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A posteriori error analysis for the normal compliance problem

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Abstract

In this work, a contact problem between a linear elastic material and a deformable obstacle is numerically analyzed. The contact is modelled using the well-known normal compliance contact condition. The weak formulation leads to a nonlinear variational equation which is approximated by using the finite element method. A priori error estimates are recalled. Then, we define an a posteriori error estimator of residual type to evaluate the accuracy of the finite element approximation of the problem. Upper and lower bounds of the discretization error are proved for this estimator.

Key words: normal compliance problem, finite element method, a posteriori error estimates, residuals
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1 Introduction and notation

The finite element method is currently used in the numerical approximation of contact problems occurring in several engineering applications (see [10,11,14,17,25]) and the a posteriori error estimators are efficient tools for evaluating numerically the quality of these finite element computations. An

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extensive review of different estimators used in various contexts can be found in [2,3,9,23,24]. Several error estimators have been chosen and studied for frictionless or frictional contact problems, in particular in [4,26] (residual approach using a penalization of the contact condition), in [12,13] (residual approach for the variational inequality and the corresponding mixed formulation for unilateral contact), in [6,7] (error in the constitutive relation for unilateral contact with or without Coulomb friction), in [19] (duality approach for the compliance model with friction) and finally in [8] (residual approach for BEM-discretizations).

In the present work we are interested in developing residual estimators for the two-dimensional normal compliance contact model in linear elasticity initially introduced in [20,21] and studied in [15,16]. As a consequence, we consider a similar study than the ones in [4,26] since the compliance law could be seen (to simplify) as a kind of penalization with a fixed penalty parameter depending on the material characteristics. In [4,26] the authors propose a first error estimator for a penalization of the unilateral contact law and prove (in [4]) that the discretization error is bounded by the estimator. In our work we also prove that the local indicators are bounded by the local discretization error for the compliance model. To our knowledge, the asymptotic equivalence between an estimator and the discretization error is a new result for the contact model with normal compliance.

The paper is organized as follows. In section 2 we introduce the equations modelling the contact problem between an elastic body and a deformable foundation. We write the problem using a nonlinear variational equation which is well posed. In the third section, we choose a classical discretization involving continuous finite elements of degree one, we write the discrete problem and we recall the corresponding a priori error estimates. Section 4 is concerned with the study of an a posteriori error estimator: we propose a residual error estimator and we prove that the discretization error is bounded by the estimator. In addition we prove that the local estimators are bounded by local discretization errors.

As usual, we denote by $(L^2(.))^d$ and by $(H^s(.))^d$, $s \geq 0, d = 1, 2$ the Lebesgue and Sobolev spaces in one and two space dimensions (see [1]). The usual norm of $(H^s(D))^d$ is denoted by $\|\cdot\|_{s,D}$ and we keep the same notation when $d = 1$ or $d = 2$. For shortness the $(L^2(D))^d$-norm will be denoted by $\|\cdot\|_D$ when $d = 1$ or $d = 2$. In the sequel the symbol $|\cdot|$ will either denote the euclidian norm in $\mathbb{R}^2$, or the length of a line segment, or the area of a plane domain. Finally the notation $a \lesssim b$ means that there exists a positive constant C independent of a and b (and of the meshsize of the triangulation) such that $a \leq C\, b$. The notation $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$ hold simultaneously. Next, bold letters like $\boldsymbol{u}, \boldsymbol{v}$, indicate vector valued quantities, while the capital ones (e.g., $\mathbf{V}, \mathbf{V}_h \ldots$) represent functional sets involving vector fields.

2 The normal compliance problem in elasticity

Let Ω represent an elastic body in $\mathbb{R}^2$ where plane strain assumptions are assumed. The boundary $\partial\Omega$ is supposed to be polygonal, i.e., it is the union of a finite number of linear segments. Moreover we suppose that the boundary consists in three nonoverlapping parts Γ_D, Γ_N and Γ_C with $\text{meas}(\Gamma_D) > 0$ and $\text{meas}(\Gamma_C) > 0$. The normal unit outward vector on $\partial\Omega$ is denoted $\boldsymbol{\nu} = (n_1, n_2)$ and we choose as unit tangential vector $\boldsymbol{t} = (-n_2, n_1)$. In its initial configuration, the body is in contact on Γ_C with normal compliance conditions. The body is clamped on Γ_D, for the sake of simplicity, and it is subjected to volume forces $\boldsymbol{f} = (f_1, f_2) \in (L^2(\Omega))^2$ and to surface forces $\boldsymbol{g} = (g_1, g_2) \in (L^2(\Gamma_N))^2$.

Let us denote by $\boldsymbol{u} = (u_i)_{1\le i\le 2}$ the displacement vector and by $\boldsymbol{\sigma} = (\sigma_{ij})_{1\le i,j\le 2}$ the stress tensor such that

$$\sigma_{ij}(\boldsymbol{u}) = a_{ijhk}\frac{\partial u_h}{\partial x_k}, \quad i, j, h, k \in \{1, ..., 2\},$$

where the summation convention of repeated indices is adopted. The functions $a_{ijhk} \in L^\infty(\Omega)$ are the coefficients of a fourth order tensor, representing the elastic properties of the material. As usual we assume that $a_{ijhk} = a_{jihk} = a_{hkij}$ and the ellipticity condition $a_{ijhk}\xi_{ij}\xi_{hk} \ge \alpha|\xi|^2$, $\forall\, \xi_{ij} = \xi_{ji}$, for some $\alpha > 0$.

