Comparaison between conformal invariants for conformally Einstein metrics : some counter-example from the metric developed by Pedersen

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Abstract

The study of asymptotically hyperbolic Einstein metric is a rich field in theoretical physic and geometry. Pedersen introduced a family of example for the dimension 4, and we look in this paper into some of it's conformal invariant, namely renormalized volume and Yamabe-type energy, as their comparaison give couter-example to the converse of well-known theorem, as well as other propositions of same nature.

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Introduction

The study of conformal invariant for asymptotically hyperbolic manifold has been of great importance in physics since the introduction by J Maldacena in [1] of the AdS-CFT correspondance. A great idea is to try to extract information from conformal structure at infinity. An example of such a result is proved by Quing in [9] (see also [10], and another proof from Han-Gursky in [11]) [1] :

Theorem 1. *Let $(\overline{M}^{n+1}, \partial M^n, g_+)$ be a conformally compact Poincaré-Einstein manifold of class C^2 satisfying that either the dimension $3 \leqslant n+1 \leqslant 5$ or that the dimension $n+1 \geqslant 6$ and $\overline{M}$ is spin.*

If we let $(\partial M, [\gamma])$ denote its conformal infinity and if it Yamabe energy satisfy : $Y(\partial M, [\gamma]) > 0$, then the first type of Yamabe-Escobar energy must be positive :

$$Y^1(M, \partial M, g_+) > 0$$

Natural question are then if the converse could be true, and if we could get the same type of result for the renormalized volume instead of the first type of Yamabe-Escobar energy. This paper show that it both are not possible with an explicit calculation with a well known family of metric.

Pedersen introduced in [2] a family of conformally hyperbolic Einstein metrics on the 4-ball with non isometric conformal infinity (the Berger sphere), which has given a great deal of examples or counter-examples in the study of such riemannian metrics.

The aim of the present paper is to give a general comprehension of some conformal invariants for these metrics, namely the renormalized volume, the sign of the infinity's Yamabe energy and the sign of the Yamabe-Escobar energy. More precisely, we will prove :

Theorem 2. *Let m>-1 and g_m be the Pedersen metric of parameter m,*

(a) The renormalised volume of this Pedersen metric is :

$$V(B_1(\mathbb{R}^4), g_m) = \frac{4\pi^2}{3}\left(1 - \frac{m^2}{(1+m)^2}\right),$$

and it is positive for $-\frac{1}{2} \leqslant m$ and negative otherwise,

(b) The infinity's conformal Yamabe energy is positive for $-\frac{3}{4} \leqslant m$ and negative otherwise

(c) There exist $m_0 \leqslant -\frac{9}{10}$ such that the conformal Yamabe-Escobar energy is positive for $m_0 \leqslant m$ and negative otherwise.
Furthermore, m_0 is the only solution in]-1,0[of the equation $4\sqrt{-m} = \ln\left(\frac{1+\sqrt{-m}}{1-\sqrt{-m}}\right)$.

In part 1, we recall the definition of the metric constructed by Pedersen, before recalling the riemannian curvature of such metric in an adapted orthonormal frame in 2, which allow us to calculate the L^2 norm of the Weyl curvature. Then in part 3 we prove the point (a), in 4 the point (b) and in 5 the point (c).

1. the definition of the Yamabe energy is recalled in section4 and the one of the Yamabe-Escobar energy is in section 5

1 Definition of the metrics

Let's remember that on $S^3(\mathbb{R})$ there is a continuous family of Riemannian non locally isometric metrics called Berger sphere. They are defined via the natural isomorphism which links $S^3(\mathbb{R}) \subset \mathbb{C}^2$ to the Lie group $SU(2)$

$$(z,w) \mapsto \begin{pmatrix} z & -w \\ \bar{w} & \bar{z} \end{pmatrix},$$

The Lie algebra $su(2)$ at identity is then spawned by $\tilde{X_1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\tilde{X_2} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$, and $\tilde{X_3} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. We then consider the unique left-invariant vector field (we abusively keep the same notation).

We then define the Berger metric h_λ on the sphere by fixing that these fields are orthogonal with $\tilde{X_1}$ and $\tilde{X_2}$ normed and $\tilde{X_3}$ of norm λ.
As X_3 is tangent to the Hopf orbit, the Berger spheres are created from the canonic metric by multiplying the Hopf fiber by λ.
We can notice that when $\lambda = 1$ we get the usual sphere.

