Generating Entanglement
from Frustration-Free Dissipation

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Before we start, a couple of things on...

Attainability of Quantum Cooling, Third Law, and all that...

Open System Dynamics

Bipartition:
S: system of interest (finite dimensional);
B: environment/bath

Unitary joint dynamics:
\[ \rho_{SB}(t) = U(t)\rho_S(0) \otimes \rho_B(0)U(t)^\dagger \]

Assume the joint system is controllable/ \( U \) is arbitrary.

How well can we cool (or purify) the system? Are there intrinsic limits?

Note: with controllability, purification and ground-state cooling are equivalent.

Def: By \( \varepsilon \)-purification at time \( t \) we mean that exists \( U \) and a pure state such that:
\[ \rho'_S = \text{Tr}_B(\rho_{SB}(t)) \text{ satisfies } \|\rho'_S, |\psi\rangle \langle \psi|\|_1 \leq \varepsilon, \forall \rho_S \]
Subsystem Principle for Purification


Most general subsystem: associated to a tensor factor of a subspace,

\[ \mathcal{H}_B = (\mathcal{H}_{S'} \otimes \mathcal{H}_F) \oplus \mathcal{H}_R \]

✓ [Thm] If the joint system is completely controllable and initially factorized:

1. \( \mathcal{E} \) - purification can be achieved if \( \| \rho_B - \tilde{\rho}_B \| \leq \varepsilon \)

   for some:

   \[ \tilde{\rho}_B = (|\psi\rangle\langle\psi| \otimes \rho_F) \oplus 0_R \]

2. Exact (\( \varepsilon = 0 \)) purification if and only if \( \rho_B = (|\psi\rangle\langle\psi| \otimes \rho_F) \oplus 0_R \)

3. \( \mathcal{E} \) - purification is possible if

   \[ \varepsilon \geq \tilde{\varepsilon}(\rho_B) \equiv \tilde{\varepsilon} = 1 - \sum_{j=1}^{d_F} \lambda_j(\rho_B) \geq 0 \]

Strategy: Swap the state of the system with the subsystem one.
Claim: (1) is actually “if and only if”, i.e. either swap works or nothing does.
Example: Thermal Bath States

✓ In [Wu, Segal & Brumer, No-go theorem for ground state cooling given initial system-thermal bath factorization. Sci.Rep. 2012], it is claimed that a no-go theorem for cooling holds, under similar (actually weaker) hypothesis.

Ok, for perfect cooling, but arbitrarily good cooling is possible!

✓ E.g. Qubit target:

1) Choose a good subspace;
2) Construct a 2D subsystem;
3) Swap the state with the qubit of interest;

\[
\rho_E = \sum_{j=1}^{\infty} p_j |j\rangle \langle j|\]
What is this useful for? Why did I speak about this?
First steps towards a general/systematic construction that achieve optimal purity/ground state cooling for the target system. Other connections to thermodynamics...

It is reminiscent of the third law: attaining perfect cooling would imply using infinitely many degrees of freedom, and (likely) infinite energy.

Usual problem: finding a formulation of the third law with clear hypothesis.

It is connected to Landauer’s principle [David’s lectures]:
- Exact purification is erasure.
- Swap operations seem to be the key.
INTRODUCTION (to the main talk)

Open quantum systems, quantum dynamical semigroups and long-time behavior.

Dissipative state preparation.
Open System Dynamics

Bipartition:
S: system of interest (finite dimensional);
B: uncontrollable environment
Full description via joint Hamiltonian:
\[ H = H_S \otimes I_B + I_S \otimes H_B + H_{SB} \]

Unitary joint dynamics:
\[ \rho_{SB}(t) = U(t)\rho_S(0) \otimes \rho_B(0)U(t)^\dagger \]

Under suitable Markovian approximation (weak coupling, singular),
generating an effective memoryless, time-invariant bath,
we can obtain convenient reduced dynamics:
\[ \rho_S(t) = \mathcal{E}_t(\rho_S(0)), \quad \{ \mathcal{E}_t = e^{\mathcal{L}t} \}_{t \geq 0} \]
Quantum Dynamical Semigroups

- Assume the dynamics to be a semigroup (i.e. the environment to be memoryless). The general form of the Markovian generator is:

\[ \hat{\rho}_t = \mathcal{L}(\rho) = -i[H, \rho_t] + \sum_{k=1}^{p} L_k \rho_t L_k^\dagger - \frac{1}{2} \{ L_k^\dagger L_k, \rho_t \} \]

- Linear CPTP system: exponential convergence, well-known theory;
- Uniqueness of the equilibrium implies it is attracting.

**Question:**
Where does, or can the state asymptotically converge?
[Davies Generator, 1976]

Under weak-coupling limit, consider:

\[ H_{SB} = \sum_{\alpha} S^{\alpha} \otimes B^{\alpha} \]

\[ e^{iH_{St}} S^{\alpha} e^{-iH_{St}} = \sum_{\omega} S^{\alpha}(\omega)e^{i\omega t} \]

we get:

\[ \mathcal{L}(\rho) = -i[H_{S}, \rho] + \sum_{\omega, \alpha} g^{\alpha}(\omega)(S^{\alpha}(\omega)\rho S^{\alpha\dagger}(\omega) \]

\[ -\frac{1}{2}\{S^{\alpha\dagger}(\omega)S^{\alpha}(\omega), \rho\} \]

Let B be a bath at inverse temperature \( \beta \). Under some additional condition (irreducibility of algebra), it is possible to show that it admits the **Gibbs state** as unique equilibrium:

\[ \rho_{\beta} = \frac{e^{-\beta H_{S}}}{\text{Tr}(e^{-\beta H_{S}})} \]

Physically consistent, expected result.