The frictionless contact problem with normal compliance in elastostatics is to find the displacement field $\boldsymbol{u}$ such that equations (1)–(4) hold (see [15,16,20,21]):

$$\begin{aligned} \mathbf{div}\, \boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} &= \boldsymbol{0} \quad \text{in } \Omega, && (1)\\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \Gamma_D, && (2)\\ \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu} &= \boldsymbol{g} \quad \text{on } \Gamma_N, && (3) \end{aligned}$$

where **div** denotes the divergence operator of tensor valued functions. For each displacement field $\boldsymbol{v}$ and for each density of surface forces $\boldsymbol{\sigma}(\boldsymbol{v})\boldsymbol{\nu}$ defined on $\partial\Omega$, we adopt the following notation:

$$\boldsymbol{v} = v_n\boldsymbol{\nu} + v_t\boldsymbol{t} \quad \text{and} \quad \boldsymbol{\sigma}(\boldsymbol{v})\boldsymbol{\nu} = \sigma_n(\boldsymbol{v})\boldsymbol{\nu} + \sigma_t(\boldsymbol{v})\boldsymbol{t},$$

where the normal and tangential components of the displacement field are given by $v_n = \boldsymbol{v}\cdot\boldsymbol{\nu}$ and $v_t = \boldsymbol{v}\cdot\boldsymbol{t}$, respectively, while the normal and shear components of the stress field are defined as $\sigma_n(\boldsymbol{v}) = \boldsymbol{\sigma}(\boldsymbol{v})\boldsymbol{\nu}\cdot\boldsymbol{\nu}$ and $\sigma_t(\boldsymbol{v}) = \boldsymbol{\sigma}(\boldsymbol{v})\boldsymbol{\nu}\cdot\boldsymbol{t}$.

Then the conditions of normal compliance without friction on Γ_C are written as follows:

$$\left.\begin{aligned}\sigma_n(\boldsymbol{u}) &= -c_n(u_n)_+^{m_n},\\ \sigma_t(\boldsymbol{u}) &= 0,\end{aligned}\right\} \tag{4}$$

where $(.)_+$ stands for the positive part so that $(u_n)_+$ represents the penetration of the body into the foundation. The constant $m_n \geq 1$, as well as the non-negative function c_n in $L^\infty(\Gamma_C)$, stand for interface parameters characterizing the contact behaviour between the body and the deformable foundation.

Remark 1 *When $m_n = 1$ and $c_n = \varepsilon^{-1}$ (where $\varepsilon > 0$ is a small positive parameter) we recover the classical penalty method (see, e.g., [4,14]) used in the approximation of unilateral contact problems.*

The set of admissible displacements

$$\mathbf{V} = \left\{ \boldsymbol{v} \in (H^1(\Omega))^2 \ : \ \boldsymbol{v} = \mathbf{0} \ \text{ on } \ \Gamma_D \right\}$$

is endowed with the norm of $(H^1(\Omega))^2$. We denote by $a(\cdot,\cdot)$ the standard bilinear form of linear elasticity

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} = \int_\Omega a_{ijhk} \frac{\partial u_i}{\partial x_j} \frac{\partial v_h}{\partial x_k} \, \mathrm{d}\boldsymbol{x},$$

where $\boldsymbol{\varepsilon} = (\varepsilon_{ij})_{1\leq i,j\leq 2}$ stands for the strain tensor where $\varepsilon_{ij}(\boldsymbol{u}) = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$. If Γ_D has positive superficial measure it is well known that the bilinear form $a(\cdot,\cdot)$ is $\mathbf{V}$-elliptic,

$$a(\boldsymbol{v},\boldsymbol{v}) \geq \alpha \int_\Omega \sum_{i,j=1}^{2} \left| \frac{\partial v_i}{\partial x_j} \right|^2 \, \mathrm{d}\boldsymbol{x} \geq \beta \|\boldsymbol{v}\|_{1,\Omega}^2, \quad \forall \boldsymbol{v} \in \mathbf{V}, \quad \beta > 0.$$

The previous boundary value problem leads to the following nonlinear variational equation (see, e.g., [15,16]),

$$\boldsymbol{u} \in \mathbf{V}, \;\; a(\boldsymbol{u},\boldsymbol{v}) + j_n(\boldsymbol{u},\boldsymbol{v}) = L(\boldsymbol{v}), \;\; \forall\, \boldsymbol{v} \in \mathbf{V}, \tag{5}$$

where the linear form L is given by

$$L(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} + \int_{\Gamma_N} \boldsymbol{g}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x}) \, \mathrm{d}\gamma(\boldsymbol{x})$$

and the normal compliance functional $j_n : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ is defined as,

$$j_n(\boldsymbol{u},\boldsymbol{v}) = \int_{\Gamma_C} c_n(u_n)_+^{m_n} v_n \, \mathrm{d}\gamma(\boldsymbol{x}).$$

Afterwards we use the imbedding (see, e.g., [1]),

$$H^1(\Omega) \hookrightarrow L^q(\Gamma_C) \tag{6}$$

for each $q \in [1, +\infty[$.

It is obvious that problem (5) admits a unique solution (see, for instance, [14]). Moreover, it is easy to check that $\sigma_n(\boldsymbol{u}) \in L^2(\Gamma_C)$. Since $\sigma_n(\boldsymbol{u}) = -c_n(u_n)_+^{m_n}$, we find that

$$\begin{aligned}
\|\sigma_n(\boldsymbol{u})\|_{L^2(\Gamma_C)} &= \|c_n(u_n)_+^{m_n}\|_{L^2(\Gamma_C)} \\
&\leq \|c_n\|_{L^\infty(\Gamma_C)}\|(u_n)_+^{m_n}\|_{L^2(\Gamma_C)} \\
&\leq \|c_n\|_{L^\infty(\Gamma_C)}\|u_n^{m_n}\|_{L^2(\Gamma_C)} \\
&= \|c_n\|_{L^\infty(\Gamma_C)}\|u_n\|^{m_n}_{L^{2m_n}(\Gamma_C)} \\
&\lesssim \|c_n\|_{L^\infty(\Gamma_C)}\|\boldsymbol{u}\|^{m_n}_{1,\Omega},
\end{aligned}$$

where we use (6) and we obtain the desired regularity.

