A sign preserving mixed finite element approximation for contact problems

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Abstract: This paper is concerned with the frictionless unilateral contact problem (i.e., a Signorini problem with the elasticity operator). We consider a mixed finite element method in which the unknowns are the displacement field and the contact pressure. The particularity of the method is that it furnishes a normal displacement field and a contact pressure verifying the sign conditions of the continuous problem. The a priori error analysis of the method is closely linked with the study of a specific positivity preserving operator of averaging type which differs from the one of Chen and Nochetto. We show that this method is convergent and satisfies the same a priori error estimates as the standard approach in which the approximated contact pressure satisfies only a weak sign condition. Finally we perform some computations to illustrate and compare the sign preserving method with the standard approach.

Key words: variational inequality, positive operator, averaging operator, contact problem, Signorini problem, mixed finite element method.

Abbreviated title: Positivity preserving operator and mixed method for contact problems

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1 Introduction

Finite element methods are efficient and widespread tools in computational contact and impact mechanics (see [17, 18, 28, 29, 33]) and the mixed formulations involving a displacement field $\mathbf{u}$ in the bodies and the contact pressure $\sigma_n(\mathbf{u})$ on the contact zone are commonly used. A particularity of the contact problem lies in the so-called unilateral conditions linking, on the contact zone Γ_C, the normal displacement field u_n and the Lagrange multiplier $\lambda = -\sigma_n(\mathbf{u})$:

$$u_n \leq 0, \qquad \lambda \geq 0, \qquad \lambda\, u_n = 0 \qquad \text{on } \Gamma_C.$$

The mixed finite element method we consider, introduced in [22], furnishes an approximated normal displacement field u_{hn} and an approximated multiplier λ_h which satisfy

$$\begin{aligned} u_{hn} \leq 0, \qquad \lambda_h \geq 0 \qquad &\text{on } \Gamma_C, \\ \lambda_h\, u_{hn} = 0 \qquad &\text{at the nodes of } \Gamma_C. \end{aligned}$$

Such a method shows three interesting aspects in comparison with the standard approach in which the multiplier is only nonnegative in a weak sense (see, e.g., [3, 11, 25]):

- the nonnegative multiplier is more relevant from a mechanical point of view,
- this multiplier vanishes where the body separates (the multiplier of the standard approach may show some artificial oscillations on the separation zone),
- it allows to define a simple a posteriori error estimator whose numerical analysis gives better bounds than for the error estimator arising from the standard approach (see [22]).

Let us mention that there exist other mixed formulations leading to a priori error estimates with nonnegative multipliers and normal displacement fields which do not satisfy the nonpositivity condition (see [4, 3, 18]).

The paper is organized as follows. In Section 2 we introduce the equations modelling the frictionless unilateral contact problem between an elastic body and a rigid foundation. We write the problem using a formulation where the unknowns are the displacement field in the body and the pressure on the contact area. In the third section, we choose a discretization involving continuous finite elements of degree one for the displacements and continuous piecewise affine multipliers on the contact zone. The main particularity of this approach is that both the normal displacement and the multiplier solution to the discrete problem satisfy the same sign conditions as the normal displacement and the multiplier solving the continuous problem. More precisely the displacement field of the sign preserving method coincides with the one in the standard approach and the multipliers are linked by a linear operator which transforms the functions satisfying some "weak" nonnegativity conditions into nonnegative functions. In Section 4, we study and discuss the main basic properties of the positivity preserving averaging operator which requires minimal regularity. Section 5 is concerned with the a priori error analysis of the sign preserving method. We prove that the method is convergent when using convenient regularity assumptions on the solution to the continuous problem and we obtain in Theorem 5.5 and Corollary 5.8 similar a priori error estimates as for the standard approach. In section 6 we implement both methods and we compare them on several examples. As expected the sign preserving method furnishes more relevant multipliers and no loss of convergence is observed in comparison with the standard approach. Finally we mention that the results in this paper obviously hold for the simpler Signorini problem with the Laplace operator.

As usual, we denote by $(H^s(.))^d$, $s \in \mathbb{R}, d = 1, 2$ the Sobolev spaces in one and two space dimensions (see [1]). The usual norm of $(H^s(D))^d$ (dual norm if $s < 0$) is denoted by $\| \cdot \|_{s,D}$ and we keep the same notation when $d = 1$ or $d = 2$.

2 The unilateral contact problem in linear elasticity

We consider an elastic body Ω in $\mathbb{R}^2$ where plane strain assumptions are made. The boundary $\partial\Omega$ of Ω is polygonal and we suppose that $\partial\Omega$ consists in three nonoverlapping parts Γ_D, Γ_N and Γ_C with $\text{meas}(\Gamma_D) > 0$ and $\text{meas}(\Gamma_C) > 0$. The normal unit outward vector on $\partial\Omega$ is denoted $\mathbf{n} = (n_1, n_2)$ and we choose as unit tangential vector $\mathbf{t} = (-n_2, n_1)$. In its initial stage, the body is in contact on Γ_C which is supposed to be a straight line segment and we suppose that the unknown final contact zone after deformation will be included in Γ_C. The body is clamped on Γ_D for the sake of simplicity. It is subjected to volume forces $\mathbf{f} = (f_1, f_2) \in (L^2(\Omega))^2$ and to surface loads $\mathbf{g} = (g_1, g_2) \in (L^2(\Gamma_N))^2$.

The unilateral contact problem in linear elasticity consists in finding the displacement field $\mathbf{u} : \Omega \to \mathbb{R}^2$ verifying the equations and conditions (1)–(6):

$$\mathbf{div}\, \boldsymbol{\sigma}(\mathbf{u}) + \mathbf{f} = \mathbf{0} \qquad \text{in } \Omega, \tag{1}$$

where $\boldsymbol{\sigma} = (\sigma_{ij})$, $1 \le i, j \le 2$, stands for the stress tensor field and **div** denotes the divergence operator of tensor valued functions. The stress tensor field is obtained from the displacement field by the constitutive law of linear elasticity:

$$\boldsymbol{\sigma}(\mathbf{u}) = \mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u}) \qquad \text{in } \Omega, \tag{2}$$

where $\mathbf{A}$ is a fourth order symmetric and elliptic tensor whose coefficients lie in $C^1(\overline{\Omega})$ and $\boldsymbol{\varepsilon}(\mathbf{v}) = (\nabla\mathbf{v} + {}^t\nabla\mathbf{v})/2$ represents the linearized strain tensor field. On Γ_D and Γ_N, the conditions are as follows:

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \Gamma_D, \tag{3}$$

$$\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} = \mathbf{g} \qquad \text{on } \Gamma_N. \tag{4}$$

For any displacement field $\mathbf{v}$ and for any density of surface forces $\boldsymbol{\sigma}(\mathbf{v})\mathbf{n}$ defined on $\partial\Omega$ we adopt the following notation

$$\mathbf{v} = v_n\mathbf{n} + v_t\mathbf{t} \quad \text{and} \quad \boldsymbol{\sigma}(\mathbf{v})\mathbf{n} = \sigma_n(\mathbf{v})\mathbf{n} + \sigma_t(\mathbf{v})\mathbf{t}.$$

The three conditions describing unilateral contact on Γ_C are (see, e.g., [12, 13, 14, 15]):

$$u_n \le 0, \qquad \sigma_n(\mathbf{u}) \le 0, \qquad \sigma_n(\mathbf{u})\, u_n = 0. \tag{5}$$

Finally the equality

$$\sigma_t(\mathbf{u}) = 0 \tag{6}$$

on Γ_C means that friction is omitted.

The mixed variational formulation of (1)–(6) uses the Hilbert space

$$\mathbf{V} = \left\{ \mathbf{v} \in \left(H^1(\Omega)\right)^2 : \quad \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D \right\}.$$

The Lagrange multiplier space M is the dual of the normal trace space N of $\mathbf{V}$ restricted to Γ_C. If the end points of Γ_C belong to $\overline{\Gamma_N}$ (resp. $\overline{\Gamma_D}$) then $N = H^{\frac{1}{2}}(\Gamma_C)$ (resp. $H^{\frac{1}{2}}_{00}(\Gamma_C)$). We next define the following convex cone of multipliers on Γ_C:

$$M^+ = \Big\{\mu \in M : \ \ \langle \mu, \psi \rangle_{\Gamma_C} \geq 0 \ \ \text{for all } \psi \in N, \psi \geq 0 \ \text{ a.e. on } \Gamma_C \Big\},$$

where the notation $\langle ., . \rangle_{\Gamma_C}$ represents the duality pairing between M and N. Define

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \int_\Omega \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \, d\Omega, \qquad b(\mu, \mathbf{v}) = \langle \mu, v_n \rangle_{\Gamma_C}, \\ L(\mathbf{v}) &= \int_\Omega \mathbf{f} \cdot \mathbf{v} \, d\Omega + \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{v} \, d\Gamma, \end{aligned}$$

for any $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{V}$ and μ in M.