Remark : We have $[\tilde{X_1}, \tilde{X_2}] = 2\tilde{X_3}$, $[\tilde{X_2}, \tilde{X_3}] = 2\tilde{X_1}$, and $[\tilde{X_3}, \tilde{X_1}] = 2\tilde{X_2}$,

Definition 3. We call Berger sphere (of parameter λ) the Riemannian manifold ($S^3(\mathbb{R})$, h_λ)

Now we can introduce the family of conformally hyperbolic Einstein metrics with conformal infinity the Berger sphere that Pedersen highlighted in [2].
Let's consider the family of vector fields $(\tilde{X_1}, \tilde{X_2}, \tilde{X_3})$ over $B_1(\mathbb{R}^4) \subset \mathbb{R}^4$ that extends $\tilde{X_1}, \tilde{X_2}$ and $\tilde{X_3}$ with the same Lie bracket and staying tangent to all spheres. Let $(dr, \sigma_1, \sigma_2, \sigma_3)$ be the dual basis of $(\partial_r, \tilde{X_1}, \tilde{X_2}, \tilde{X_3})$, where r denote the usual radial coordinate. Then, Pedersen showed that :

Proposition 4. *Let m>-1, the metric*

$$g_m = \frac{4}{(1-r^2)^2}\left(\frac{1+mr^2}{1+mr^4}\mathbf{dr}^2 + r^2(1+mr^2)(\sigma_1^2+\sigma_2^2) + \frac{r^2(1+mr^4)}{1+mr^2}\sigma_3^2\right)$$

is Einstein ($Ricc(g_m) = -3g_m$), asymptotically hyperbolic with the Berger sphere $h_{\frac{1}{1+m}}$ as conformal infinity. Furthermore, this metric is the only asymptotically hyperbolic Einstein (with constant 3) metric with conformal infinity $h_{\frac{1}{1+m}}$ and self-dual conformal structure.

Remark : As the parameter can take negative values, we change the original m^2 to a simple m.
Remark 2 : m=0 is the hyperbolic metric in spherical coordinates.

Definition 5. In this paper, we call Pedersen metric with parameter m the Riemannian manifold $(B_1(\mathbb{R}^4), g_m)$

2 Riemann tensor of Pedersen metrics and norm L^2 of Weyl curvature

For the following, we fix $m > -1$.

There is a natural orthonormal frame $(X_i)_{i\in[\![0,4]\!]}$ for the Pedersen metric of parameter m , namely :

$$X_0 = \frac{1-r^2}{2}\sqrt{\frac{1+mr^4}{1+mr^2}}\partial_r \qquad X_1 = \frac{1-r^2}{2r\sqrt{1+mr^2}}\tilde{X}_1 \qquad X_2 = \frac{1-r^2}{2r\sqrt{1+mr^2}}\tilde{X}_2 \qquad X_3 = \frac{1-r^2}{2r}\sqrt{\frac{1+mr^2}{1+mr^4}}\tilde{X}_3$$

Then a lengthy calculation using the Koszul formula allows to calculate the Riemannian tensor of this metric from its Levi-Cevita connexion. We get :

Proposition 6. *Let m>-1, g_m the Pedersen metric of parameter m, and R the riemannian tensor of g_m. In the basis (X_i), we have :*

$$R_{0101} = R_{0202} = R_{1313} = R_{2323} = -1 - \frac{m}{2}\left(\frac{1-r^2}{1+mr^2}\right)^3,$$

$$R_{0303} = R_{1212} = -1 + m\left(\frac{1-r^2}{1+mr^2}\right)^3,$$

$$R_{0123} = -\frac{1}{2}R_{0312} = \frac{m}{2}\left(\frac{1-r^2}{1+mr^2}\right)^3.$$

And the terms that do not come from symmetries of these are zero.

Remark : We can from here really easily find again the result in [3] by Cortés and Saha about the parameter for which the metric has negative sectional curvature everywhere ($m \leqslant 1$).