3 Finite element approximation and a priori error estimate

We approximate problem (5) by a standard finite element method. Namely we fix a family of meshes $T_h, h > 0$, regular in Ciarlet's sense (see [5]), made of closed triangles and assumed to be subordinated to the decomposition of the boundary $\partial\Omega$ into Γ_D, Γ_N and Γ_C. For $K \in T_h$ we recall that h_K is the diameter of K and $h = \max\limits_{K \in T_h} h_K$. The regularity of the mesh implies in particular that for any edge E of K one has $h_E = |E| \sim h_K$.

Let us define E_h (resp. $\mathcal{N}_h$) as the set of edges (resp. nodes) of the triangulation and set $E_h^{int} = \{E \in E_h : E \subset \Omega\}$ the set of interior edges of T_h (the edges are supposed to be relatively open). We denote by $E_h^N = \{E \in E_h : E \subset \Gamma_N\}$ the set of exterior edges included into the part of the boundary where we impose Neumann conditions, and similarly $E_h^C = \{E \in E_h : E \subset \Gamma_C\}$ is the set of exterior edges included into the part of the boundary where we impose the contact conditions. Set $\mathcal{N}_h^D = \mathcal{N}_h \cap \overline{\Gamma_D}$ (note that the extreme nodes of $\overline{\Gamma_D}$ belong to $\mathcal{N}_h^D$).

For an element K, we will denote by E_K the set of edges of K and according to the above notation, we set $E_K^{int} = E_K \cap E_h^{int}$, $E_K^N = E_K \cap E_h^N$, $E_K^C = E_K \cap E_h^C$. For an edge E of an element K, introduce $\boldsymbol{\nu}_{K,E} = (n_1, n_2)$ the unit outward normal vector to K along E and the tangent vector $\boldsymbol{t}_{K,E} = \boldsymbol{\nu}_{K,E}^\perp = (-n_2, n_1)$. Furthermore, for each edge E, we fix one of the two normal vectors and denote it by $\boldsymbol{\nu}_E$ and we set $\boldsymbol{t}_E = \boldsymbol{\nu}_E^\perp$. The jump of some vector valued function $\boldsymbol{v}$ across an edge $E \in E_h^{int}$ at a point $\boldsymbol{y} \in E$ is defined as

$$\left[\!\left[\boldsymbol{v}\right]\!\right]_E(\boldsymbol{y}) = \lim_{\alpha \to 0^+} \boldsymbol{v}(\boldsymbol{y} + \alpha\boldsymbol{\nu}_E) - \boldsymbol{v}(\boldsymbol{y} - \alpha\boldsymbol{\nu}_E), \quad \forall E \in E_h^{int}.$$

Note that the sign of $\left[\!\left[\boldsymbol{v}\right]\!\right]_E$ depends on the orientation of $\boldsymbol{\nu}_E$. Finally we will need local subdomains (also called patches). As usual, let ω_K be the union of all elements having a nonempty intersection with K. Similarly for a node $\boldsymbol{x}$ and an edge E, let $\omega_x = \cup_{K:x\in K} K$ and $\omega_E = \cup_{x\in \overline{E}}\omega_x$.

The finite element space used in Ω is then defined by

$$\mathbf{V}_h = \left\{ \boldsymbol{v}_h \in (\mathcal{C}(\overline{\Omega}))^2 : \quad \forall K \in T_h, \quad \boldsymbol{v}_h|_K \in (\mathbb{P}_1(K))^2, \quad \boldsymbol{v}_h|_{\Gamma_D} = \mathbf{0} \right\}.$$

Thus, the discrete formulation of the normal compliance problem (5) is the following.

$$\boldsymbol{u}_h \in \mathbf{V}_h, \; a(\boldsymbol{u}_h, \boldsymbol{v}_h) + j_n(\boldsymbol{u}_h, \boldsymbol{v}_h) = L(\boldsymbol{v}_h), \; \forall\, \boldsymbol{v}_h \in \mathbf{V}_h. \tag{7}$$

Using the same arguments as in the continuous case (see [15,16]), it is straightforward that problem (7) admits a unique solution.

We now consider the quasi-interpolation operator π_h: for any $v \in L^1(\Omega)$, we define $\pi_h v$ as the unique element in $V_h = \{v_h \in \mathcal{C}(\overline{\Omega}) : \; \forall K \in T_h, \; v_h|_K \in \mathbb{P}_1(K), \; v_h|_{\Gamma_D} = 0\}$ such that:

$$\pi_h v = \sum_{x \in \mathcal{N}_h \setminus \mathcal{N}_h^D} \alpha_x(v)\lambda_x, \tag{8}$$

where for any $\boldsymbol{x} \in \mathcal{N}_h \setminus \mathcal{N}_h^D$, λ_x is the standard basis function in V_h satisfying $\lambda_x(x') = \delta_{x,x'}$, for all $\boldsymbol{x}' \in \mathcal{N}_h \setminus \mathcal{N}_h^D$, and $\alpha_x(v)$ is defined as follows:

$$\alpha_x(v) = \frac{1}{|\omega_x|}\int_{\omega_x} v(\boldsymbol{y})\, d\boldsymbol{y}, \quad \forall \boldsymbol{x} \in \mathcal{N}_h \setminus \mathcal{N}_h^D.$$

The following estimates hold (see, e.g., [24]).

Lemma 2 *For any $v \in V = \{v \in H^1(\Omega) \; : \; v = 0 \;\; on \;\; \Gamma_D\}$ we have*

$$\|v - \pi_h v\|_K \lesssim h_K \|\nabla v\|_{\omega_K}, \quad \forall K \in T_h,$$
$$\|v - \pi_h v\|_E \lesssim h_E^{1/2} \|\nabla v\|_{\omega_E}, \quad \forall E \in E_h.$$

Since we deal with vector valued functions we can define a vector valued operator (which we denote again by π_h for the sake of simplicity) whose components are defined above. Consequently we can directly state the following.