The mixed formulation of the unilateral contact problem without friction (1)–(6) consists then in finding $\mathbf{u} \in \mathbf{V}$ and $\lambda \in M^+$ such that

$$\text{(7)} \qquad \begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\lambda, \mathbf{v}) = L(\mathbf{v}), & \forall \, \mathbf{v} \in \mathbf{V}, \\ b(\mu - \lambda, \mathbf{u}) \leq 0, & \forall \, \mu \in M^+. \end{cases}$$

An equivalent formulation of (7) consists in finding $(\lambda, \mathbf{u}) \in M^+ \times \mathbf{V}$ satisfying

$$\mathcal{L}(\mu, \mathbf{u}) \leq \mathcal{L}(\lambda, \mathbf{u}) \leq \mathcal{L}(\lambda, \mathbf{v}), \qquad \forall \mathbf{v} \in \mathbf{V}, \ \forall \mu \in M^+,$$

where $\mathcal{L}(\mu, \mathbf{v}) = \frac{1}{2} a(\mathbf{v}, \mathbf{v}) - L(\mathbf{v}) + b(\mu, \mathbf{v})$. Another classical weak formulation of problem (1)–(6) is a variational inequality: find $\mathbf{u}$ such that

$$\text{(8)} \qquad \mathbf{u} \in \mathbf{K}, \qquad a(\mathbf{u}, \mathbf{v} - \mathbf{u}) \geq L(\mathbf{v} - \mathbf{u}), \qquad \forall \mathbf{v} \in \mathbf{K},$$

where $\mathbf{K}$ denotes the closed convex cone of admissible displacement fields satisfying the non-penetration conditions:

$$\mathbf{K} = \big\{ \mathbf{v} \in \mathbf{V} : \quad v_n \leq 0 \text{ on } \Gamma_C \big\}.$$

The existence and uniqueness of solution $(\lambda, \mathbf{u})$ to (7) has been stated in [18]. Moreover, the first argument $\mathbf{u}$ solution to (7) is also the unique solution of problem (8) and $\lambda = -\sigma_n(\mathbf{u})$.

3 Finite element approximation

A regular family of triangulations denoted T_h is associated with the body Ω (see [7, 9]). The closed triangles $K \in T_h$ are of diameter h_K and we set $h = \max_{K \in T_h} h_K$. In order to use inverse inequalities on the contact area, we suppose that the monodimensional mesh inherited on Γ_C is uniformly regular and we denote by h_C a parameter representing the size of the elements on the contact zone (if the entire mesh is uniformly regular as it will be the case in the computations we can merely choose $h_C = h$).

The finite dimensional space involving continuous affine finite elements is

$$\mathbf{V}_h = \left\{ \mathbf{v}_h \in (C(\overline{\Omega}))^2 : \quad \forall \kappa \in T_h, \quad \mathbf{v}_{h_{|\kappa}} \in (P_1(\kappa))^2, \quad \mathbf{v}_{h_{|\Gamma_D}} = \mathbf{0} \right\}.$$

The normal trace space on the contact zone is defined as

$$W_h = \left\{ \mu_h \in C(\overline{\Gamma_C}) : \exists \mathbf{v}_h \in \mathbf{V}_h \text{ s.t. } \mathbf{v}_h \cdot \mathbf{n} = \mu_h \text{ on } \Gamma_C \right\},$$

and the nonnegative functions of W_h become

$$W_h^+ = \left\{ \mu_h \in W_h : \mu_h \geq 0 \right\}.$$

The discrete problem approximating (7) is: find $\mathbf{u}_h \in \mathbf{V}_h$ and $\lambda_h \in W_h^+$ such that

$$\text{(9)} \qquad \left\{ \begin{array}{l} a(\mathbf{u}_h, \mathbf{v}_h) + \displaystyle\int_{\Gamma_C} I_h(\lambda_h v_{hn}) \, d\Gamma = L(\mathbf{v}_h), \qquad \forall \, \mathbf{v}_h \in \mathbf{V}_h, \\ \displaystyle\int_{\Gamma_C} I_h((\mu_h - \lambda_h) u_{hn}) \, d\Gamma \leq 0, \qquad \forall \, \mu_h \in W_h^+, \end{array} \right.$$

where I_h stands for the standard Lagrange interpolation operator of degree one defined at the nodes of $\overline{\Gamma_C}$: $\forall v \in C(\Gamma_C) : I_h v \in C(\Gamma_C), I_h v(\mathbf{x}) = v(\mathbf{x})$ for any node $\mathbf{x}$ in $\overline{\Gamma_C}$ and $I_h v$ is an affine function between two nodes. The following proposition proves the existence of a unique solution to problem (9). It gives also some elementary properties of the solution and describes the links with a standard variational inequality.

Proposition 3.1 (i) *Problem* (9) *admits a unique solution* $(\lambda_h, \mathbf{u}_h) \in W_h^+ \times \mathbf{V}_h$.
(ii) *One has* $u_{hn} \leq 0, \lambda_h \geq 0$ *on* Γ_C; $\lambda_h \, u_{hn} = 0$ *at the nodes of* Γ_C.
(iii) *The displacement field* $\mathbf{u}_h$ *solving* (9) *is the unique solution to problem: find* $\mathbf{u}_h \in \mathbf{K}_h = \{\mathbf{v}_h \in \mathbf{V}_h : v_{hn} \leq 0$ *on* $\Gamma_C\}$ *such that*

$$\text{(10)} \qquad a(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) \geq L(\mathbf{v}_h - \mathbf{u}_h), \qquad \forall \mathbf{v}_h \in \mathbf{K}_h.$$

Proof: (i) Since we are in the finite dimensional case we only need to check (see [18], Theorem 3.9 and Example 3.8) that

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h, \mathbf{v}_h \neq \mathbf{0}} \frac{\displaystyle\int_{\Gamma_C} I_h(\mu_h v_{hn}) \, d\Gamma}{\|\mathbf{v}_h\|_{1,\Omega}}$$

is a norm on W_h. So we have to verify that

$$\left\{ \mu_h \in W_h : \int_{\Gamma_C} I_h(\mu_h v_{hn}) \, d\Gamma = 0, \; \forall \mathbf{v}_h \in \mathbf{V}_h \right\} = \{0\},$$

which is satisfied according to the definition of W_h. So Problem (9) admits a unique solution $(\lambda_h, \mathbf{u}_h) \in W_h^+ \times \mathbf{V}_h$.

(ii) Set

$$c(\mu_h, \mathbf{v}_h) = \int_{\Gamma_C} I_h(\mu_h v_{hn})\, d\Gamma, \quad \forall \mu_h \in W_h, \forall \mathbf{v}_h \in \mathbf{V}_h.$$

Taking $\mu_h = 0$ and $\mu_h = 2\lambda_h$ in (9) leads to

$$c(\lambda_h, \mathbf{u}_h) = 0 \quad \text{ and } \quad c(\mu_h, \mathbf{u}_h) \le 0, \quad \forall\, \mu_h \in W_h^+.$$

Taking $\mu_h = \psi_x \in W_h^+$ in the previous inequality where ψ_x is the scalar basis function of W_h (defined on $\overline{\Gamma_C}$) at node $x \in \overline{\Gamma_C}$ verifying $\psi_x(x') = \delta_{x,x'}$ for any node $x' \in \overline{\Gamma_C}$, we deduce that $u_{hn}(x) \le 0$. Hence $u_{hn} \le 0$ on Γ_C.

From $\lambda_h u_{hn} \le 0$ on Γ_C and since $c(\lambda_h, \mathbf{u}_h) = 0$ we come to the conclusion that $I_h(\lambda_h u_{hn}) = 0$ on Γ_C. That proves point (ii).

