and

$$L(v) := \int_\Omega f v, \ \ \forall\, v \in H^1_0(\Omega),$$

with $f \in L^2(\Omega)$. The notation K_1 represents the nonempty closed convex set of admissible stress potentials:

$$K_1 := \left\{ v \in H^1_0(\Omega) \ : \ |\nabla v| \leq 1 \text{ a.e. in } \Omega \right\},$$

where $|\cdot|$ denotes the euclidian norm in $\mathbb{R}^n$. From Stampacchia's theorem we deduce that Problem (1) admits a unique solution (see also, *e.g.*, [12, 16, 17, 22]).

Remark 2.1. *We recall some regularity results for (1): if $\Omega \subset \mathbb{R}^n$ is open, bounded and convex, with Lipschitz boundary, and for $f \in L^p(\Omega)$ with $n < p < +\infty$, then $u \in W^{2,p}(\Omega) \cap \mathscr{C}^{1,\alpha}(\overline{\Omega})$, where $\alpha = 1 - n/p$ [4]. When the domain is nonconvex the $W^{2,p}(\Omega)$ regularity can be obtained but the boundary needs to be more regular ($\mathscr{C}^{1,1}$ more precisely, see [15]) so reentrant corners of polytopes are not allowed. When reentrant corners of polytopes are considered, the loss of $W^{2,p}$-regularity is only located near these corners [6].*

Next we suppose that $f = \mathcal{C}$ is a constant function. In this case and according to [3] (see also [21]) the problem (1) can be rewritten as follows: find the stress potential u such that

$$u \in \overline{K} \ : \qquad a(u, v-u) \geq \mathcal{C} \int_\Omega (v-u) \qquad \forall v \in \overline{K}, \tag{2}$$

with

$$\overline{K} := \left\{ v \in H_0^1(\Omega) \ : \ |v| \leq d_{\partial\Omega} \text{ a.e. in } \Omega \right\},$$

and $d_{\partial\Omega}$ denotes the (interior) distance function with respect to the boundary $\partial\Omega$:

$$d_{\partial\Omega}(x) := \inf_{y\in\partial\Omega} |x-y|, \ \forall\, x \in \overline{\Omega}.$$

Note that (2) still admits a unique solution from Stampacchia's theorem. To lighten the discussion we can suppose without loss of generality that $\mathcal{C} > 0$ (see [9, Remark 2.2]), so problem (2) can be rewritten as follows: find the stress potential u such that

$$u \in K \ : \qquad a(u, v-u) \geq \mathcal{C} \int_\Omega (v-u) \qquad \forall\, v \in K, \tag{3}$$

with

$$K := \left\{ v \in H_0^1(\Omega) \ : \ v \leq d_{\partial\Omega} \text{ a.e. in } \Omega \right\}.$$

Again (3) admits a unique solution from Stampacchia's theorem. Moreover, Problem (2) and Problem (3) are equivalent, when $\mathcal{C} > 0$ [9, Proposition 2.1 and Remark 2.3]. So the torsion problem can be seen as an obstacle problem where the distance function plays the role of the obstacle. Generally speaking, for a polytope, the distance function does not lie in $H^2(\Omega)$. This implies that the classical finite element error analysis for the obstacle problem can not be directly applied.

Problem (3) in strong form, reads: find $u : \Omega \longrightarrow \mathbb{R}$ solution to:

$$\begin{cases} -\Delta u \leq \mathcal{C} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ u \leq d_{\partial\Omega} & \text{in} \quad \Omega, \\ (u - d_{\partial\Omega})(\Delta u + \mathcal{C}) = 0 & \text{in} \quad \Omega. \end{cases} \tag{4}$$

We reformulate (4) using a Lagrange multiplier λ, and get:

$$\begin{cases} -\Delta u + \lambda = \mathcal{C} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ \lambda \geq 0 & \text{in} \quad \Omega, \\ u \leq d_{\partial\Omega} & \text{in} \quad \Omega, \\ (u - d_{\partial\Omega})\lambda = 0 & \text{in} \quad \Omega. \end{cases} \tag{5}$$

We introduce the following notation for the positive part: $[a]_+ := \max(0, a)$, for $a \in \mathbb{R}$, and recall the relationship

$$([a]_+ - [b]_+)(a-b) \geq ([a]_+ - [b]_+)^2, \tag{6}$$

for $a, b \in \mathbb{R}$.

Following [5], the Kuhn-Tucker condition $(5)_{3-5}$ can equivalently be reformulated as

$$\lambda = \gamma\big[-d_{\partial\Omega} + u + \gamma^{-1}\lambda \big]_+, \tag{7}$$

with γ an arbitrary positive function on the domain Ω.

3. A Nitsche finite element method

Let V_h^k be a family of Lagrange finite element spaces of degree $k \geq 1$ indexed by h, and coming from a family T_h of simplicial meshes of the domain Ω ($h := \max_{T\in T_h} h_T$ where h_T is the diameter of $T \in T_h$). The family of meshes is assumed regular. More precisely we have:

$$V_h^k = \{v_h \in \mathscr{C}(\overline{\Omega}) \cap H_0^1(\Omega) : v_h|_T \in \mathbb{P}_k(T) \ \forall\, T \in T_h\}.$$

Each simplex T of the mesh T_h is supposed to be closed, and we denote by $\mathring{T}$ the interior of T. We define a piecewise polynomial discrete Laplacian as follows, for every v_h in V_h^k, and every simplex $T \in T_h$:

$$(\Delta_h v_h)|_{\mathring{T}} := \Delta(v_h|_{\mathring{T}}).$$

The value of $\Delta_h v_h$ on the facets of the mesh is of no importance, and can be set in practice to 0, for instance. We define also:

$$R_h(v_h) := \Delta_h v_h + \mathcal{C}.$$

Remark that, for $k = 1$, $\Delta_h v_h = 0$ and $R_h(v_h) = \mathcal{C}$. The Nitsche-type method proposed for the discretization of the elastoplastic torsion problem (5) reads: find $u_h \in V_h^k$ such that

$$a(u_h, v_h) + \big(\gamma_h\big[-d_{\partial\Omega} + u_h + \gamma_h^{-1} R_h(u_h)\big]_+, v_h\big) = (\mathcal{C}, v_h) \tag{8}$$

for all $v_h \in V_h^k$. Above the notation $(\cdot,\cdot)$ stands for the $L^2(\Omega)$-scalar product, and the function γ_h is defined cell-wise as follows:

$$\gamma_h|_{\mathring{T}} := \frac{\gamma_0}{h_T^2},$$

where $\gamma_0 > 0$ is the Nitsche parameter. Again, the value of γ_h on the facets of the mesh is of no importance, and can be set in practice to 0, for instance.