Lemma 3 *For any $\boldsymbol{v} \in \mathbf{V}$ we have*

$$\|\boldsymbol{v} - \pi_h \boldsymbol{v}\|_K \lesssim h_K \|\boldsymbol{v}\|_{1,\omega_K}, \quad \forall K \in T_h, \tag{9}$$
$$\|\boldsymbol{v} - \pi_h \boldsymbol{v}\|_E \lesssim h_E^{1/2} \|\boldsymbol{v}\|_{1,\omega_E}, \quad \forall E \in E_h. \tag{10}$$

We next recall a standard result dealing with the a priori error estimate (see [18] for the early studies).

Theorem 4 *Let $\boldsymbol{u}$ be the solution to problem (5) and let $\boldsymbol{u}_h$ be the solution to discrete problem (7). If we assume that $\boldsymbol{u} \in (H^2(\Omega))^2$, then we have*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} \lesssim h\|\boldsymbol{u}\|_{2,\Omega}.$$

PROOF. Let $\boldsymbol{v}_h \in \mathbf{V}_h$. From the $\mathbf{V}$-ellipticity of $a(\cdot,\cdot)$ and the equations in (5) and (7) we obtain:

$$\begin{aligned}\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega}^2 &\lesssim a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{u}_h)\\ &= a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{v}_h) + a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h)\\ &= a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{v}_h) + j_n(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) - j_n(\boldsymbol{u}, \boldsymbol{v}_h - \boldsymbol{u}_h)\\ &= a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{v}_h) + j_n(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}) - j_n(\boldsymbol{u}, \boldsymbol{v}_h - \boldsymbol{u})\\ &\quad + j_n(\boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{u}_h) - j_n(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}_h).\end{aligned} \tag{11}$$

Using a standard monotonicity argument, we get

$$\begin{aligned}&j_n(\boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{u}_h) - j_n(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}_h)\\ &\quad = -\int_{\Gamma_C} c_n((u_n)_+^{m_n} - (u_{hn})_+^{m_n})(u_n - u_{hn})\, \mathrm{d}\gamma(\boldsymbol{x}) \le 0.\end{aligned} \tag{12}$$

From the inequality

$$\begin{aligned}|(a)_+^m - (b)_+^m| &\le m|(a)_+ - (b)_+|\left((a)_+^{m-1} + (b)_+^{m-1}\right)\\ &\le m|a - b|\left(|a|^{m-1} + |b|^{m-1}\right) \qquad a, b \in \mathbb{R}, m \ge 1,\end{aligned} \tag{13}$$

we deduce for each $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v} \in (H^1(\Omega))^2$

$$\begin{aligned}&|j_n(\boldsymbol{u}_1, \boldsymbol{v}) - j_n(\boldsymbol{u}_2, \boldsymbol{v})|\\ &\lesssim \int_{\Gamma_C} |u_{1n} - u_{2n}|\left(|u_{1n}|^{m_n-1} + |u_{2n}|^{m_n-1}\right)|v_n|\, \mathrm{d}\gamma(\boldsymbol{x})\\ &\le \|u_{1n} - u_{2n}\|_{L^r(\Gamma_C)}\left(\|u_{1n}\|_{L^{q(m_n-1)}(\Gamma_C)}^{m_n-1} + \|u_{2n}\|_{L^{q(m_n-1)}(\Gamma_C)}^{m_n-1}\right)\|v_n\|_{L^r(\Gamma_C)}\\ &\lesssim \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{1,\Omega}\left(\|\boldsymbol{u}_1\|_{1,\Omega}^{m_n-1} + \|\boldsymbol{u}_2\|_{1,\Omega}^{m_n-1}\right)\|\boldsymbol{v}\|_{1,\Omega}.\end{aligned} \tag{14}$$

In the previous bounds we use Hölder inequalities with $1/r + 1/q + 1/r = 1$ and $q(m_n - 1) \ge 1$ together with embedding (6). Note that the case $m_n = 1$ is straightforward. Combining (11), (12) and (14), we get

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}^2 &\lesssim \|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}\|\boldsymbol{u}-\boldsymbol{v}_h\|_{1,\Omega} \\ &\quad +\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}\left(\|\boldsymbol{u}\|_{1,\Omega}^{m_n-1}+\|\boldsymbol{u}_h\|_{1,\Omega}^{m_n-1}\right)\|\boldsymbol{u}-\boldsymbol{v}_h\|_{1,\Omega} \\ &\lesssim \|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}\|\boldsymbol{u}-\boldsymbol{v}_h\|_{1,\Omega}.\end{aligned}$$

In the last expression we used the straightforward property resulting from (5) and (7) that $\|\boldsymbol{u}\|_{1,\Omega}$ and $\|\boldsymbol{u}_h\|_{1,\Omega}$ are bounded by constants depending on the loads.

Therefore, we conclude the proof of the theorem by choosing $\boldsymbol{v}_h = I_h\boldsymbol{u}$, where I_h stands for the Lagrange interpolation operator mapping onto $\mathbf{V}_h$.

4 A residual a posteriori error estimator

4.1 Definition of the residual error estimator

The element residual of the equilibrium equation (1) is defined by

$$\mathbf{div}\boldsymbol{\sigma}(\boldsymbol{u}_h) + \boldsymbol{f} = \boldsymbol{f} \text{ on } K.$$

As usual this element residual can be replaced by some finite dimensional approximation, called approximate element residual (see, e.g., [2])

$$\boldsymbol{f}_K \in (\mathbb{P}_k(K))^2.$$

A current choice is to take $\boldsymbol{f}_K = \int_K \boldsymbol{f}(\boldsymbol{x})/|K|\,d\boldsymbol{x}$ since for $\boldsymbol{f} \in (H^1(\Omega))^2$, scaling arguments yield $\|\boldsymbol{f}-\boldsymbol{f}_K\|_K \lesssim h_K\|\boldsymbol{f}\|_{1,K}$ and it is then negligible with respect to the estimator η defined hereafter. In the same way $\boldsymbol{g}$ is approximated by a computable quantity denoted $\boldsymbol{g}_E$ on any $E \in E_h^N$.