Corolary 7. *Let m>-1, g_m the Pedersen metric of parameter m, and W the Weyl curvature tensor of g_m. If we denote dV_{g_m} the volume form associated to g_m, we have :*

$$\int_M ||W||^2_{g_m} dV_{g_m} = 8\pi^2\frac{m^2}{(1+m)^2}.$$

Proof. First, let's note that since g_m is Einstein, if we note $\owedge$ the Kulkarni–Nomizu product, we have $W = Riem(g_m) + g_m \owedge g_m$. So the norm of the Weyl curvature as an element of $S^2\Lambda^2M$ is easy to compute, particularly using the orthonormal basis (X_i) :

$$\begin{aligned}
||W||^2_{g_m} &= W^2_{0101} + 2W^2_{0123} + W^2_{0202} + 2W^2_{0213} + W^2_{0303} + 2W^2_{0312} + W^2_{1212} + W^2_{1313} + W^2_{2323},\\
&= 4\left(\frac{m(1-r^2)^3}{2(1+mr^2)^3}\right)^2 + 2\left(\frac{m(1-r^2)^3}{(1+mr^2)^3}\right)^2 + 12\left(\frac{m(1-r)^3}{2(1+mr^2)^3}\right)^2,\\
&= \frac{6m^2(1-r^2)^6}{(1+mr^2)^6}.
\end{aligned}$$

Now to integrate with ease, we choose to work with spherical coordinates, and note that

$$dV_{g_m} = \left(\frac{2}{1-r^2}\right)^4 (1+mr^2)r^3\sin^2(\psi)\sin(\theta)dr\wedge d\psi\wedge d\theta\wedge d\phi,$$

to conclude that

$$\int_M ||W||^2_{g_m} dV_{g_m} = 8\pi^2\int_0^1 24m^2\frac{r^3(1-r^2)^2}{(1+mr^2)^5}dr.$$

and from here, as :

$$m^2r^3(1-r^2) = \frac{r}{m}(1+mr^2)^3 - \frac{(2m+3)r}{m}(1+mr^2)^2 + \frac{(m^2+4m+3)r}{m}(1+mr^2) - \frac{m^2+2m+1}{m}r,$$

a simple computation allows to conclude that

$$\int_M ||W||^2_{g_m} dV_{g_m} = 8\pi^2 \frac{m^2}{(1+m)^2}.$$

□

3 Renormalized volume of Pedersen metrics

From the theorem 0.1 of [4], we have that for (M,g) a complete AH Einstein 4-manifold,

$$\frac{1}{8\pi^2}\int_M ||W||^2_{g_m} dV_{g_m} = \chi(M) - \frac{3}{4\pi^2}V(M,g),$$

where we have denoted $\chi(M)$ the Euler characteristic and $V(M,g)$ the renormalised volume of g (see again [4] for the definition), and W the weyl curvature of g.

Now since the Pedersen metrics are complete AH Einstein manifold on the 4-ball, we immediately get :

Theorem 8. *Let m>-1 and g_m be the Pedersen metric of parameter m, the renormalised volume of this Pedersen metric is :*

$$V(B_1(\mathbb{R}^4), g_m) = \frac{4\pi^2}{3}\left(1 - \frac{m^2}{(1+m)^2}\right).$$

Remark :This gives here a result which could be found again in a more general but less explicit way in the following preprint [8].

4 Sign of the conformal Yamabe energy of the conformal infinity

Let's recall that for a riemannian compact manifold without boundary (N,$\tilde{g}$) of dimension n>2, we have a conformal invariant : the conformal Yamabe energy defined as :

$$Q(g) := \frac{\int_N Scal_g dV_g}{\left(\int_M dV_g\right)^{\frac{n-2}{n}}}, \qquad Y([\tilde{g}]) := \inf_{g\in[\tilde{g}]} Q(g).$$

Remark :A first point to notice is that if the metric $\tilde{g}$ has a scalar curvature of constant sign, then the sign of the conformal Yamabe energy is the same as the sign of the scalar curvature . In the case $Scal_{\tilde{g}} \leqslant 0$, this is direct as Y is an infimum and in the case $Scal_{\tilde{g}} \geqslant 0$ it results of the transformation (with $c_n = \frac{4(n-1)}{n-2}$)

$$Q(u^{\frac{4}{n-2}}\tilde{g}) = \frac{\int_N c_n|\nabla u|^2_{\tilde{g}} + u^2 Scal_{\tilde{g}} dV_{\tilde{g}}}{\left(\int_M u^{\frac{2n}{n-2}} dV_{\tilde{g}}\right)^{\frac{n-2}{n}}}$$

We want to know the sign of the conformal Yamabe energy for the conformal infinity of the Pedersen metrics, the Berger spheres.