Remark 3.1. *As in [10, 20], for any parameter $\theta \in \mathbb{R}$, we can write a whole family of methods:*

$$\begin{aligned} & a(u_h, v_h) - \theta(\gamma_h^{-1}\Delta_h u_h, \Delta_h v_h) \\ & + \big(\gamma_h\big[-d_{\partial\Omega} + u_h + \gamma_h^{-1} R_h(u_h)\big]_+, v_h + \theta\gamma_h^{-1}\Delta_h v_h\big) = (\mathcal{C}, v_h + \theta\gamma_h^{-1}\Delta_h v_h). \end{aligned} \tag{9}$$

Method (8) corresponds to $\theta = 0$ and can be called an incomplete method, using the terminology widespread for discontinuous Galerkin methods. This method involves less terms and is the easiest to extend to more complex problems [24]. A symmetric method is recovered when $\theta = 1$, that corresponds to the Galerkin Least Squares technique of [5] and the Nitsche method of [19]: this symmetric method can be recovered thanks to a minimization argument, and the tangent system has a symmetric Jacobian. Provided that the Nitsche parameter γ_0 is large enough, the analysis below, for $\theta = 0$, can be extended without difficulty to other values of θ.

Remark 3.2. *Remark that, for the distance function, there holds $d_{\partial\Omega} \in H^1(\Omega) \cap \mathscr{C}^{0,1}(\overline{\Omega})$, see [9] and references therein. In [5] the assumption made on the obstacle function is stronger and this function is supposed to be $\mathscr{C}^{1,1}(\overline{\Omega})$. In fact, such Nitsche or Galerkin Least Squares formulations do not require so much regularity on the obstacle function.*

The following local inverse inequality will be helpful in the sequel, that holds for an arbitrary $v_h \in V_h^k$ and every $T \in T_h$:

$$\|\nabla v_h\|_{0,T} \leq C_{\mathrm{I}} h_T^{-1} \|v_h\|_{0,T}, \tag{10}$$

where $C_{\mathrm{I}} > 0$ is a constant that depends on the shape regularity of the mesh and of the polynomial order k, but not of h and $T \in T_h$. See, e.g., [1, 13] for the proof.

4. Numerical analysis: well-posedness and error estimate

We first state a preliminary consistency result:

$$a(u, v_h) + \big(\gamma_h\big[-d_{\partial\Omega} + u + \gamma_h^{-1}\lambda\big]_+, v_h\big) = (\mathcal{C}, v_h) \quad \forall\, v_h \in V_h^k. \tag{11}$$

The above result is a direct consequence of (5)–(7) and the inclusion $V_h^k \subset H_0^1(\Omega)$.

4.1. Well-posedness

We make use of the results from Brezis (see, *e.g.*, [2]) for M-type and pseudo-monotone operators in vector spaces. It consists in showing that the operator associated to Problem (8) is one-to-one.

Theorem 4.1. *For $\gamma_0 \ge C_{\rm I}^2$, Problem* (8) *admits one unique solution $u_h \in V_h^k$.*

Proof: We introduce the nonlinear operator $B_h : V_h^k \to V_h^k$ defined as follows:

$$\big(B_h z_h, v_h\big)_{1,\Omega} \stackrel{\text{def}}{=} a(z_h, v_h) + \big(\gamma_h\big[-d_{\partial\Omega} + z_h + \gamma_h^{-1}R_h(z_h)\big]_+, v_h\big) \tag{12}$$

for all $z_h \in V_h^k$ and where $(\cdot,\cdot)_{1,\Omega}$ is the scalar product in $H^1(\Omega)$. Note that the right-hand side of (12) is linear with respect to v_h and, hence, B_h is well defined thanks to Riesz' Theorem. The existence and uniqueness of solution for problem (8) is equivalent to the property of B_h to be one-to-one. To this purpose, according to [2], it suffices to show that by B_h is monotone and hemicontinuous.

For the monotonicity we first note that, owing to (12), we have

$$\begin{aligned}
&\big(B_h z_h - B_h v_h, z_h - v_h\big)_{1,\Omega}\\
=&\|\nabla(z_h - v_h)\|_{0,\Omega}^2\\
&+ \underbrace{\big(\gamma_h\big[-d_{\partial\Omega} + z_h + \gamma_h^{-1}R_h(z_h)\big]_+ - \gamma_h\big[-d_{\partial\Omega} + v_h + \gamma_h^{-1}R_h(v_h)\big]_+, z_h - v_h\big)}_{T_1}
\end{aligned} \tag{13}$$

for all $z_h, v_h \in V_h^k$. On the other hand, by adding and subtracting suitable terms into T_1, we get

$$\begin{aligned}
&T_1\\
=&\big(\gamma_h\big[-d_{\partial\Omega} + z_h + \gamma_h^{-1}R_h(z_h)\big]_+ - \gamma_h\big[-d_{\partial\Omega} + v_h + \gamma_h^{-1}R_h(v_h)\big]_+,\\
&-d_{\partial\Omega} + z_h + \gamma_h^{-1}R_h(z_h) - (-d_{\partial\Omega} + v_h + \gamma_h^{-1}R_h(v_h))\big)\\
&+ \big(\gamma_h^{\frac12}\big[-d_{\partial\Omega} + z_h + \gamma_h^{-1}R_h(z_h)\big]_+ - \gamma_h^{\frac12}\big[-d_{\partial\Omega} + v_h + \gamma_h^{-1}R_h(v_h)\big]_+, \gamma_h^{-\frac12}(\Delta_h v_h - \Delta_h z_h)\big).
\end{aligned}$$