Definition 5 *The local error estimators η_K and the the global estimator η are defined by*

$$\begin{aligned}\eta_K &= \left(\sum_{i=1}^{4}\eta_{iK}^2\right)^{1/2}, \\ \eta_{1K} &= h_K\|\boldsymbol{f}_K\|_K, \\ \eta_{2K} &= h_K^{1/2}\left(\sum_{E\in E_K^{int}\cup E_K^N}\|J_{E,n}(\boldsymbol{u}_h)\|_E^2\right)^{1/2},\end{aligned}$$

$$\eta_{3K} = h_K^{1/2} \left(\sum_{E \in E_K^C} \|\sigma_t(\boldsymbol{u}_h)\|_E^2 \right)^{1/2},$$

$$\eta_{4K} = h_K^{1/2} \left(\sum_{E \in E_K^C} \|c_n(u_{hn})_+^{m_n} + \sigma_n(\boldsymbol{u}_h)\|_E^2 \right)^{1/2},$$

$$\eta = \left(\sum_{K \in T_h} \eta_K^2 \right)^{1/2},$$

where $J_{E,n}(\boldsymbol{u}_h)$ *means the constraint jump of* $\boldsymbol{u}_h$ *in the normal direction, i.e.,*

$$J_{E,n}(\boldsymbol{u}_h) = \begin{cases} \left[\!\left[\boldsymbol{\sigma}(\boldsymbol{u}_h)\boldsymbol{\nu}_E\right]\!\right]_E, & \forall E \in E_h^{int}, \\ \boldsymbol{\sigma}(\boldsymbol{u}_h)\boldsymbol{\nu}_E - \boldsymbol{g}_E, & \forall E \in E_h^N. \end{cases} \tag{15}$$

The local and global approximation terms are given by

$$\zeta_K = \left(h_K^2 \sum_{K' \subset \omega_K} \|\boldsymbol{f} - \boldsymbol{f}_{K'}\|_{K'}^2 + h_E \sum_{E \subset E_K^N} \|\boldsymbol{g} - \boldsymbol{g}_E\|_E^2 \right)^{1/2},$$

$$\zeta = \left(\sum_{K \in T_h} \zeta_K^2 \right)^{1/2}.$$

4.2 Upper error bound

We first study the upper error bound of the discretization error.

Theorem 6 *Let* $\boldsymbol{u}$ *be the solution to nonlinear variational equation (5) and let* $\boldsymbol{u}_h$ *be the solution to the corresponding discrete problem (7). Then we have*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} \lesssim \eta + \zeta.$$

PROOF. Afterwards we adopt the following notation for the displacement error term:

$$\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{u}_h.$$

Let $\boldsymbol{v}_h \in \mathbf{V}_h$. From the $\mathbf{V}$-ellipticity of $a(.,.)$ and the equations in (5) and (7) we obtain:

$$
\begin{aligned}
\|\boldsymbol{e}\|_{1,\Omega}^2 &\lesssim a(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{u}-\boldsymbol{u}_h) \\
&= a(\boldsymbol{u}, \boldsymbol{u}-\boldsymbol{u}_h) - a(\boldsymbol{u}_h, \boldsymbol{u}-\boldsymbol{u}_h) \\
&= L(\boldsymbol{u}-\boldsymbol{u}_h) - j_n(\boldsymbol{u}, \boldsymbol{u}-\boldsymbol{u}_h) - a(\boldsymbol{u}_h, \boldsymbol{u}-\boldsymbol{u}_h) \\
&= L(\boldsymbol{u}-\boldsymbol{v}_h) - j_n(\boldsymbol{u}, \boldsymbol{u}-\boldsymbol{u}_h) - a(\boldsymbol{u}_h, \boldsymbol{u}-\boldsymbol{v}_h) \\
&\qquad + j_n(\boldsymbol{u}_h, \boldsymbol{v}_h-\boldsymbol{u}) + j_n(\boldsymbol{u}_h, \boldsymbol{u}-\boldsymbol{u}_h) \\
&\le L(\boldsymbol{u}-\boldsymbol{v}_h) + j_n(\boldsymbol{u}_h, \boldsymbol{v}_h-\boldsymbol{u}) - a(\boldsymbol{u}_h, \boldsymbol{u}-\boldsymbol{v}_h)
\end{aligned}
$$

where we use the monotonicity argument (12).

Integrating by parts on each triangle K and using the definition of $J_{E,n}(\boldsymbol{u}_h)$ in (15) yields:

$$
\begin{aligned}
\|\boldsymbol{e}\|_{1,\Omega}^2 \lesssim & \sum_{K\in T_h} \int_K \boldsymbol{f}\cdot(\boldsymbol{u}-\boldsymbol{v}_h)\,\mathrm{d}\boldsymbol{x} + \sum_{E\in E_h^C} \int_E c_n (u_{hn})_+^{m_n} (v_{hn}-u_n)\,\mathrm{d}\gamma(\boldsymbol{x}) \\
& - \sum_{E\in E_h^C} \int_E (\boldsymbol{\sigma}(\boldsymbol{u}_h)\boldsymbol{\nu})\cdot(\boldsymbol{u}-\boldsymbol{v}_h)\,\mathrm{d}\gamma(\boldsymbol{x}) \\
& - \sum_{E\in E_h^{int}\cup E_h^N} \int_E J_{E,n}(\boldsymbol{u}_h)\cdot(\boldsymbol{u}-\boldsymbol{v}_h)\,\mathrm{d}\gamma(\boldsymbol{x}) \\
& + \sum_{E\in E_h^N} \int_E (\boldsymbol{g}-\boldsymbol{g}_E)\cdot(\boldsymbol{u}-\boldsymbol{v}_h)\,\mathrm{d}\gamma(\boldsymbol{x}).
\end{aligned}
$$