(iii) From the equation in (9) and $c(\lambda_h, \mathbf{u}_h) = 0$ we get

$$a(\mathbf{u}_h, \mathbf{u}_h) = L(\mathbf{u}_h) \tag{11}$$

and for any $\mathbf{v}_h \in \mathbf{K}_h$, we obtain

$$a(\mathbf{u}_h, \mathbf{v}_h) - L(\mathbf{v}_h) = -\int_{\Gamma_C} I_h(\lambda_h v_{hn})\, d\Gamma \ge 0. \tag{12}$$

Putting together (11) and (12) implies that $\mathbf{u}_h$ is solution of the variational inequality (10) which admits a unique solution according to Stampacchia's theorem.

■

The standard approach (see, e.g., [3, 11, 25]) consists of solving the following discrete problem (using the same arguments as in the previous proposition, it admits a unique solution): find $\mathbf{w}_h \in \mathbf{V}_h$ and $\theta_h \in M_h^+$ such that

$$\left\{ \begin{array}{l} a(\mathbf{w}_h, \mathbf{v}_h) + b(\theta_h, \mathbf{v}_h) = L(\mathbf{v}_h), \qquad \forall\, \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mu_h - \theta_h, \mathbf{w}_h) \le 0, \qquad \forall\, \mu_h \in M_h^+, \end{array} \right. \tag{13}$$

where

$$M_h^+ = \left\{ \mu_h \in W_h : \int_{\Gamma_C} \mu_h \psi_h\, d\Gamma \ge 0,\ \forall \psi_h \in W_h^+ \right\}. \tag{14}$$

Remark 3.2 *We have $W_h^+ \subset M^+$ and $W_h^+ \subset M_h^+ \not\subset M^+$.*

The next proposition establishes the link between the solutions of Problems (9) and (13).

Proposition 3.3 *The solutions $(\lambda_h, \mathbf{u}_h)$ and $(\theta_h, \mathbf{w}_h)$ of Problems (9) and (13) satisfy:*
(i) $\mathbf{u}_h = \mathbf{w}_h$,

(ii) $\lambda_h = \pi_h \theta_h$ *where* $\pi_h : L^1(\Gamma_C) \mapsto W_h$ *is the quasi-interpolation operator defined for any function* v *in* $L^1(\Gamma_C)$ *by:*

$$\pi_h v = \sum_{x \in N_h} \alpha_x(v) \psi_x,$$

where N_h *represents the set of nodes of* $\overline{\Gamma_C}$, ψ_x *is the scalar basis function of* W_h *(defined on* $\overline{\Gamma_C}$*) at node* x *verifying* $\psi_x(x') = \delta_{x,x'}$ *for all* $x' \in N_h$ *and*

$$\alpha_x(v) = \left(\int_{\Gamma_C} v \psi_x \, d\Gamma \right) \left(\int_{\Gamma_C} \psi_x \, d\Gamma \right)^{-1}.$$

Proof: (i) The same discussion as in points (ii) and (iii) of Proposition 3.1 and some polarity arguments (see, e.g., [20, 22]) which we describe hereafter prove that $\mathbf{w}_h$ is also the unique solution of the variational inequality (10). Let us briefly summarize the result: choosing $\mu_h = 0$ and $\mu_h = 2\theta_h$ in (13) implies $b(\theta_h, \mathbf{w}_h) = 0$ and $b(\mu_h, \mathbf{w}_h) = \int_{\Gamma_C} \mu_h w_{hn} \, d\Gamma \le 0, \quad \forall \mu_h \in M_h^+$. Consequently $w_{hn} \in -(M_h^+)^*$ (the notation X^* stands for the positive polar cone of X for the inner product on W_h induced by $b(.,.)$, see [23], p. 119). We have $(M_h^+)^* = ((W_h^+)^*)^* = W_h^+$ since W_h^+ is a closed convex cone. Hence $w_{hn} \in -W_h^+$ and $\mathbf{w}_h \in \mathbf{K}_h$. Besides (13) and $b(\theta_h, \mathbf{w}_h) = 0$ lead to $a(\mathbf{w}_h, \mathbf{w}_h) = L(\mathbf{w}_h)$ and for any $\mathbf{v}_h \in \mathbf{K}_h$, we get

$$a(\mathbf{w}_h, \mathbf{v}_h) - L(\mathbf{v}_h) = -\int_{\Gamma_C} \theta_h v_{hn} \, d\Gamma \ge 0,$$

since $\theta_h \in M_h^+ = (W_h^+)^*$ and $v_{hn} \in -W_h^+$. Hence $\mathbf{w}_h$ is the unique solution of the variational inequality (10) and point (iii) of Proposition 3.1 establishes the result.

(ii) From (i), and the equalities in (9) and (13) we deduce that

$$\int_{\Gamma_C} \theta_h v_{hn} \, d\Gamma = \int_{\Gamma_C} I_h(\lambda_h v_{hn}) \, d\Gamma, \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{15}$$

We choose $\mathbf{v}_h$ such that $v_{hn} = \psi_x$ where ψ_x is the scalar basis function of W_h at node $x \in \overline{\Gamma_C}$. As a consequence

$$\int_{\Gamma_C} \theta_h \psi_x \, d\Gamma = \lambda_h(x) \int_{\Gamma_C} \psi_x \, d\Gamma.$$

This proves that $\lambda_h = \pi_h \theta_h$ where π_h is the linear operator defined above. ∎

4 The positivity preserving averaging operator: basic properties

Now, we intend to study the basic properties of the operator π_h defined in Proposition 3.3, (ii). It is obvious that π_h is a linear averaging operator (see [6, 8, 10, 19, 30, 31] for other averaging operators) and that it preserves not only the nonnegative functions but satisfies also $\pi_h(M_h^+) = W_h^+$ which means that it transforms finite element type functions with a weak nonnegativity condition into nonnegative functions (such a property is also satisfied by the operator in [8]; for a detailed discussion concerning positivity preserving finite element approximation, we refer the reader to [27]). Obviously $\pi_h v_h \neq v_h$ in the general case when $v_h \in W_h$. Moreover it is easy to see that $\pi_h(W_h) = W_h$. Finally it is straightforward to check that any locally constant function is reproduced locally by π_h (this is not the case for locally affine functions, since the meshes on Γ_C have not the same length), and that

$$\int_{\Gamma_C} v - \pi_h v \, d\Gamma = 0, \tag{16}$$

for any $v \in L^1(\Gamma_C)$ which means that the operator preserves globally the average (note that a local average preserving property does not hold). In the following proofs we denote by C a positive generic constant independent of the discretization parameter h. Now we show the L^2-stability property of π_h.

Lemma 4.1 *There is a positive constant C independent of h such that for any $v \in L^2(\Gamma_C)$ and any $E \in E_h^C$ (E_h^C denotes the set of closed edges lying in $\overline{\Gamma_C}$):*

$$\|\pi_h v\|_{0,E} \leq C\|v\|_{0,\gamma_E},$$

where $\gamma_E = \cup_{\{F \in E_h^C :\, F \cap E \neq \emptyset\}} F$.

Proof: Let γ_x be the support of the basis function ψ_x in Γ_C. Using the definition of $\alpha_x(v)$ in Proposition 3.3, Cauchy-Schwarz inequality, and the uniform regularity, we get

$$|\alpha_x(v)| \leq \|v\|_{0,\gamma_x}\|\psi_x\|_{0,\gamma_x}\|\psi_x\|^{-1}_{L^1(\gamma_x)} \leq Ch_C^{-\frac{1}{2}}\|v\|_{0,\gamma_x}.$$

Denoting by N_h the set of nodes of $\overline{\Gamma_C}$, we obtain by a triangular inequality

$$\|\pi_h v\|_{0,E} = \left\| \sum_{x \in N_h \cap E} \alpha_x(v)\psi_x \right\|_{0,E} \leq C\|v\|_{0,\gamma_E}.$$

■

The next lemma is concerned with the L^2-approximation properties of π_h.

Lemma 4.2 *There is a positive constant C independent of h such that for any $v \in H^\eta(\Gamma_C)$, $0 \leq \eta \leq 1$, and any $E \in E_h^C$ (E_h^C denotes the set of closed edges lying in $\overline{\Gamma_C}$):*

$$\|v - \pi_h v\|_{0,E} \leq Ch^\eta\|v\|_{\eta,\gamma_E},$$

where $\gamma_E = \cup_{\{F \in E_h^C :\, F \cap E \neq \emptyset\}} F$.