Hence, using the inequality (6), Cauchy-Schwarz, Young inequalities and the inverse inequality (10), it follows that

$$\begin{aligned}
&T_1\\
\ge&\left\|\gamma_h^{\frac12}\left(\big[-d_{\partial\Omega} + z_h + \gamma_h^{-1}R_h(z_h)\big]_+ - \big[-d_{\partial\Omega} + v_h + \gamma_h^{-1}R_h(v_h)\big]_+\right)\right\|_{0,\Omega}^2\\
&-\frac12\left\|\gamma_h^{\frac12}\left(\big[-d_{\partial\Omega} + z_h + \gamma_h^{-1}R_h(z_h)\big]_+ - \big[-d_{\partial\Omega} + v_h + \gamma_h^{-1}R_h(v_h)\big]_+\right)\right\|_{0,\Omega}^2\\
&-\frac12\|\gamma_h^{-\frac12}(\Delta_h z_h - \Delta_h v_h)\|_{0,\Omega}^2\\
\ge&\frac12\left\|\gamma_h^{\frac12}\left(\big[-d_{\partial\Omega} + z_h + \gamma_h^{-1}R_h(z_h)\big]_+ - \big[-d_{\partial\Omega} + v_h + \gamma_h^{-1}R_h(v_h)\big]_+\right)\right\|_{0,\Omega}^2\\
&-\frac{C_{\rm I}^2}{2\gamma_0}\|\nabla(z_h - v_h)\|_{0,\Omega}^2.
\end{aligned}$$

The monotonicity of B_h then follows by inserting this estimate into (13) under the condition $\gamma_0 \geq C_{\mathrm{I}}^2$.

For the hemicontinuity, we must show that the real function $\varphi : [0,1] \to \mathbb{R}$ defined as

$$\varphi(t) \stackrel{\text{def}}{=} (B_h(z_h - tv_h), v_h)$$

is continuous for all $z_h, v_h \in V_h^k$.

We bound $\varphi(t) - \varphi(s)$ using the inequality $|[a]_+ - [b]_+| \leq |a - b|$, for all $a, b \in \mathbb{R}$. This gives

$$\begin{aligned}
&|\varphi(t) - \varphi(s)| \\
=&\big|(s-t)a(v_h, v_h) + \big(\gamma_h\big[-d_{\partial\Omega} + (z_h - tv_h) + \gamma_h^{-1}R_h(z_h - tv_h)\big]_+ \\
&- \gamma_h\big[-d_{\partial\Omega} + (z_h - sv_h) + \gamma_h^{-1}R_h(z_h - sv_h)\big]_+, v_h\big)\big| \\
\leq&|t-s|a(v_h, v_h) + \big(\gamma_h\big|(z_h - tv_h) + \gamma_h^{-1}R_h(z_h - tv_h) - (z_h - sv_h) - \gamma_h^{-1}R_h(z_h - sv_h)\big|, \big|v_h\big|\big) \\
=&|t-s|\Big[a(v_h, v_h) + \big(\gamma_h\big|v_h + \gamma_h^{-1}\Delta_h v_h)\big|, |v_h|\big)\Big]
\end{aligned}$$

which means that φ is Lipschitz and, thus, B_h is hemicontinuous. □

4.2. A priori error estimate

We provide first an abstract estimate, as in [5].

Theorem 4.2. *Assume that the solution (u, λ) ($\lambda = \Delta u + \mathcal{C}$) to the elastoplastic torsion problem (5) belongs to $K \times L^2(\Omega)$. For $\gamma_0 > 0$ large enough, the approximation u_h provided by (8) satisfies the following error estimate:*

$$\begin{aligned}
&\|u - u_h\|_{1,\Omega} \\
&+ \Big\|\gamma_h^{\frac{1}{2}}\Big(\big[-d_{\partial\Omega} + u + \gamma_h^{-1}\lambda\big]_+ - \big[-d_{\partial\Omega} + u_h + \gamma_h^{-1}R_h(u_h)\big]_+\Big)\Big\|_{0,\Omega} + \Big\|\gamma_h^{-\frac{1}{2}}(\Delta u - \Delta_h u_h)\Big\|_{0,\Omega} \\
&\leq C \inf_{v_h \in V_h^k}\Big\{\|u - v_h\|_{1,\Omega} + \Big\|\gamma_h^{\frac{1}{2}}(u - v_h)\Big\|_{0,\Omega} + \Big\|\gamma_h^{-\frac{1}{2}}(\Delta u - \Delta_h v_h)\Big\|_{0,\Omega}\Big\}. \quad (14)
\end{aligned}$$

Proof: Let $v_h \in V_h^k$. We first use the V-ellipticity and the continuity of $a(\cdot,\cdot)$, as well as Young's inequality, to obtain:

$$\begin{aligned}
\alpha\|u - u_h\|_{1,\Omega}^2 &\leq a(u - u_h, u - u_h) \\
&= a(u - u_h, (u - v_h) + (v_h - u_h)) \\
&\leq C\|u - u_h\|_{1,\Omega}\|u - v_h\|_{1,\Omega} + a(u - u_h, v_h - u_h) \\
&\leq \frac{\alpha}{2}\|u - u_h\|_{1,\Omega}^2 + \frac{C^2}{2\alpha}\|u - v_h\|_{1,\Omega}^2 + a(u - u_h, v_h - u_h),
\end{aligned} \quad (15)$$

with $\alpha > 0$ the ellipticity constant. We can transform the last term using the consistency property (11) for u and the finite element formulation (8) for u_h. This yields

$$\begin{aligned}
&a(u - u_h, v_h - u_h) \\
=&a(u, v_h - u_h) - a(u_h, v_h - u_h) \\
=&\big(\gamma_h\big[-d_{\partial\Omega} + u + \gamma_h^{-1}\lambda\big]_+ - \gamma_h\big[-d_{\partial\Omega} + u_h + \gamma_h^{-1}R_h(u_h)\big]_+, u_h - v_h\big).
\end{aligned} \quad (16)$$