Splitting up the integrals on Γ_C into normal and tangential components gives:

$$
\begin{aligned}
\|\boldsymbol{e}\|_{1,\Omega}^2 \lesssim & \sum_{K\in T_h} \int_K \boldsymbol{f}\cdot(\boldsymbol{u}-\boldsymbol{v}_h)\,\mathrm{d}\boldsymbol{x} \\
& + \sum_{E\in E_h^C} \int_E (c_n (u_{hn})_+^{m_n} + \sigma_n(\boldsymbol{u}_h))(v_{hn}-u_n)\,\mathrm{d}\gamma(\boldsymbol{x}) \\
& + \sum_{E\in E_h^C} \int_E \sigma_t(\boldsymbol{u}_h)(v_{ht}-u_t)\,\mathrm{d}\gamma(\boldsymbol{x}) \\
& - \sum_{E\in E_h^{int}\cup E_h^N} \int_E J_{E,n}(\boldsymbol{u}_h)\cdot(\boldsymbol{u}-\boldsymbol{v}_h)\,\mathrm{d}\gamma(\boldsymbol{x}) \\
& + \sum_{E\in E_h^N} \int_E (\boldsymbol{g}-\boldsymbol{g}_E)\cdot(\boldsymbol{u}-\boldsymbol{v}_h)\,\mathrm{d}\gamma(\boldsymbol{x}). \qquad (16)
\end{aligned}
$$

We now need to estimate each term of this right-hand side. For that purpose, we take

$$
\boldsymbol{v}_h = \boldsymbol{u}_h + \pi_h(\boldsymbol{u}-\boldsymbol{u}_h), \qquad (17)
$$

where π_h is the quasi-interpolation operator defined in Lemma 3.

We start with the integral term. Cauchy-Schwarz's inequality implies

$$\sum_{K\in T_h}\int_K \boldsymbol{f}\cdot(\boldsymbol{u}-\boldsymbol{v}_h)\,\mathrm{d}\boldsymbol{x} \le \sum_{K\in T_h}\|\boldsymbol{f}\|_K\|\boldsymbol{u}-\boldsymbol{v}_h\|_K,$$

and it suffices to estimate $\|\boldsymbol{u}-\boldsymbol{v}_h\|_K$ for any triangle K. From the definition of $\boldsymbol{v}_h$ and (9) we get:

$$\|\boldsymbol{u}-\boldsymbol{v}_h\|_K = \|\boldsymbol{e}-\pi_h\boldsymbol{e}\|_K \lesssim h_K\|\boldsymbol{e}\|_{1,\omega_K}.$$

As a consequence

$$\left|\int_\Omega \boldsymbol{f}\cdot(\boldsymbol{u}-\boldsymbol{v}_h)\,\mathrm{d}\boldsymbol{x}\right| \lesssim (\eta+\zeta)\|\boldsymbol{e}\|_{1,\Omega}.$$

We now consider the interior and Neumann boundary terms in (16). As we previously noticed, the application of Cauchy-Schwarz's inequality leads to

$$\left|\sum_{E\in E_h^{int}\cup E_h^N}\int_E J_{E,n}(\boldsymbol{u}_h)\cdot(\boldsymbol{u}-\boldsymbol{v}_h)\,\mathrm{d}\gamma(\boldsymbol{x})\right| \le \sum_{E\in E_h^{int}\cup E_h^N}\|J_{E,n}(\boldsymbol{u}_h)\|_E\|\boldsymbol{u}-\boldsymbol{v}_h\|_E.$$

Therefore using expression (17) and estimate (10), we obtain

$$\|\boldsymbol{u}-\boldsymbol{v}_h\|_E = \|\boldsymbol{e}-\pi_h\boldsymbol{e}\|_E \lesssim h_E^{1/2}\|\boldsymbol{e}\|_{1,\omega_E}.$$

Inserting this estimate in the previous one we deduce that

$$\left|\sum_{E\in E_h^{int}\cup E_h^N}\int_E J_{E,n}(\boldsymbol{u}_h)\cdot(\boldsymbol{u}-\boldsymbol{v}_h)\,\mathrm{d}\gamma(\boldsymbol{x})\right| \lesssim \eta\|\boldsymbol{e}\|_{1,\Omega}.$$

Moreover,

$$\left|\sum_{E\in E_h^N}\int_E(\boldsymbol{g}-\boldsymbol{g}_E)\cdot(\boldsymbol{u}-\boldsymbol{v}_h)\,\mathrm{d}\gamma(\boldsymbol{x})\right| \lesssim \zeta\|\boldsymbol{e}\|_{1,\Omega}.$$

The two following terms are handled in a similar way as the previous ones so that

$$\left|\sum_{E\in E_h^C}\int_E(c_n(u_{hn})_+^{m_n}+\sigma_n(\boldsymbol{u}_h))(v_{hn}-u_n)\,\mathrm{d}\gamma(\boldsymbol{x})\right| \lesssim \eta\|\boldsymbol{e}\|_{1,\Omega},$$

and

$$\left|\sum_{E\in E_h^C}\int_E\sigma_t(\boldsymbol{u}_h)(v_{ht}-u_t)\,\mathrm{d}\gamma(\boldsymbol{x})\right| \lesssim \eta\|\boldsymbol{e}\|_{1,\Omega}.$$

Putting together the previous estimates we come to the conclusion that

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega} \lesssim \eta+\zeta.$$

4.3 Lower error bound

We now consider the local lower error bounds of the discretization error terms.

Theorem 7 *For all elements $K \in T_h$, the following local lower error bounds hold:*

$$\eta_{1K} \lesssim \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,K} + \zeta_K, \tag{18}$$
$$\eta_{2K} \lesssim \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\omega_K} + \zeta_K. \tag{19}$$

For all elements K such that $K \cap E_h^C \neq \emptyset$, the following local lower error bounds (with $p > 2$) hold:

$$\eta_{3K} \lesssim \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,K} + \zeta_K, \tag{20}$$
$$\eta_{4K} \lesssim \sum_{E \in E_K^C} C(p) h_K^{1/2} \|u_n - u_{hn}\|_{L^p(E)} + h_K^{1/2} \|\sigma_n(\boldsymbol{u} - \boldsymbol{u}_h)\|_E. \tag{21}$$

PROOF. The estimates of η_{1K} and η_{2K} in (18) and (19) are standard (see, e.g., [23]).