From the result in part 3.4.2 of the Petersen's book ([5]), it is easy to get that $h_{\frac{1}{1+m}}$ (see 3 for the definition) has a scalar curvature of $8 - \frac{2}{1+m}$, which is positive for $m \geqslant -\frac{3}{4}$ and negative otherwise. It immediately follows that the sign of the conformal Yamabe energy $Y(g_m)$ has the same distinction.

5 Sign of the conformal Yamabe-Escobar energy

The generalization of the conformal Yamabe energy is the conformal Yamabe-Escobar (first type) energy, defined for a riemannian compact manifold with boundary $(\overline{M}, \partial M, \tilde{g})$ as :

$$Q^1(g) := \frac{\int_{\overline{M}} Scal_g dV_g + 2\oint_{\partial M} H_g dV_{g_{|\partial M}}}{\left(\int_M dV_g\right)^{\frac{n-2}{n}}}, \qquad Y^1([\tilde{g}]) := \inf_{g\in[\tilde{g}]} Q(g).$$

Where H_g is the mean curvature of ∂M (with inward normal vector).

It was shown that in the case were the conformal Yamabe energy of the boundary is positive, then the first Yamabe-Escobar energy is also positive, see [6] and citation within (in particular [7]). In this section we find a better estimation of the parameters for which the first Yamabe-Escobar energy is positive than the one we get from this propriety ($m \geqslant -\frac{3}{4}$).

We have the following :

Proposition 9. *Let 0>m>-1, g_m the Pedersen metric of parameter m, and $\overline{g_m} := (\frac{1-r^2}{2})^2 g_m$ the compactified metric associated.*
Then $Y^1([\overline{g_m}])$ has the same sign as

$$f(m) := 4 - \frac{1}{\sqrt{-m}} \ln\left(\frac{1+\sqrt{-m}}{1-\sqrt{-m}}\right),$$

where f is an increasing function with $f(-0.92) \approx -0.03$ and $f(-0.91) \approx 0.07$

Proof. Similarly to the case without boundary, we have the law of transformation under conformal change :

$$Q(u^{\frac{4}{n-2}}\tilde{g}) = \frac{\int_{\overline{M}} c_n|\nabla u|^2_{\tilde{g}} + u^2 Scal_{\tilde{g}} dV_{\tilde{g}} + 2\oint_{\partial M} H_{\tilde{g}} u^2 dV_{\tilde{g}_{|\partial M}} dV_{\tilde{g}}}{\left(\int_M u^{\frac{2n}{n-2}} dV_{\tilde{g}}\right)^{\frac{n-2}{n}}}$$

With the same argument as in section 4, if $g \in [\tilde{g}]$ is such that $R_g = 0$ and with mean curvature of constant sign, then the first Yamabe-Escobar conformal energy has the same sign as the mean curvature.

Let's recall the transformation law : if h is a riemannian metric on a manifold of dimension n , with ν the inward normal vector, and u is a positive function, and if we define $\tilde{h} := u^{\frac{2}{n-2}} h$, then

$$\Delta_h u + \frac{n-2}{4(n-1)} R_h u = \frac{n-2}{4(n-1)} u^{\frac{n+2}{n-2}} R_{\tilde{h}} \tag{5.1}$$

$$H_{\tilde{h}} := u^{-\frac{n}{n-2}} \left(-2\frac{n-1}{n-2}\frac{\partial u}{\partial \nu} + H_h u\right) \tag{5.2}$$

Applying 5.1 to $\overline{g_m} := (\frac{1-r^2}{2})^2 g_m$ and using the fact that g_m is Einstein with $R_{g_m} = -12$, we get that $R_{\overline{g_m}} = -6\frac{8mr^2}{1+mr^2}$.