We now transform the expression $u_h - v_h$:

$$\begin{aligned}
u_h - v_h =&u_h - v_h + \gamma_h^{-1}R_h(u_h) - \gamma_h^{-1}R_h(u_h) \\
&- \big(-d_{\partial\Omega} - u - \gamma_h^{-1}\lambda + d_{\partial\Omega} + u + \gamma_h^{-1}\lambda\big) \\
=&-\big[(-d_{\partial\Omega} + u + \gamma_h^{-1}\lambda) - (-d_{\partial\Omega} + u_h + \gamma_h^{-1}R_h(u_h))\big] - v_h + u - \gamma_h^{-1}(R_h(u_h) - \lambda).
\end{aligned}$$

Hence, by inserting this expression into (16), we get

$$\begin{aligned}
&a(u-u_h, v_h-u_h)\\
&=\underbrace{-\big(\gamma_h\big[-d_{\partial\Omega}+u+\gamma_h^{-1}\lambda\big]_+ - \gamma_h\big[-d_{\partial\Omega}+u_h+\gamma_h^{-1}R_h(u_h)\big]_+,}_{T_1}\\
&\underbrace{(-d_{\partial\Omega}+u+\gamma_h^{-1}\lambda)-(-d_{\partial\Omega}+u_h+\gamma_h^{-1}R_h(u_h))\big)}_{T_1}\\
&+\underbrace{\big(\gamma_h\big[-d_{\partial\Omega}+u+\gamma_h^{-1}\lambda\big]_+ - \gamma_h\big[-d_{\partial\Omega}+u_h+\gamma_h^{-1}R_h(u_h)\big]_+, -v_h+u-\gamma_h^{-1}(R_h(u_h)-\lambda)\big)}_{T_2}.
\end{aligned} \tag{17}$$

The first term is estimated by using the inequality (6), which yields

$$T_1 \le -\left\|\gamma_h^{\frac{1}{2}}\left(\big[-d_{\partial\Omega}+u+\gamma_h^{-1}\lambda\big]_+ - \big[-d_{\partial\Omega}+u_h+\gamma_h^{-1}R_h(u_h)\big]_+\right)\right\|_{0,\Omega}^2. \tag{18}$$

For the term T_2 we use Cauchy-Schwarz and Young's inequality, to obtain

$$\begin{aligned}
T_2 \le &\frac{1}{2}\left\|\gamma_h^{\frac{1}{2}}\left(\big[-d_{\partial\Omega}+u+\gamma_h^{-1}\lambda\big]_+ - \big[-d_{\partial\Omega}+u_h+\gamma_h^{-1}R_h(u_h)\big]_+\right)\right\|_{0,\Omega}^2\\
&+\left\|\gamma_h^{\frac{1}{2}}(u-v_h)\right\|_{0,\Omega}^2+\left\|\gamma_h^{-\frac{1}{2}}(\lambda-R_h(u_h))\right\|_{0,\Omega}^2.
\end{aligned} \tag{19}$$

There remains to bound the last term above. Let $T \in T_h$ be a mesh cell. Using a triangular inequality, we bound first

$$\|R_h(u_h)-\lambda\|_{0,T} = \|\Delta_h u_h - \Delta u\|_{0,T} \le \|\Delta_h u_h - \Delta_h v_h\|_{0,T} + \|\Delta u - \Delta_h v_h\|_{0,T}.$$

With the inverse inequality (10) and triangular inequality, we bound

$$\|\Delta_h u_h - \Delta_h v_h\|_{0,T} \le C_{\mathrm{I}} h_T^{-1}\|\nabla u_h - \nabla v_h\|_{0,T} \le C_{\mathrm{I}} h_T^{-1}(\|\nabla u_h - \nabla u\|_{0,T} + \|\nabla u - \nabla v_h\|_{0,T}).$$

Therefore the last term in (19) can be bounded as

$$\begin{aligned}
\left\|\gamma_h^{-\frac{1}{2}}(R_h(u_h)-\lambda)\right\|_{0,\Omega}^2 \quad \le \quad & C\,\gamma_0^{-1}\left(\|\nabla u_h - \nabla u\|_{0,\Omega}^2 + \|\nabla u - \nabla v_h\|_{0,\Omega}^2\right)\\
&+C\left\|\gamma_h^{-\frac{1}{2}}(\Delta u - \Delta_h v_h)\right\|_{0,\Omega}^2.
\end{aligned} \tag{20}$$

We combine estimates (15)–(17)–(18)–(19)–(20), which yields

$$\begin{aligned}
&\left(\frac{\alpha}{2}-\frac{C}{\gamma_0}\right)\|u-u_h\|_{1,\Omega}^2\\
&+\frac{1}{2}\left\|\gamma_h^{\frac{1}{2}}\left(\big[-d_{\partial\Omega}+u+\gamma_h^{-1}\lambda\big]_+ - \big[-d_{\partial\Omega}+u_h+\gamma_h^{-1}R_h(u_h)\big]_+\right)\right\|_{0,\Omega}\\
\le &\left(\frac{C^2}{2\alpha}+\frac{C}{\gamma_0}\right)\|u-v_h\|_{1,\Omega}^2 + \left\|\gamma_h^{\frac{1}{2}}(u-v_h)\right\|_{0,\Omega} + C\left\|\gamma_h^{-\frac{1}{2}}(\Delta u - \Delta_h v_h)\right\|_{0,\Omega}.
\end{aligned} \tag{21}$$

Choosing γ_0 large enough, we obtain the desired bound on the first two terms in (14). The bound on the error on the multiplier λ comes from combination of (20) and (21). □

The optimal convergence of the method for $\mathbb{P}_1$ and $\mathbb{P}_2$ Lagrange finite elements is stated below, for a Sobolev regularity $\alpha \le 2$.