We now estimate η_{3K}. Writing $\boldsymbol{w}_E = w_{En}\boldsymbol{\nu} + w_{Et}\boldsymbol{t}$ on $E \in E_K^C$ and denoting by b_E the edge bubble function associated with E (i.e., $b_E = 4\lambda_{a_1}\lambda_{a_2}$, when a_1, a_2 are the two extremities of E; we recall that λ_x is the standard basis function at node x in V_h satisfying $\lambda_x(x') = \delta_{x,x'}$ for any node x', see (8)), we choose $w_{En} = 0$ and $w_{Et} = \sigma_t(\boldsymbol{u}_h) b_E$ in the element K containing E (here we made a slight abuse of notation to simplify the writing) and $\boldsymbol{w}_E = \boldsymbol{0}$ in $\overline{\Omega} \setminus K$. Therefore,

$$\begin{aligned}
\|\sigma_t(\boldsymbol{u}_h)\|_E^2 &\sim \int_E \sigma_t(\boldsymbol{u}_h) w_{Et} \, \mathrm{d}\gamma(\boldsymbol{x}) \\
&= \int_K \boldsymbol{\sigma}(\boldsymbol{u}_h) : \boldsymbol{\varepsilon}(\boldsymbol{w}_E) \, \mathrm{d}\boldsymbol{x} \\
&= \int_K \boldsymbol{\sigma}(\boldsymbol{u}_h - \boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{w}_E) \, \mathrm{d}\boldsymbol{x} + \int_K \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{w}_E) \, \mathrm{d}\boldsymbol{x} \\
&= L(\boldsymbol{w}_E) + \int_K \boldsymbol{\sigma}(\boldsymbol{u}_h - \boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{w}_E) \, \mathrm{d}\boldsymbol{x} \\
&\lesssim \|\boldsymbol{f}\|_K \|\boldsymbol{w}_E\|_K + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,K} \|\boldsymbol{w}_E\|_{1,K} \\
&\lesssim \|\boldsymbol{f}\|_K \|\boldsymbol{w}_E\|_K + h_K^{-1} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,K} \|\boldsymbol{w}_E\|_K,
\end{aligned}$$

where we use an inverse inequality in the last bound. Another inverse inequality and estimate (18) imply

$$h_K^{1/2}\|\sigma_t(\boldsymbol{u}_h)\|_E \lesssim \|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,K} + h_K\|\boldsymbol{f}\|_K$$
$$\lesssim \|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,K} + \zeta_K.$$

This estimate gives the estimate of η_{3K} in (20). The bound of η_{4K} in (21) can not be obtained as previously by choosing $w_{En} = (c_n(u_{hn})_+^{m_n} + \sigma_n(\boldsymbol{u}_h))b_E$ and $w_{Et} = 0$ since we have in general (due to the positive part)

$$\|c_n(u_{hn})_+^{m_n} + \sigma_n(\boldsymbol{u}_h)\|_E^2 \not\lesssim \int_E (c_n(u_{hn})_+^{m_n} + \sigma_n(\boldsymbol{u}_h))w_{En}\, \mathrm{d}\gamma(\boldsymbol{x}).$$

So we use the identity $c_n(u_n)_+^{m_n} + \sigma_n(\boldsymbol{u}) = 0$ and, keeping in mind that $\sigma_n(\boldsymbol{u}) \in L^2(\Gamma_C)$, we write

$$\begin{aligned}
&\|c_n(u_{hn})_+^{m_n} + \sigma_n(\boldsymbol{u}_h)\|_E \\
&\quad\le \|c_n((u_{hn})_+^{m_n} - (u_n)_+^{m_n})\|_E + \|\sigma_n(\boldsymbol{u}-\boldsymbol{u}_h)\|_E \\
&\quad\le \|c_n\|_{L^\infty(E)}\|((u_{hn})_+^{m_n} - (u_n)_+^{m_n})\|_E + \|\sigma_n(\boldsymbol{u}-\boldsymbol{u}_h)\|_E \\
&\quad\le m_n\|c_n\|_{L^\infty(E)}\|(u_{hn}-u_n)(|u_{hn}|^{m_n-1} + |u_n|^{m_n-1})\|_E + \|\sigma_n(\boldsymbol{u}-\boldsymbol{u}_h)\|_E \\
&\quad\le m_n\|c_n\|_{L^\infty(E)}\|u_{hn}-u_n\|_{L^p(E)}\left(\|u_{hn}\|^{m_n-1}_{L^{q(m_n-1)}(E)} + \|u_n\|^{m_n-1}_{L^{q(m_n-1)}(E)}\right) \\
&\qquad\quad + \|\sigma_n(\boldsymbol{u}-\boldsymbol{u}_h)\|_E \\
&\quad\le C(p)m_n\|c_n\|_{L^\infty(E)}\|u_{hn}-u_n\|_{L^p(E)}\left(\|\boldsymbol{u}_h\|^{m_n-1}_{1,\Omega} + \|\boldsymbol{u}\|^{m_n-1}_{1,\Omega}\right) \\
&\qquad\quad + \|\sigma_n(\boldsymbol{u}-\boldsymbol{u}_h)\|_E \\
&\quad\lesssim \|u_n-u_{hn}\|_{L^p(E)} + \|\sigma_n(\boldsymbol{u}-\boldsymbol{u}_h)\|_E,
\end{aligned} \tag{22}$$

where $1/p+1/q = 1/2$, $q(m_n-1) \ge 1$ (the case $m_n = 1$ is straightforward) and we have used the identity (13) and the boundedness of $\|\boldsymbol{u}\|_{1,\Omega}$ and $\|\boldsymbol{u}_h\|_{1,\Omega}$.