Proof: When $\eta = 0$ the bound results from the previous lemma. Note that π_h preserves the constant functions on Γ_C. Let be given an arbitrary constant function $c(x) = c$, $\forall x \in \Gamma_C$. From the definition of π_h, we may write for any $v \in H^\eta(\Gamma_C)$:

$$v - \pi_h v = v - c - \pi_h(v - c).$$

Therefore by Lemma 4.1 we get

$$\|v - \pi_h v\|_{0,E} \leq C\left(\|v - c\|_{0,E} + \|v - c\|_{0,\gamma_E}\right) \leq C\|v - c\|_{0,\gamma_E}, \forall c \in \mathbb{R}. \tag{17}$$

We then choose $c = \int_{\gamma_E} v(x)\,dx/|\gamma_E|$ in (17) where $|\gamma_E|$ denotes the length of γ_E. Then if $x \in \gamma_E$ and $0 < \eta < 1$ we have

$$\begin{aligned} v(x) - c &= |\gamma_E|^{-1} \int_{\gamma_E} v(x) - v(y) dy \\ &= |\gamma_E|^{-1} \int_{\gamma_E} \frac{v(x) - v(y)}{|x - y|^{\frac{1+2\eta}{2}}} |x - y|^{\frac{1+2\eta}{2}} dy. \end{aligned}$$

Using Cauchy-Schwarz inequality we deduce

$$\begin{aligned} \int_{\gamma_E} (v(x) - c)^2 dx &= |\gamma_E|^{-2} \int_{\gamma_E} \left(\int_{\gamma_E} \frac{v(x) - v(y)}{|x - y|^{\frac{1+2\eta}{2}}} |x - y|^{\frac{1+2\eta}{2}} dy \right)^2 dx \\ &\leq |\gamma_E|^{-2} \int_{\gamma_E} \left(\int_{\gamma_E} \frac{(v(x) - v(y))^2}{|x - y|^{1+2\eta}} dy \int_{\gamma_E} |x - y|^{1+2\eta} dy \right) dx \\ &\leq |\gamma_E|^{2\eta} \int_{\gamma_E} \int_{\gamma_E} \frac{(v(x) - v(y))^2}{|x - y|^{1+2\eta}} dy dx \\ &\leq Ch^{2\eta} \|v\|^2_{\eta,\gamma_E}. \end{aligned}$$

Hence the result.

If $x \in \gamma_E$ and $\eta = 1$ we have

$$v(x) - c = |\gamma_E|^{-1} \int_{\gamma_E} v(x) - v(y) dy = |\gamma_E|^{-1} \int_{\gamma_E} \int_y^x v'(t)\, dt dy,$$

where the notation v' stands for the derivative of v. Hence

$$|v(x) - c| \leq |\gamma_E|^{\frac{1}{2}} \|v'\|_{0,\gamma_E}.$$

The result is then straightforward. ■

A open question is concerned with the optimal approximation properties of π_h in dual Sobolev spaces (typically $H^{-\frac{1}{2}}(\Gamma_C)$). It is straightforward that the $L^2(\Gamma_C)$-projection operator onto continuous and piecewise affine functions as well as the $L^2(\Gamma_C)$-projection operator onto piecewise constant functions satisfy such properties. On the contrary it can be shown that the Lagrange interpolation operator as well as the $L^2(\Gamma_C)$-projection operator applied to nonnegative functions and mapping onto W_h^+ do not fulfill such properties. Unfortunately the counter examples for the last two operators use the fact that the average of the function is not preserved and this is not the case for π_h (see (16)).

5 A priori error estimates

Now we intend to analyze the convergence of the finite element problem (9). In the forthcoming error analysis we suppose that $\mathbf{u} \in (H^{\frac{3}{2}+\eta}(\Omega))^2$ with $0 < \eta \leq 1/2$ which implies that u_n is continuous on Γ_C (which is a straight line segment). Set

$$\begin{aligned} \gamma_c &= \{x \in \Gamma_C : u_n(x) = 0\}, \\ \gamma_s &= \Gamma_C \setminus \gamma_c. \end{aligned}$$

In order to obtain an optimal convergence rate we need to use the assumption:

(18) the number of points in $\overline{\overset{\circ}{\gamma}_c} \cap \overline{\gamma_s}$ is finite.

The case where (18) is not assumed is considered in Corollary 5.8. Let us first recall the recent result established in [25].

Lemma 5.1 [25] *Let $(\lambda, \mathbf{u})$ be the solution of (7) and let $(\theta_h, \mathbf{u}_h)$ be the solution of (13). Assume that (18) holds. Let the regularity assumption $\mathbf{u} \in (H^{\frac{3}{2}+\eta}(\Omega))^2$ with $0 < \eta \leq 1/2$ hold. Then, there exists a positive constant C independent of h satisfying*

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|\lambda - \theta_h\|_{-\frac{1}{2},\Gamma_C} \leq Ch^{\frac{1}{2}+\eta}\|\mathbf{u}\|_{\frac{3}{2}+\eta,\Omega}.$$

This result and a triangular inequality imply the bound in the next lemma.

Lemma 5.2 *Let $(\lambda, \mathbf{u})$ be the solution of (7), let $(\lambda_h, \mathbf{u}_h)$ be the solution of (9) and let $(\theta_h, \mathbf{u}_h)$ be the solution of (13). Assume that (18) holds. Let the regularity assumption $\mathbf{u} \in (H^{\frac{3}{2}+\eta}(\Omega))^2$ with $0 < \eta \leq 1/2$ hold. Then, there exists a positive constant C independent of h satisfying*

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|\lambda - \lambda_h\|_{-\frac{1}{2},\Gamma_C} \leq Ch^{\frac{1}{2}+\eta}\|\mathbf{u}\|_{\frac{3}{2}+\eta,\Omega} + \|\lambda_h - \theta_h\|_{-\frac{1}{2},\Gamma_C}.$$

Now we need to estimate the term $\|\lambda_h - \theta_h\|_{-\frac{1}{2},\Gamma_C}$. A first bound is given hereafter.

Lemma 5.3 *Assume that the hypotheses of Lemma 5.2 hold. Then there exists a positive constant C independent of h satisfying*

$$\|\lambda_h - \theta_h\|_{-\frac{1}{2},\Gamma_C} \leq C\left(h^{\frac{1}{2}+\eta}\|\mathbf{u}\|_{\frac{3}{2}+\eta,\Omega} + h_C^{\frac{1}{2}}\|\lambda_h - \theta_h\|_{0,\Gamma_C}\right).$$

Proof: From the discrete inf-sup condition (see, e.g., [11])

$$0 < C \leq \inf_{\mu_h \in W_h} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mu_h, \mathbf{v}_h)}{\|\mu_h\|_{-\frac{1}{2},\Gamma_C}\|\mathbf{v}_h\|_{1,\Omega}}$$

and (15), we get

$$\begin{aligned} \|\lambda_h - \theta_h\|_{-\frac{1}{2},\Gamma_C} &\leq C \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\lambda_h - \theta_h, \mathbf{v}_h)}{\|\mathbf{v}_h\|_{1,\Omega}} \\ &= C \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{\int_{\Gamma_C} \lambda_h v_{hn} - I_h(\lambda_h v_{hn})\, d\Gamma}{\|\mathbf{v}_h\|_{1,\Omega}}. \end{aligned} \tag{19}$$

Besides we have

$$\int_{\Gamma_C} \lambda_h v_{hn} - I_h(\lambda_h v_{hn})\, d\Gamma = \sum_{E \in E_h^C} \int_E \lambda_h v_{hn} - I_h(\lambda_h v_{hn})\, d\Gamma,$$

where E_h^C denotes the set of closed edges (of triangles) lying in $\overline{\Gamma_C}$. From numerical integration (trapezoidal formula), and Cauchy-Schwarz inequality we get

$$\begin{aligned}
\int_E \lambda_h v_{hn} - I_h(\lambda_h v_{hn})\, d\Gamma &\leq Ch_E^3 |(\lambda_h v_{hn})''_{|_E}| \\
&\leq Ch_E^3 |(\lambda'_h v'_{hn})_{|_E}| \\
&\leq Ch_E^2 \|\lambda'_h\|_{0,E} \|v'_{hn}\|_{0,E} \\
&= Ch_E^2 \|(\lambda_h - \bar{\lambda})'\|_{0,E} \|v'_{hn}\|_{0,E}
\end{aligned}$$

where h_E denotes the length of the edge E, $\bar{\lambda} = (\int_E \lambda\, d\Gamma)/h_E$ and v', v'' denote the derivatives of first and second order of v. An inverse inequality implies

$$\int_E \lambda_h v_{hn} - I_h(\lambda_h v_{hn})\, d\Gamma \leq Ch_E \|\lambda_h - \bar{\lambda}\|_{0,E} \|v'_{hn}\|_{0,E}.$$