Theorem 4.3. *Suppose that $k = 1, 2$, and that the solution u belongs to $H^1_0(\Omega) \cap H^\alpha(\Omega)$, with the Sobolev regularity α that satisfies $\max(1, n/2) < \alpha \leq 2$. Suppose that λ belongs to $L^2(\Omega)$. Suppose finally that the Nitsche parameter γ_0 is large enough. The solution u_h to Problem (8) satisfies the following error estimate:*

$$\begin{aligned}
&\|u - u_h\|_{1,\Omega} \\
&+ \left\|\gamma_h^{\frac{1}{2}}\left(\left[-d_{\partial\Omega} + u + \gamma_h^{-1}\lambda\right]_+ - \left[-d_{\partial\Omega} + u_h + \gamma_h^{-1}R_h(u_h)\right]_+\right)\right\|_{0,\Omega} + \left\|\gamma_h^{-\frac{1}{2}}\left(\Delta u - \Delta_h u_h\right)\right\|_{0,\Omega} \\
&\leq C(h^{\alpha-1}|u|_{\alpha,\Omega} + h\|\Delta u\|_{0,\Omega})
\end{aligned} \tag{22}$$

with $C > 0$ a constant, independent of h and u, but not of γ_0.

Proof: We consider first the case $k = 1$. We start from the estimate (14). We take $v_h = \mathcal{I}^1_h u$, the Lagrange interpolant of u onto V^1_h. So, first, from standard interpolation estimates, we obtain

$$\left\|u - \mathcal{I}^1_h u\right\|_{1,\Omega} \leq Ch^{\alpha-1}|u|_{\alpha,\Omega},$$

and

$$\left\|\gamma_h^{\frac{1}{2}}\left(u - \mathcal{I}^1_h u\right)\right\|_{0,\Omega} \leq Ch^{\alpha-1}|u|_{\alpha,\Omega}.$$

And then, for $k = 1$, we can simply proceed as follows:

$$\left\|\gamma_h^{-\frac{1}{2}}\left(\Delta u - \Delta_h(\mathcal{I}^1_h u)\right)\right\|_{0,\Omega} \leq \gamma_0^{-\frac{1}{2}} h\|\Delta u - \underbrace{\Delta_h(\mathcal{I}^1_h u)}_{=0}\|_{0,\Omega} = \gamma_0^{-\frac{1}{2}} h\|\Delta u\|_{0,\Omega}.$$

For $k = 2$, we proceed exactly as above. We start from the estimate (14). Since $V^1_h \subset V^2_h$, we take $v_h = \mathcal{I}^1_h u \in V^1_h (\subset V^2_h)$, the Lagrange interpolant of u onto V^1_h, and get the same estimates as above for the three terms. This ends the proof. □

The next statement is for $\mathbb{P}_2$ Lagrange finite elements, where a better convergence rate than $\mathcal{O}(h)$ can be expected if the solution u is regular enough.

Theorem 4.4. *Suppose that $k = 2$, and that the solution u belongs to $H^1_0(\Omega) \cap H^\alpha(\Omega)$, with the Sobolev regularity α that satisfies $\max(2, n/2) < \alpha \leq 3$. Suppose that the Nitsche parameter γ_0 is large enough. The solution u_h to Problem (8) satisfies the following error estimate:*

$$\begin{aligned}
&\|u - u_h\|_{1,\Omega} \\
&+ \left\|\gamma_h^{\frac{1}{2}}\left(\left[-d_{\partial\Omega} + u + \gamma_h^{-1}\lambda\right]_+ - \left[-d_{\partial\Omega} + u_h + \gamma_h^{-1}R_h(u_h)\right]_+\right)\right\|_{0,\Omega} + \left\|\gamma_h^{-\frac{1}{2}}\left(\Delta u - \Delta_h u_h\right)\right\|_{0,\Omega} \\
&\leq Ch^{\alpha-1}|u|_{\alpha,\Omega},
\end{aligned} \tag{23}$$

with $C > 0$ a constant, independent of h and u, but not of γ_0.

Proof: We proceed as previously, in Theorem 4.3, but with the following modifications. We start from the estimate (14). We take $v_h = \mathcal{I}^2_h u$, the Lagrange interpolant of u onto V^2_h. Still from standard interpolation estimates, we obtain

$$\|u - \mathcal{I}^2_h u\|_{1,\Omega} \leq Ch^{\alpha-1}|u|_{\alpha,\Omega},$$

and

$$\left\|\gamma_h^{\frac{1}{2}}\left(u - \mathcal{I}^2_h u\right)\right\|_{0,\Omega} \leq Ch^{\alpha-1}|u|_{\alpha,\Omega}.$$

For the last term, we need the following local error bound, for each $T \in T_h$:

$$\left\|\gamma_h^{-\frac{1}{2}}\left(\Delta u - \Delta_h(\mathcal{I}_h^2 u)\right)\right\|_{0,T} = \gamma_0^{-\frac{1}{2}} h_T \|\Delta u - \Delta(\mathcal{I}_h^2 u|_T)\|_{0,T} \leq C\gamma_0^{-\frac{1}{2}} h_T \|u - \mathcal{I}_h^2 u|_T\|_{2,T}$$

which is possible since $\alpha > 2$. Then we use standard (local) interpolation error estimates to get:

$$\left\|\gamma_h^{-\frac{1}{2}}\left(\Delta u - \Delta_h(\mathcal{I}_h^2 u)\right)\right\|_{0,T} \leq C\gamma_0^{-\frac{1}{2}} h_T h_T^{\alpha-2} |u|_{\alpha,T} \leq C h_T^{\alpha-1} |u|_{\alpha,T}.$$