Corollary 8 *Assume that $\boldsymbol{u} \in (H^2(\Omega))^2$ and define:*

$$\eta_i = \left(\sum_{K\in T_h} \eta_{iK}^2\right)^{1/2}, \quad 1 \le i \le 4.$$

Then we have $\eta_i \lesssim h$ for $1 \le i \le 4$. So

$$\eta \lesssim h.$$

PROOF. From Theorem 7, we deduce by addition $\eta_1 \lesssim \|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega} + \zeta$ and the latter quantity is bounded by $h\|\boldsymbol{u}\|_{2,\Omega}$ according to Theorem 4. The bounds for η_2 and η_3 are the same. We now consider η_4:

$$\begin{aligned}
\eta_4^2 &\lesssim \sum_{E\in E_h^C} h_E(\|u_n-u_{hn}\|_{L^p(E)} + \|\sigma_n(\boldsymbol{u}-\boldsymbol{u}_h)\|_E)^2 \\
&\lesssim \sum_{E\in E_h^C} h_E\|u_n-u_{hn}\|^2_{L^p(E)} + \sum_{E\in E_h^C} h_E\|\sigma_n(\boldsymbol{u}-\boldsymbol{u}_h)\|_E^2
\end{aligned} \tag{23}$$

where $p > 2$. The first term is roughly bounded as follows by using (6)

$$h_E\|u_n - u_{hn}\|^2_{L^p(E)} \leq h_E\|u_n - u_{hn}\|^2_{L^p(\Gamma_C)} \lesssim h_E\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{1,\Omega}. \tag{24}$$

By using the identity

$$\sum_{E\in E_h^C} h_E = |\Gamma_C| \tag{25}$$

and Theorem 4 we conclude that

$$\sum_{E\in E_h^C} h_E\|u_n - u_{hn}\|^2_{L^p(E)} \lesssim h^2\|\boldsymbol{u}\|^2_{2,\Omega}. \tag{26}$$

The second term is bounded by using the scaled trace inequality

$$\|v\|_E \lesssim h_E^{-1/2}\|v\|_{\omega_E} + h_E^{1/2}\|\nabla v\|_{\omega_E}, \quad \forall E \in E_h, \forall v \in H^1(\omega_E).$$

Hence, supposing without loss of generality that Γ_C is a straight line segment parallel to the $x-$axis and using the latter inequality, we obtain:

$$\begin{aligned} h_E\|\sigma_n(\boldsymbol{u} - \boldsymbol{u}_h)\|^2_E &= h_E\|\sigma_{yy}(\boldsymbol{u} - \boldsymbol{u}_h)\|^2_E \\ &\lesssim \|\sigma_{yy}(\boldsymbol{u} - \boldsymbol{u}_h)\|^2_{\omega_E} + h_E^2\|\nabla\sigma_{yy}(\boldsymbol{u} - \boldsymbol{u}_h)\|^2_{\omega_E} \\ &\leq \|\boldsymbol{\sigma}(\boldsymbol{u} - \boldsymbol{u}_h)\|^2_{\omega_E} + h_E^2\|\nabla\boldsymbol{\sigma}(\boldsymbol{u})\|^2_{\omega_E}. \end{aligned}$$

By summation, we find that

$$\sum_{E\in E_h^C} h_E\|\sigma_n(\boldsymbol{u} - \boldsymbol{u}_h)\|^2_E \lesssim \|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{1,\Omega} + h^2\|\boldsymbol{u}\|^2_{2,\Omega} \lesssim h^2\|\boldsymbol{u}\|^2_{2,\Omega}. \tag{27}$$

The corollary is proved by putting together (23), (26) and (27).

Remark 9 *We redefine the discretization error $e = \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega}$ as follows:*

$$\tilde{e} = \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} + \left(\sum_{E\in E_h^C} h_E\|\sigma_n(\boldsymbol{u} - \boldsymbol{u}_h)\|^2_E\right)^{1/2}.$$

Such an approach with a modified error has already been used successfully for the obstacle problem in [22]. Note that in our reference the additional norm term is mesh dependent contrary to [22]. With this new definition, we can prove its equivalence with the error estimator η:

$$\tilde{e} \lesssim \eta + \zeta, \tag{28}$$

$$\eta \lesssim \tilde{e} + \zeta. \tag{29}$$

We first prove the upper bound (28). Let $E \in E_h^C$. Using the estimate $(a+b)^2 \leq 2(a^2+b^2)$, it gives:

$$h_E\|\sigma_n(\boldsymbol{u}-\boldsymbol{u}_h)\|_E^2 \lesssim h_E\|-c_n(u_n)_+^{m_n}+c_n(u_{hn})_+^{m_n}\|_E^2 \\ +h_E\|c_n(u_{hn})_+^{m_n}+\sigma_n(\boldsymbol{u}_h)\|_E^2. \tag{30}$$

In (22) we prove

$$\|c_n((u_{hn})_+^{m_n}-(u_n)_+^{m_n})\|_E \lesssim \|u_n-u_{hn}\|_{L^p(E)},$$

for any $E \in E_h^C$ $(p>2)$. As a consequence we deduce from (6) that

$$\|c_n((u_{hn})_+^{m_n}-(u_n)_+^{m_n})\|_E \lesssim \|u_n-u_{hn}\|_{L^p(\Gamma_C)} \lesssim \|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}. \tag{31}$$

Denoting by K the element containing E, using (30) and (31) and the definition of η_{4K} we deduce

$$h_E\|\sigma_n(\boldsymbol{u}-\boldsymbol{u}_h)\|_E^2 \lesssim h_E\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}^2+\eta_{4K}^2.$$

Therefore,

$$\left(\sum_{E\in E_h^C} h_E\|\sigma_n(\boldsymbol{u}-\boldsymbol{u}_h)\|_E^2\right)^{1/2} \lesssim \|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}+\eta,$$

which together with Theorem 6 yields (28).

Finally, the lower bound (29) is a straightforward consequence of the proof of Corollary 8 since $\eta_i \lesssim \|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}+\zeta$, $1 \leq i \leq 3$ and $\eta_4 \lesssim \tilde{e}$ according to (23), (24) and (25).

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