Writing $\mathbf{v}_h = (v_{hx}, v_{hy})$, we can suppose without loss of generality that Γ_C is parallel to the horizontal $x-$axis (the $y-$axis is vertical). Using the scaled trace theorem (see, e.g., [16]):

$$\|v\|_{0,E} \leq C\left(h_E^{-\frac{1}{2}} \|v\|_{0,K} + h_E^{\frac{1}{2}} \|\nabla v\|_{0,K}\right), \quad \forall E \in E_K, \forall v \in H^1(K),$$

(E_K represents the set of the three edges belonging to the triangle K) we deduce that

$$\|v'_{hn}\|_{0,E} = \left\|\frac{\partial v_{hy}}{\partial x}\right\|_{0,E} \leq Ch_E^{-\frac{1}{2}} \left\|\frac{\partial v_{hy}}{\partial x}\right\|_{0,K} \leq Ch_E^{-\frac{1}{2}} \|v_{hy}\|_{1,K} \leq Ch_E^{-\frac{1}{2}} \|\mathbf{v}_h\|_{1,K}.$$

Hence

$$\int_E \lambda_h v_{hn} - I_h(\lambda_h v_{hn})\, d\Gamma \leq Ch_E^{\frac{1}{2}} \|\lambda_h - \bar{\lambda}\|_{0,E} \|\mathbf{v}_h\|_{1,K}.$$

Therefore denoting again by $\bar{\lambda}$ the piecewise constant function defined on Γ_C such that $\bar{\lambda}_{|_E} = (\int_E \lambda\, d\Gamma)/h_E$, we obtain by addition

$$\int_{\Gamma_C} \lambda_h v_{hn} - I_h(\lambda_h v_{hn})\, d\Gamma \leq Ch_C^{\frac{1}{2}} \|\bar{\lambda} - \lambda_h\|_{0,\Gamma_C} \|\mathbf{v}_h\|_{1,\Omega}.$$

According to (19) we deduce

$$\begin{aligned}
\|\lambda_h - \theta_h\|_{-\frac{1}{2},\Gamma_C} &\leq Ch_C^{\frac{1}{2}} \|\bar{\lambda} - \lambda_h\|_{0,\Gamma_C} \\
&\leq Ch_C^{\frac{1}{2}} \left(\|\lambda - \theta_h\|_{0,\Gamma_C} + \|\theta_h - \lambda_h\|_{0,\Gamma_C} + \|\lambda - \bar{\lambda}\|_{0,\Gamma_C}\right).
\end{aligned}$$

Then we use the standard estimate $\|\lambda-\bar{\lambda}\|_{0,\Gamma_C} \le Ch^{\eta}\|\lambda\|_{\eta,\Gamma_C}$ (the latter result is obtained in the proof of Lemma 4.2) together with the trace theorem (the coefficients in the elasticity operator are supposed to lie in $C^1(\overline{\Omega})$). The term $\|\lambda-\theta_h\|_{0,\Gamma_C}$ is estimated by using an inverse inequality, Lemma 5.1 and the optimal approximation properties in $H^{-\frac{1}{2}}(\Gamma_C)$ of the $L^2(\Gamma_C)$-projection operator p_h mapping onto W_h. We recall that p_h is defined for any $v \in L^2(\Gamma_C)$ by:

$$p_h v \in W_h, \qquad \int_{\Gamma_C} (v-p_h v)\psi_h \, d\Gamma = 0, \qquad \forall \psi_h \in W_h. \tag{20}$$

More precisely we have

$$\begin{aligned}
\|\lambda-\theta_h\|_{0,\Gamma_C} &\le \|\lambda-p_h\lambda\|_{0,\Gamma_C} + \|p_h\lambda-\theta_h\|_{0,\Gamma_C} \\
&\le C\left(h^{\eta}\|\lambda\|_{\eta,\Gamma_C} + h_C^{-\frac{1}{2}}\|p_h\lambda-\theta_h\|_{-\frac{1}{2},\Gamma_C}\right) \\
&\le C\left(h^{\eta}\|\mathbf{u}\|_{\frac{3}{2}+\eta,\Omega} + h_C^{-\frac{1}{2}}\|p_h\lambda-\lambda\|_{-\frac{1}{2},\Gamma_C} + h_C^{-\frac{1}{2}}\|\lambda-\theta_h\|_{-\frac{1}{2},\Gamma_C}\right)
\end{aligned}$$

and

$$h_C^{\frac{1}{2}}\|\lambda-\theta_h\|_{0,\Gamma_C} \le Ch^{\frac{1}{2}+\eta}\|\mathbf{u}\|_{\frac{3}{2}+\eta,\Omega}. \tag{21}$$

Finally

$$\|\lambda_h-\theta_h\|_{-\frac{1}{2},\Gamma_C} \le C\left(h^{\frac{1}{2}+\eta}\|\mathbf{u}\|_{\frac{3}{2}+\eta,\Omega} + h_C^{\frac{1}{2}}\|\lambda_h-\theta_h\|_{0,\Gamma_C}\right).$$

■

Lemma 5.4 *Assume that the hypotheses of Lemma 5.2 hold. Then there exists a positive constant C independent of h satisfying*

$$h_C^{\frac{1}{2}}\|\lambda_h-\theta_h\|_{0,\Gamma_C} \le Ch^{\frac{1}{2}+\eta}\|\mathbf{u}\|_{\frac{3}{2}+\eta,\Omega}.$$

Proof: We write

$$\begin{aligned}
\|\lambda_h-\theta_h\|_{0,\Gamma_C} &= \|\theta_h-\pi_h\theta_h\|_{0,\Gamma_C} \\
&\le \|(\theta_h-\lambda)-\pi_h(\theta_h-\lambda)\|_{0,\Gamma_C} + \|\lambda-\pi_h\lambda\|_{0,\Gamma_C}.
\end{aligned}$$

Using Lemma 4.2 when adding the local estimates gives

$$\|\lambda_h-\theta_h\|_{0,\Gamma_C} \le C\left(\|\lambda-\theta_h\|_{0,\Gamma_C} + h^{\eta}\|\lambda\|_{\eta,\Gamma_C}\right),$$

and bound (21) yields

$$h_C^{\frac{1}{2}}\|\lambda_h-\theta_h\|_{0,\Gamma_C} \le Ch^{\frac{1}{2}+\eta}\|\mathbf{u}\|_{\frac{3}{2}+\eta,\Omega}.$$

■

We finally obtain the optimal a priori error estimate for the sign preserving method.

Theorem 5.5 *Let* $(\lambda, \mathbf{u})$ *be the solution of* (7) *and let* $(\lambda_h, \mathbf{u}_h)$ *be the solution of* (9). *Assume that* (18) *holds. Let the regularity assumption* $\mathbf{u} \in (H^{\frac{3}{2}+\eta}(\Omega))^2$ *with* $0 < \eta \le 1/2$ *hold. Then, there exists a positive constant* C *independent of* h *satisfying*

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|\lambda - \lambda_h\|_{-\frac{1}{2},\Gamma_C} \le Ch^{\frac{1}{2}+\eta}\|\mathbf{u}\|_{\frac{3}{2}+\eta,\Omega}.$$

Proof: It suffices to put together the results of Lemmas 5.2, 5.3 and 5.4. ∎

Remark 5.6 *If one supposes that the operator* π_h *satisfies optimal approximation properties in dual Sobolev spaces (as* $H^{-\frac{1}{2}}(\Gamma_C)$*) then the proof of Theorem 5.5 would be straightforward (in this case one could avoid Lemma 5.3) since it suffices to write* $\|\lambda_h - \theta_h\|_{-\frac{1}{2},\Gamma_C} = \|\theta_h - \pi_h\theta_h\|_{-\frac{1}{2},\Gamma_C} \le \|(\lambda - \theta_h) - \pi_h(\lambda - \theta_h)\|_{-\frac{1}{2},\Gamma_C} + \|\lambda - \pi_h\lambda\|_{-\frac{1}{2},\Gamma_C}$ *and these properties (together with some inverse estimates) would end the proof. Unfortunately such properties are not available (see also the discussion at the end of Section 4).*

Remark 5.7 *A deeper insight into the estimates shows that the direct error analysis of the finite element method* (9) *by circumventing the standard approximation* (13) *would be not trivial (at least not shorter than the present analysis).*

The assumption (18) is concerned with the finite number of transition points between contact and separation zones. Actually we can not prove that such an assumption is satisfied in practice. Without this hypothesis we can obtain a convergence result for the finite element method (9). This is achieved in the next corollary.