By summation on the simplices T of the mesh T_h, we get finally

$$\left\|\gamma_h^{-\frac{1}{2}}\left(\Delta u - \Delta_h(\mathcal{I}_h^2 u)\right)\right\|_{0,\Omega} \leq C h^{\alpha-1} |u|_{\alpha,\Omega}.$$

This ends the proof. □

5. Numerical experiments

The following results are computed with the help of scikit-fem [18] for finite element assembly and autograd [23] for automatic differentiation. We consider two experiments introduced in [17] with convex and nonconvex domains. The parameters are chosen as $\mathcal{C} = \gamma_0 = 10$. First, a convergence study is performed using linear elements and the mesh sequences are depicted in Figure 1. Error in the stress potential u is computed against a reference solution which is obtained using quadratic finite elements and a sufficiently refined mesh. The resulting discrete solutions u_h and the magnitudes of the gradient $|\nabla u_h|$ are depicted in Figures 2 and 4, and the error in Figures 3 and 5 for the convex and nonconvex examples, respectively. The convergence rate of the H^1 error is $\mathcal{O}(h)$ which is also an expected consequence of Theorem 4.3. Moreover, the convergence rate of the L^2-error is $\mathcal{O}(h^2)$, a rate we are not able to prove. It is well-known that L^2-error estimates are difficult to prove for problems modelled by variational inequalities.

We continue by solving the same experiments using quadratic elements. For this purpose, a reference solution is computed using cubic elements and a suitably refined mesh. The resulting discrete solutions are given in Figure 6 with the convergence rates in Figure 7. This time we observe that the H^1 error is approximately $\mathcal{O}(h^{1.5})$. This is in agreement with Theorem 4.4. However this is lower than the rate $\mathcal{O}(h^2)$ for quadratic elements and a smooth solution. The reduction in the convergence rate is explained by the Sobolev regularity of the exact solution as $u \in H^2(\Omega)$ while $u \notin H^3(\Omega)$. In one dimension, it is easy to check that the second derivative is expected to have a step discontinuity at the free boundary between elastic and plastic zones which suggests $u \in H^s(\Omega)$, $s < 5/2$ (see also Remark 2.1). This also means that adaptive techniques could be used to improve the convergence rate with respect to the number of degrees-of-freedom.

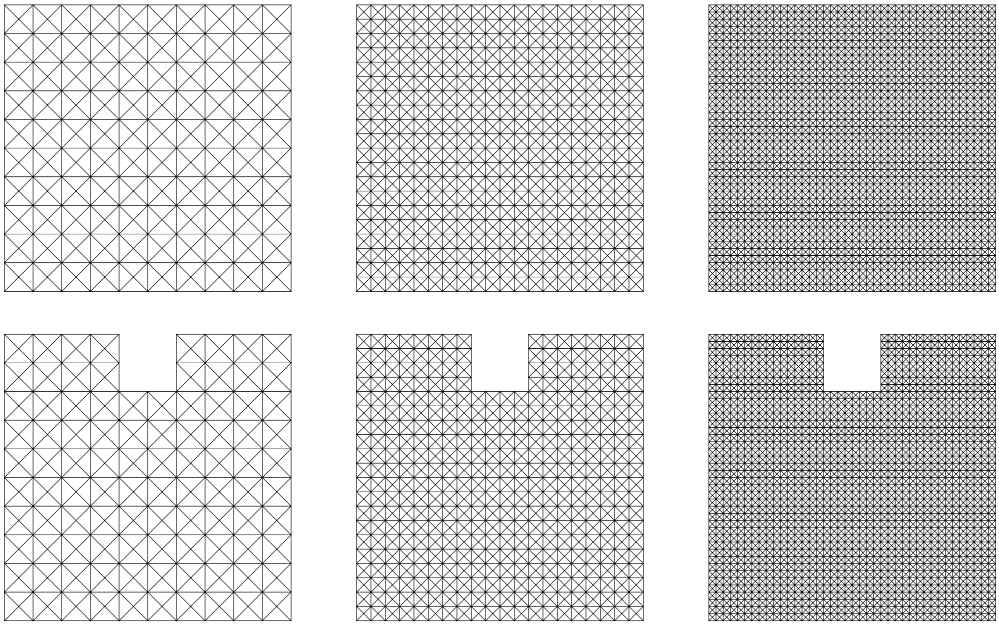

Figure 1: The first three meshes for the convex (top) and nonconvex (bottom) examples from the uniform mesh sequences. The side length of the larger square is 1 while the smaller square in the nonconvex example has a side length of 0.2.