To get the mean curvature, recall that :

$$\overline{g_m} = \frac{1+mr^2}{1+mr^4}\mathbf{dr}^2 + r^2(1+mr^2)(\sigma_1^2+\sigma_2^2) + \frac{r^2(1+mr^4)}{1+mr^2}\sigma_3^2$$

so we have again a natural orthonormal basis :

$$X_0 = -\sqrt{\tfrac{1+mr^4}{1+mr^2}}\partial_r := f_0\partial_r \qquad X_1 = \tfrac{1}{r\sqrt{1+mr^2}}\tilde{X}_1 := f_1\tilde{X}_1$$
$$X_2 = \tfrac{1}{r\sqrt{1+mr^2}}\tilde{X}_2 := f_2\tilde{X}_2 \qquad X_3 = \tfrac{1}{r}\sqrt{\tfrac{1+mr^2}{1+mr^4}}\tilde{X}_3 := f_3\tilde{X}_3,$$

and again with the Koszul formula, we get the second form fundamental of the boundary as

$$\Pi(X_i, X_j) := \delta_{ij}\delta_{i\neq 0} f_0 \frac{\partial_r f_i}{f_i} X_0,$$

and in particular,

$$H_{\overline{g_m}} = 3\frac{1+2m}{1+m}$$

We will look for an analytic $u := \sum a_k r^k$ such that $\tilde{g_m} := u^{\frac{2}{n-2}}\overline{g_m}$ has a null scalar curvature $R_{\tilde{g_m}}$.

The fact that u must be a solution to

$$\begin{aligned} 0 &= \Delta_{\overline{g_m}}(u) + \frac{1}{6}R_{\overline{g_m}}u, \\ &= -\frac{1}{r^3(1+mr^2)}\partial_r\left(r^3(1+mr^2)\frac{1+mr^4}{1+mr^2}\partial_r u\right) - \frac{8mr^2}{1+mr^2}u, \end{aligned}$$

tells us that $a_1 = a_2 = a_3 = 0$ and for $k \geqslant 0$, $a_{k+4}(k+4)(k+6) = (-m) \times (k(k+6)+8))\, a_k$, so that such a solution must be :

$$u(r) := a_0 \sum_{k\geqslant 0} \frac{(-m)^k}{2k+1} r^{4k}.$$

For $a_0 = 1$ this solution is well defined and positive on [0,1] for $-1 < m < 0$, and in what follow, we will call $u_m(r) := \sum \frac{(-m)^k}{2k+1} r^{4k}$.

As said in the beginning of the proof, we just have to look at the sign of the mean curvature of $\tilde{g_m} := u_m^{\frac{2}{n-2}}\overline{g_m}$ to conclude. From 5.2, this is the same (since m>-1) as looking at the sign of

$$(1+2m)u_m(1) + (1+m)u_m'(1)$$

$$\begin{aligned} (1+2m)u_m(1) + (1+m)u_m'(1) &= (1+2m)\sum_{k\geqslant 0}\frac{(-m)^k}{2k+1} + (1+m)\sum_{k\geqslant 0}\frac{4k(-m)^k}{2k+1} \\ &= 1 + \sum_{k\geqslant 1}\left(\frac{1}{2k+1} - 2\frac{1}{2k-1} + \frac{4k}{2k+1} - \frac{4k-4}{2k-1}\right)(-m)^k \\ &= 1 - \sum_{k\geqslant 1}\frac{(-m)^k}{2k+1} \\ &= 1 - \frac{1}{\sqrt{-m}}\sum_{k\geqslant 1}\frac{1}{2k+1}(\sqrt{-m})^{2k+1} \end{aligned}$$

We recognise in the sum a primitive of the analytic fonction $\sum_{k\geqslant 1} X^{2k} = \frac{X^2}{1-X^2}$. Since such a primitive is of the form $x \mapsto -x + \frac{1}{2}\ln(\frac{1+x}{1-x}) + c$ and there is no constant (as a fonction of $\sqrt{-m}$) term, we get that :

$$(1+2m)u_m(1) + (1+m)u_m'(1) = 2 - \frac{1}{2\sqrt{-m}} ln(\frac{1+\sqrt{-m}}{1-\sqrt{-m}}).$$

Since the last affirmation of the proposition easily follows from a simple calculation of the derivative of the sum, we have proven our claim. □

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