Corollary 5.8 *Let* $(\lambda, \mathbf{u})$ *be the solution of* (7) *and let* $(\lambda_h, \mathbf{u}_h)$ *be the solution of* (9). *Assume that* $\mathbf{u} \in (H^{\frac{3}{2}+\eta}(\Omega))^2$ *with* $0 < \eta \le 1/2$. *Then, there exists a positive constant* C *independent of* h *satisfying*

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|\lambda - \lambda_h\|_{-\frac{1}{2},\Gamma_C} \le Ch^{\frac{1+\eta}{2}}\|\mathbf{u}\|_{\frac{3}{2}+\eta,\Omega}.$$

Proof: The result is straightforward by noting that the solution $(\theta_h, \mathbf{u}_h)$ of (13) satisfies under the $(H^{\frac{3}{2}+\eta}(\Omega))^2$ regularity hypothesis (see, e.g., [2]):

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|\lambda - \theta_h\|_{-\frac{1}{2},\Gamma_C} \le Ch^{\frac{1+\eta}{2}}\|\mathbf{u}\|_{\frac{3}{2}+\eta,\Omega},$$

and that the proofs of Lemmas 5.2–5.4 remain the same when removing assumption (18). ∎

Remark 5.9 *Using the same techniques as in* (21), *it becomes possible to obtain the same bounds as in Theorem 5.5 and Corollary 5.8 for the error with a weighted L2-norm on the multipliers:* $\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + h_C^{\frac{1}{2}}\|\lambda - \lambda_h\|_{0,\Gamma_C}$.

6 Numerical experiments

This section is concerned with the numerical implementation of the finite element method (9) and its comparison with the standard approach (13). We suppose that the contacting bodies are homogeneous isotropic so that Hooke's law (2) becomes

$$\boldsymbol{\sigma}(\mathbf{v}) = \frac{E\nu}{(1-2\nu)(1+\nu)} tr(\boldsymbol{\varepsilon}(\mathbf{v}))\mathbf{I} + \frac{E}{1+\nu}\boldsymbol{\varepsilon}(\mathbf{v})$$

where $\mathbf{I}$ represents the identity matrix, tr is the trace operator, E and ν denote Young's modulus and Poisson's ratio, respectively with $E > 0$ and $0 \leq \nu < 1/2$. Hereafter we denote by N_C the number of elements on the contact area Γ_C.

In the first test we compute the values of the standard and nonstandard multipliers θ_h and λ_h and we discuss the convergence rate of $\|\lambda_h - \theta_h\|_{0,\Gamma_C}$. The second example deals with Hertzian contact where the exact multiplier λ is known: this allows us to compare the accuracy of both discrete multipliers. A case with two contacting bodies and nonmatching meshes on the contact area is considered in the third example. We show how the sign preserving approach can be extended to this framework, at least numerically.

6.1 A first example with slow variation of the contact pressure

We study a realistic physical example also considered in [22] (see Figure 1). We choose the domain $\Omega =]0,1[\times]0,1[$ and we suppose that the body is an iron square of $1m^2$ whose material characteristics are $E = 2.1\,10^{11} Pa, \nu = 0.3$ and $\rho = 7800 kg.m^{-3}$. The body is clamped on its right side, it is initially in contact on its left side and no forces are applied on the upper and lower boundary parts of Ω. Moreover the body is acted on by its own weight only (with $g = 9.81 m.s^{-2}$). We consider quasi-uniform unstructured meshes. A first configuration with 51 nodes on the contact area is depicted in Figure 2. We see that Γ_C is divided into two parts: an upper part where the body remains in contact with the axis $x = 0$ and the lower part of Γ_C where it separates from this axis.

The nodes on Γ_C are numbered from 1 (up) to 51 (bottom) and $u_{hn} = 0$ at nodes 1 to 16 whereas $u_{hn} < 0$ at the other nodes. The corresponding standard (resp. nonstandard) multipliers θ_h (resp. λ_h) are reported in Table 1. As expected we observe that θ_h is sometimes negative and that it shows some artificial (from a mechanical point of view) oscillations on the separation part (nodes 16 to 51). These oscillations weaken when moving away from the transition point (node 16).

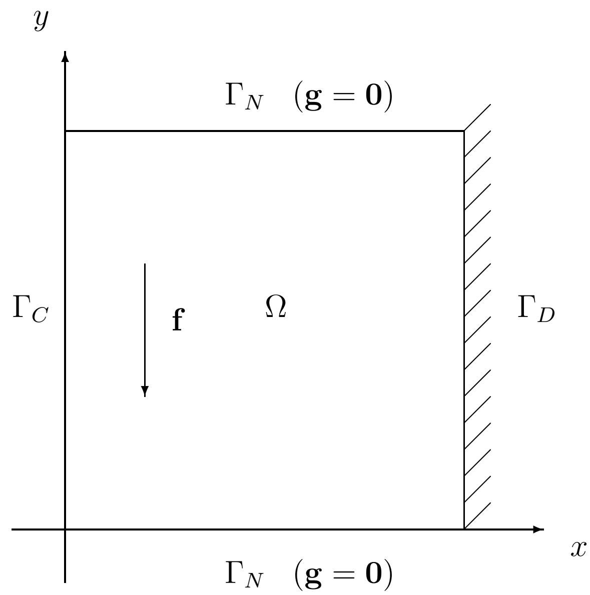

Figure 1: The geometry of the body Ω

Node	λ_h	θ_h	Node	λ_h	θ_h
1	$1.20010E+05$	$1.26478E+05$	27	0	$-\ 6.82686E-03$
2	$1.09245E+05$	$1.07075E+05$	28	0	$1.82925E-03$
3	$1.00097E+05$	$1.00693E+05$	29	0	$-\ 4.90147E-04$
4	$9.10860E+04$	$9.07382E+04$	30	0	$1.31334E-04$
5	$8.29212E+04$	$8.28702E+04$	31	0	$-\ 3.51910E-05$
6	$7.53913E+04$	$7.53082E+04$	32	0	$9.42939E-06$
7	$6.83133E+04$	$6.82448E+04$	33	0	$-\ 2.52660E-06$
8	$6.16557E+04$	$6.15925E+04$	34	0	$6.77000E-07$
9	$5.53683E+04$	$5.53196E+04$	35	0	$-\ 1.81401E-07$
10	$4.93761E+04$	$4.93389E+04$	36	0	$4.86064E-08$
11	$4.36050E+04$	$4.35813E+04$	37	0	$-\ 1.30240E-08$
12	$3.79678E+04$	$3.79658E+04$	38	0	$3.48978E-09$
13	$3.23468E+04$	$3.23622E+04$	39	0	$-\ 9.35084E-10$
14	$2.65571E+04$	$2.66665E+04$	40	0	$2.50555E-10$
15	$2.02133E+04$	$2.03146E+04$	41	0	$-\ 6.71360E-11$
16	$1.16927E+04$	$1.33550E+04$	42	0	$1.79890E-11$
17	0	$-\ 3.57846E+03$	43	0	$-\ 4.82015E-12$
18	0	$9.58847E+02$	44	0	$1.29155E-12$
19	0	$-\ 2.56922E+02$	45	0	$-\ 3.46071E-13$
20	0	$6.88421E+01$	46	0	$9.27297E-14$
21	0	$-\ 1.84462E+01$	47	0	$-\ 2.48474E-14$
22	0	$4.94264E+00$	48	0	$6.66014E-15$
23	0	$-\ 1.32438E+00$	49	0	$-\ 1.79311E-15$
24	0	$3.54866E-01$	50	0	$5.12319E-16$
25	0	$-\ 9.50859E-02$	51	0	$-\ 2.56159E-16$
26	0	$2.54782E-02$			

Table 1: Behavior of the nonstandard and standard multipliers λ_h and θ_h.