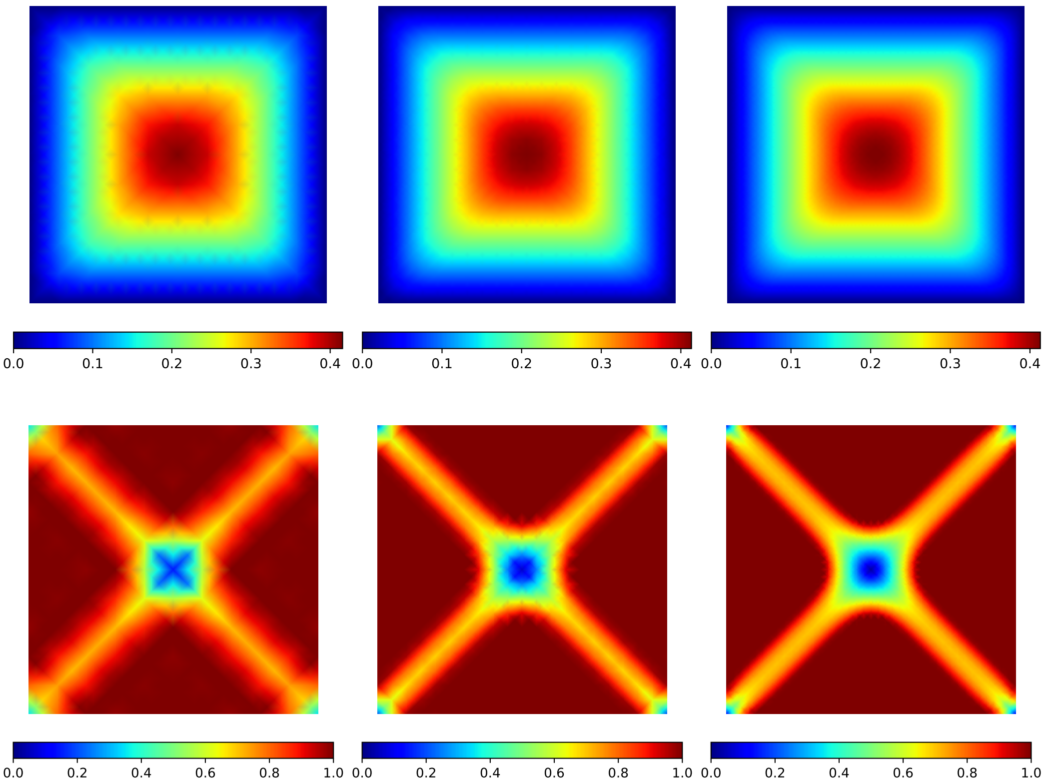

Figure 2: The first three discrete solutions u_h (top) and gradient magnitudes $|\nabla u_h|$ (bottom) for the convex example.

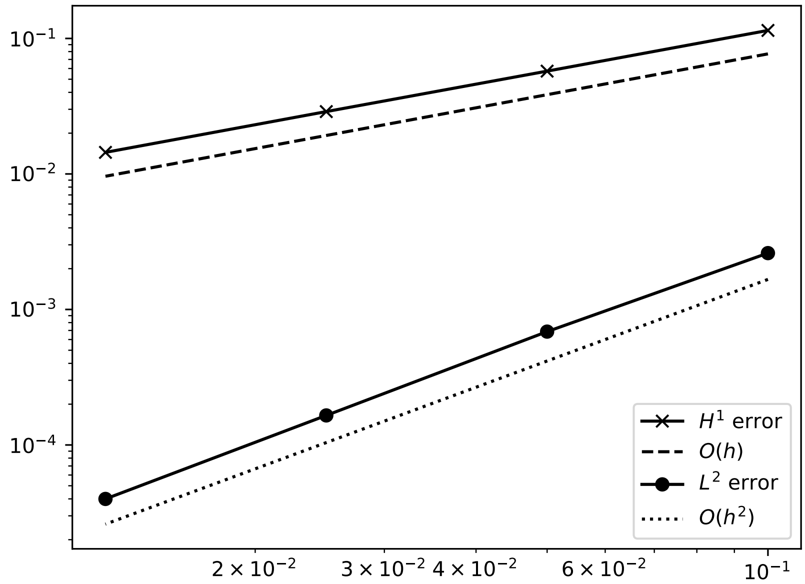

Figure 3: The error between u_h and a reference solution as a function of the mesh parameter h for the convex example. The reference solution is obtained using a quadratic finite element method on a suitably refined mesh.

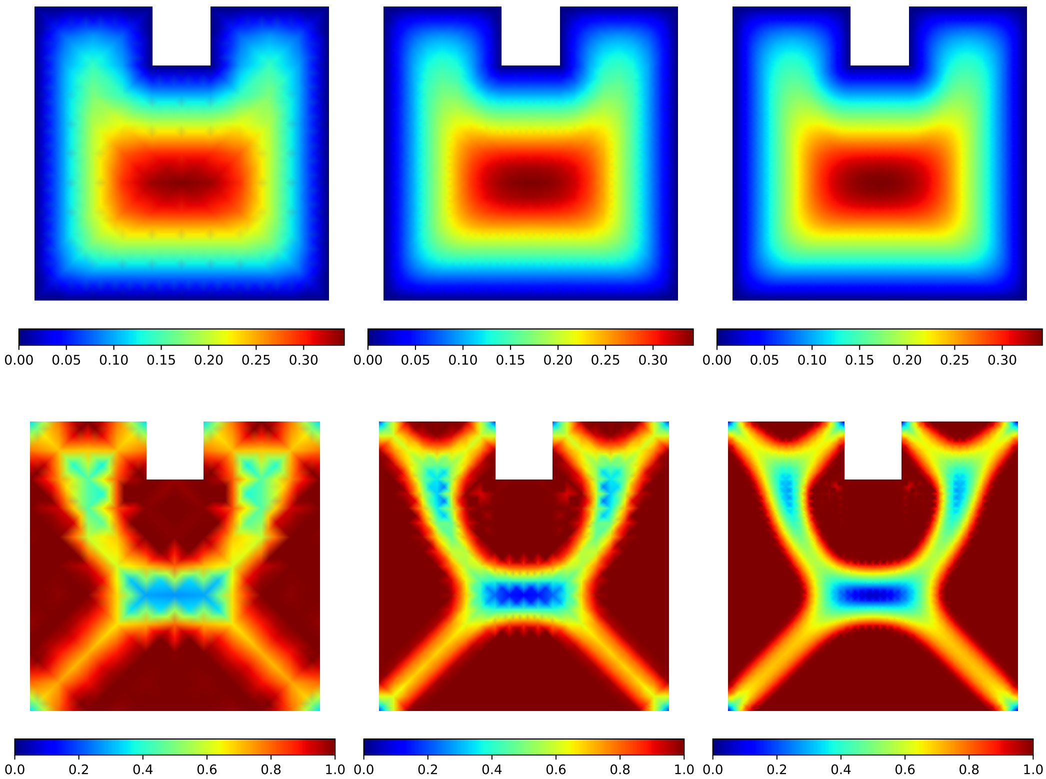

Figure 4: The first three discrete solutions u_h (top) and gradient magnitudes $|\nabla u_h|$ (bottom) for the nonconvex example.

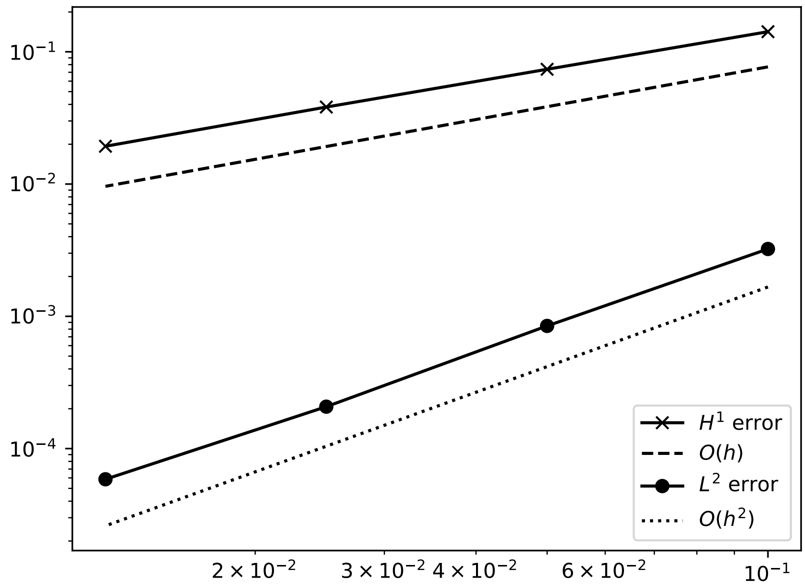

Figure 5: The error between u_h and a reference solution as a function of the mesh parameter h for the nonconvex example. The reference solution is obtained using a quadratic finite element method on a suitably refined mesh.

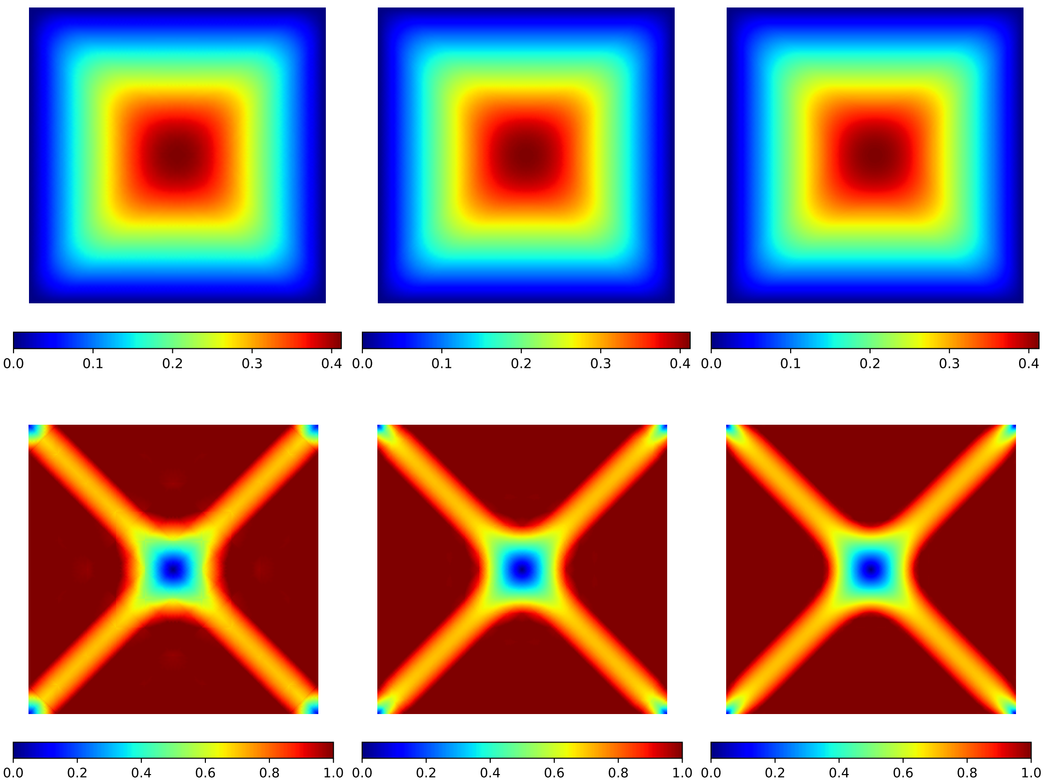

Figure 6: The first three discrete solutions u_h (top) and gradient magnitudes $|\nabla u_h|$ (bottom) for the convex example using quadratic elements.

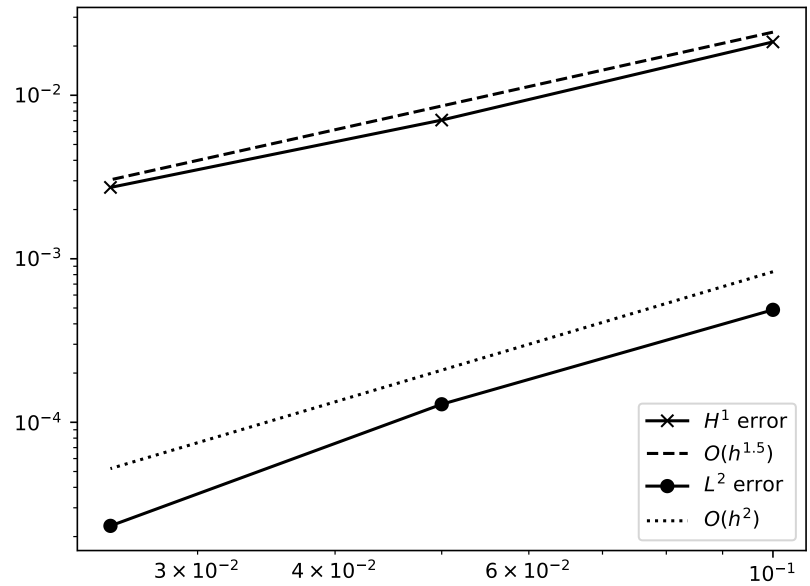

Figure 7: The error between u_h and a reference solution as a function of the mesh parameter h for the convex example using quadratic elements. The reference solution is obtained using a cubic finite element method on a suitably refined mesh.

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