We then compute the convergence rate of $\|\lambda_h - \theta_h\|_{0,\Gamma_C}$ in order to illustrate Lemma 5.4. The results are reported in Table 2 where this expression is computed

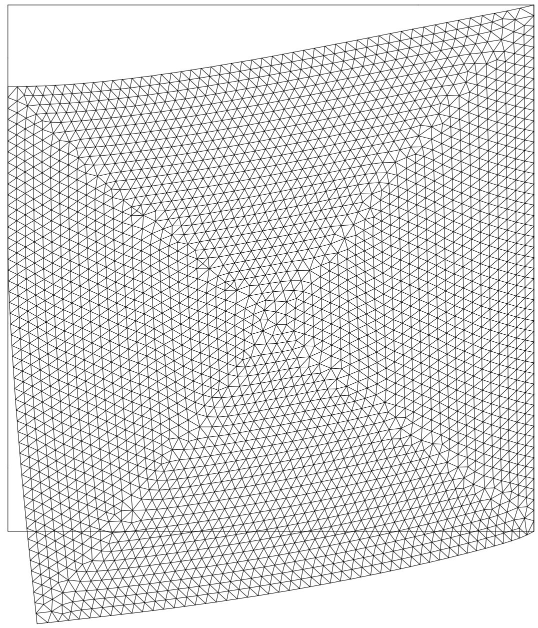

Figure 2: Initial and deformed configuration with $N_C = 50$ (deformation is amplified by a factor $2.\,10^5$).

from $N_C = 1$ to $N_C = 128$. The average convergence rate (between $N_C = 8$ and $N_C = 128$) is 1.25 and a limit rate near 1.24 is observed. In this example we avoid computing the convergence rates of $\|\lambda - \lambda_h\|_{0,\Gamma_C}$ and $\|\lambda - \theta_h\|_{0,\Gamma_C}$ since problem (7) does not admit an explicit solution $(\lambda, \mathbf{u})$ in this case and the choice of a reference multiplier would require to choose one of both methods (9) or (13). This study will be performed in the next example where the exact expression of the multiplier λ is known. Of course such a phenomenon does not occur for the reference displacement since they coincide for both finite element methods (9) and (13) according to Proposition 3.3. So we compute a reference displacement denoted by $\mathbf{u}_{ref}$ corresponding to a mesh which is as fine as possible. The most refined mesh corresponds to 129 nodes on the contact area and it furnishes the reference solution $\mathbf{u}_{ref}$ which is the chosen approximation for $\mathbf{u}$.

	$\|\lambda_h - \theta_h\|_{0,\Gamma_C}$	$(a(\mathbf{u}_{ref} - \mathbf{u}_h, \mathbf{u}_{ref} - \mathbf{u}_h))^{1/2}$
$N_C = 1$	17063	0.12778
$N_C = 2$	17299	$9.72400\,10^{-2}$
$N_C = 4$	13355	$7.01423\,10^{-2}$
$N_C = 8$	6181.4	$4.40570\,10^{-2}$
$N_C = 16$	2789	$2.54805\,10^{-2}$
$N_C = 32$	1121	$1.40710\,10^{-2}$
$N_C = 64$	453.27	–
$N_C = 128$	191.29	–
Limit rate	1.24	0.86

Table 2: Difference between the multipliers and error on the displacements

Since the limit convergence rate of $(a(\mathbf{u}_{ref} - \mathbf{u}_h, \mathbf{u}_{ref} - \mathbf{u}_h))^{1/2}$ (which is a norm equivalent to $\|\mathbf{u}_{ref} - \mathbf{u}_h\|_{1,\Omega}$) is near 0.86 one could merely believe that the convergence rate of $\|\lambda_h - \theta_h\|_{0,\Gamma_C}$ would be around 0.36. In fact the computed rate (of 1.25) is much higher, a phenomenon that we cannot explain.

From this example we conclude as expected, that the multiplier λ_h is more relevant from a mechanical point of view than θ_h.

6.2 Example of Hertzian contact

The next example is concerned with the Hertzian contact problem of an elastic ball with an infinite half plane. The material characteristics of the ball of radius $r = 1mm$ are chosen as in [25]: $\nu = 0.3$, $E = 7000\ MPa$ and a force of $(0, -f)$ with $f = 100\ N$ is applied at the top of the ball. Since the analytical expression of the contact pressure is

$$(22) \qquad \lambda(x) = \frac{2f}{\pi b^2}\sqrt{b^2 - x^2}, \quad -b \le x \le b, \qquad b = 2\sqrt{\frac{fr(1-\nu^2)}{E\pi}},$$

we have at our disposal a useful analytical solution for a comparison of λ_h and θ_h. Here $b \approx 0.1286mm$ and $\lambda(x) \approx 494.8\sqrt{1-(x/b)^2}\ N/mm, -b \le x \le b$. In our computations we choose quasi-uniform unstructured meshes (we do not symmetrize the problem and the mesh is not symmetric). The results are reported in Table 3.

Nodes on $\partial\Omega$	$\|\lambda - \lambda_h\|_{0,\Gamma_C}$	$\|\lambda - \theta_h\|_{0,\Gamma_C}$	$(\max_{\Gamma_C} \lambda_h, \min_{\Gamma_C} \lambda_h)$	$(\max_{\Gamma_C} \theta_h, \min_{\Gamma_C} \theta_h)$
24	47.080	100.23	$(381.97, 0)$	$(663.98, -178.76)$
48	54.651	80.604	$(511.47, 0)$	$(769.27, -3.0305)$
96	23.704	30.822	$(503.82, 0)$	$(535.93, -19.902)$
192	8.9620	15.223	$(498.20, 0)$	$(501.49, -40.091)$
384	1.8805	9.8732	$(496.93, 0)$	$(498.27, -59.165)$
768	1.2057	4.4893	$(496.43, 0)$	$(496.77, -45.345)$
Average rate	1.057	0.896	–	–

Table 3: Errors and comparison of the multipliers

We first observe that the convergence rates of $\|\lambda - \lambda_h\|_{0,\Gamma_C}$ and $\|\lambda - \theta_h\|_{0,\Gamma_C}$ are not constant when h decreases: the average rates are 1.057 and 0.896, respectively, so that the terms $\|\lambda - \lambda_h\|_{0,\Gamma_C}$ remain smaller than $\|\lambda - \theta_h\|_{0,\Gamma_C}$ as h vanishes. From the expression (22), we see that $(\max_{\Gamma_C} \lambda, \min_{\Gamma_C} \lambda)$ is approximately $(494.8, 0)$. The first argument is reached by the two approaches but the value 0 is not obtained in a satisfactory way by θ_h.

From this example we conclude that the sign preserving approach involving the nonnegative multiplier λ_h is more accurate than the standard method.

6.3 An example with two contacting bodies and nonmatching meshes

As third example we choose a problem of two contacting bodies Ω^1 and Ω^2 with nonmatching meshes on the common contact zone $\Gamma_C = \overline{\Omega^1} \cap \overline{\Omega^2}$. The dimensions of Ω^1 and Ω^2 are $1mm \times 0.05mm$. A Poisson's ratio $\nu = 0.2$ for both solids, Young's modulus $E_1 = 13000$ Mpa for the upper body and $E_2 = 30000$ Mpa for the lower body are assumed. There are two applied boundary loads on Ω^1, of value 100 N/mm (see Figure 3) : $\mathbf{g}_1$ (on the upper half of the left side) and $\mathbf{g}_2$ (on the right half of the upper side). Symmetry conditions are applied on the lower and right parts of the structure. The mesh of Ω^1 (resp. Ω^2) divides Γ_C into 119 (resp. 120) identical segments.

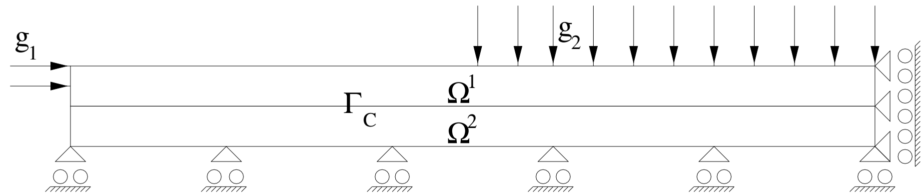

Figure 3: Setting of the problem.

In order to handle with nonmatching meshes we consider a global contact condition of mortar type. For error estimates dealing with mortar methods for contact problems we refer the reader to e.g., [2, 11, 20, 24, 25, 32]. Such a contact condition furnishes a multiplier denoted θ_h^1 which does not satisfy the nonnegativity condition. Our aim is to extend, at least numerically, the range of applicability of the sign preserving method to a configuration with nonmatching meshes.

We denote by $\mathbf{V}_h^1$ and $\mathbf{V}_h^2$ the finite element spaces associated with Ω^1 and Ω^2 and by M_h^{1+} the positive polar cone of W_h^{1+} (see the definition in (14)). Note that the set W_h^{1+} involves functions defined on Γ_C which are continuous, nonnegative and piecewise of degree one on the mesh of Ω^1. Of course one could also choose a symmetrical definition (e.g., M_h^{2+}) using the mesh of Ω^2. The standard approach is to find $\mathbf{u}_h = (\mathbf{u}_h^1, \mathbf{u}_h^2) \in \mathbf{V}_h^1 \times \mathbf{V}_h^2$ and $\theta_h^1 \in M_h^{1+}$ satisfying (see [11, 25]):

$$\left\{\begin{array}{ll} a(\mathbf{u}_h, \mathbf{v}_h) + \displaystyle\int_{\Gamma_C} \theta_h^1 (v_{hn}^1 + v_{hn}^2)\, d\Gamma = L(\mathbf{v}_h), & \forall\, \mathbf{v}_h \in \mathbf{V}_h^1 \times \mathbf{V}_h^2, \\ \displaystyle\int_{\Gamma_C} (\mu_h^1 - \theta_h^1)(u_{hn}^1 + u_{hn}^2)\, d\Gamma \leq 0, & \forall\, \mu_h^1 \in M_h^{1+}, \end{array}\right.$$

where $a(.,.)$ and $L(.)$ denote the bilinear and linear forms involving both bodies Ω^1 and Ω^2. The sign preserving approach is to find $\mathbf{u}_h = (\mathbf{u}_h^1, \mathbf{u}_h^2) \in \mathbf{V}_h^1 \times \mathbf{V}_h^2$ and $\lambda_h^1 \in W_h^{1+}$ satisfying

$$\left\{\begin{array}{ll} a(\mathbf{u}_h, \mathbf{v}_h) + \displaystyle\int_{\Gamma_C} I_h^1(\lambda_h^1 (v_{hn}^1 + p_h^1(v_{hn}^2)))\, d\Gamma = L(\mathbf{v}_h), & \forall\, \mathbf{v}_h \in \mathbf{V}_h^1 \times \mathbf{V}_h^2, \\ \displaystyle\int_{\Gamma_C} I_h^1((\mu_h^1 - \lambda_h^1)(u_{hn}^1 + p_h^1(u_{hn}^2)))\, d\Gamma \leq 0, & \forall\, \mu_h^1 \in W_h^{1+}, \end{array}\right.$$

where I_h^1 denotes the Lagrange interpolation operator of degree one at the nodes of Ω^1 on Γ_C and p_h^1 stands for the $L^2(\Gamma_C)$-projection operator onto W_h^1 (see (20)).

As expected the deformed configuration shows a separation area on the left part of Γ_C and a contact area on the right part of Γ_C (see Figure 4). The multiplier

θ_h^1, representing the contact pressure is depicted in Figure 5; as already noticed the multiplier is not always nonnegative and it shows some artificial oscillations near the transition point from contact to separation. Besides the multiplier λ_h^1 is represented in Figure 6 and we observe that it is more relevant from a mechanical point of view. We observe that the multiplier value is close to 100 on the contact zone which corresponds to the value of $\mathbf{g}_2$. Finally the difference $\theta_h^1 - \lambda_h^1$ is depicted in Figure 7 and we see that θ_h^1 and λ_h^1 differ in a significant way near the transition point. Again we conclude that the new approach involving λ_h^1 seems to be more accurate than the standard one when handling with nonmatching meshes.

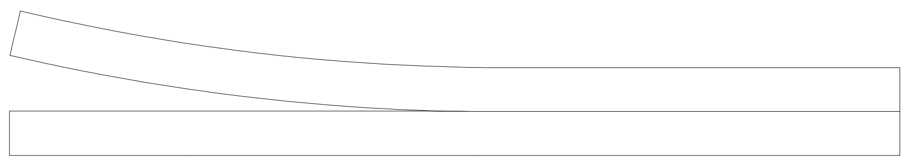

Figure 4: Deformed configuration.

7 Conclusion

In this work we consider a mixed finite element method which furnishes primal and dual variables with a good sign in opposition to the already known mixed methods for contact problems (in particular the classical approach). The study of the method uses an averaging positivity preserving operator which is analyzed and discussed. The convergence analysis in this paper leads to the same error estimates as the standard approach. The numerical experiments obtained with the new method seem to be more relevant and efficient in comparison with the standard method. Finally, the friction (see e.g., [21]) or the crack problems (see e.g., [5, 26]) are some possible applications of the method.

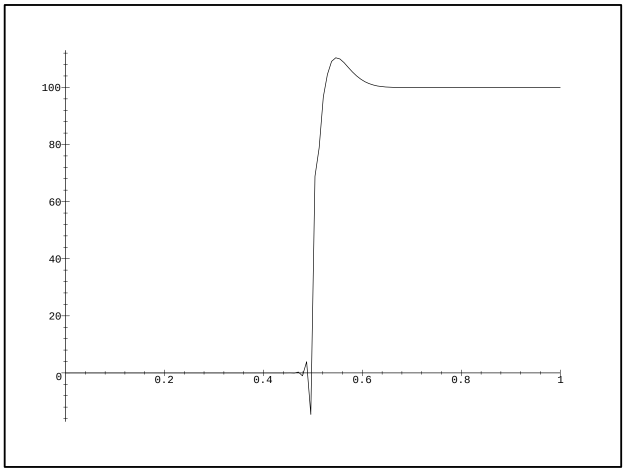

Figure 5: The multiplier θ_h^1.

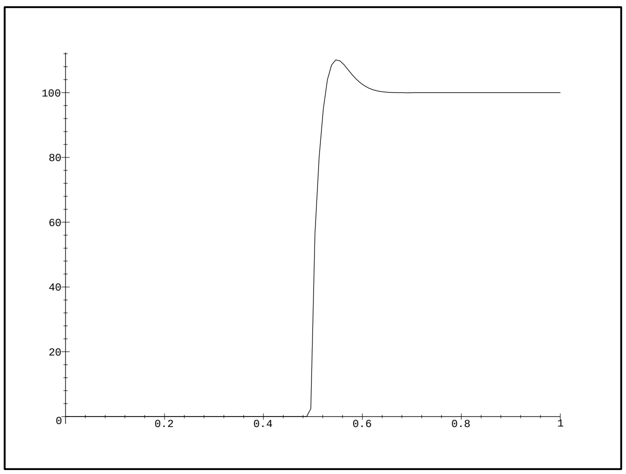

Figure 6: The multiplier λ_h^1.

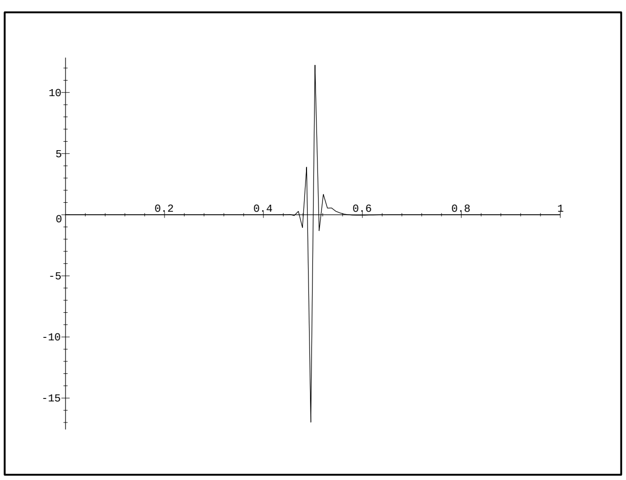

Figure 7: Difference between the multipliers: $\theta_h^1 - \lambda_h^1$.

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