Why keep looking into it?
New challenge:

**Engineering of open quantum dynamics**

S: system of interest;
E: environment, including possibly:
   B: uncontrollable environment
   A: auxiliary, engineered system (quantum and/or classical controller)

**Full description via Joint Hamiltonian:**

\[ H = (H_S \otimes I_E + I_S \otimes H_E + H_{SE}) + H_c(t) \]

**Reduced description via controlled generator** (not just weak coupling!):

\[ \mathcal{L}_t(\rho) = -i[H_S + H_C(t), \rho] + \sum_k \lambda_k(t)(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\}) \]
Two Prevailing & Complementary Approaches:

I. **Environment as Enemy**: we want to “remove” the coupling.
   Noise suppression methods, active and passive, including
   *hardware engineering, noiseless subsystems, quantum error correction, dynamical decoupling*;

II. **Environment as Resource**: we want to “engineer” the coupling.
   *Needed* for state preparation, open-system simulation, and much more...
Dissipation for Information Engineering

- Dissipation allows for:
  - Entanglement Generation
  - Computing
  - Open System Simulator

Quantum computation and quantum-state engineering driven by dissipation
Frank Verstraete, Michael M. Wolf, and J. Ignacio Cirac

An open-system quantum simulator with trapped ions
Julio T. Barreiro, Markus Müller, Philipp Schmidtler, Daniel Nigg, Thomas Monz, Michael Chwalla, Markus Hennrich, Christian F. Roos, Peter Zoller, and Rainer Blatt

Autonomously stabilized entanglement between two superconducting quantum bits

Scalable dissipative preparation of many-body entanglement
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(Dated: January 28, 2015)
Focus: Dissipative State Preparation

- Can we design an environment that “prepares” a desired state?
  
  Naive Answer: **YES!**
  
  **Mathematically easy:**
  
  \[
  \dot{\rho} = \mathcal{L}(\rho) = \mathcal{E}(\rho) - \rho, \quad \mathcal{E}(\rho) = \rho_{\text{target}} \text{trace}(\rho)
  \]

- Choice is non-unique: “simple” Markov evolutions that do the job always exist:
  - **Pure state**: generator with single L is enough, with ladder-type operator;
    
  
  - **Mixed state**: generator with H and a single L (tri-diagonal matrices);
    
    [T-Schirmer-Wang, IEEE T.A.C., 2010]

- However...
  
  **Can we do it with experimentally-available controls?** **Typically NOT.**
  We need to take into account:
  
  - The control method [open-loop, switching, feedback, coherent feedback,...]
  - Limits on speed and strength of the control actions;
  - Faulty controls;
  - **Locality constraints.**

**Physical relevance; Key limitation for large-scale entanglement generation**
Main Task

Understanding the role of locality constraints and providing general design rules for dissipative state preparation
Multipartite Systems and Locality

- Consider $n$ finite-dimensional systems, indexed:

$$\mathcal{H}_Q = \bigotimes_{a=1}^{n} \mathcal{H}_a$$

- **Locality notion:** from the start, we specify subsets of indexes, or neighborhoods, corresponding to group of subsystems:

$$\mathcal{N}_1 = \{1, 2\}$$
$$\mathcal{N}_2 = \{1, 3\}$$
$$\mathcal{N}_3 = \{2, 3, 4\}$$

...on which “we can act simultaneously”: how?

- **Neighborhood operator:** $M_k = M_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k}$
- A Hamiltonian is said **Quasi-Local (QL)** if:

$$H = \sum_k H_k, \quad H_k = H_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k}$$

Neighborhood operators will model the allowed interactions.
Constraints: Frustration-Freeness & Locality

• Consider $n$ finite-dimensional systems, and a fixed locality notion.

- Quasi-Local (QL) if

  \begin{align*}
  \mathcal{L} &= \sum_{k} \mathcal{L}_{\mathcal{N}_k} \otimes I_{\mathcal{N}_k} \\
  H &= \sum_{k} H_k, \quad H_k = H_{\mathcal{N}_k} \otimes I_{\mathcal{\bar{N}}_k} \quad L_{k,j} = L_{\mathcal{N}_k(j)} \otimes I_{\mathcal{\bar{N}}_k}
  \end{align*}

  or, explicitly:

- Frustration-Free (FF) [Kastoryano, Brandao, 2014; Johnson-T-Viola, 2015] if it is QL and

  \[ \mathcal{L}(\rho) = 0 \implies \mathcal{L}_{\mathcal{N}_k} \otimes I_{\mathcal{\bar{N}}_k}(\rho) = 0 \]

• A state is a global equilibrium if and only if it is so for the local generators.
Frustration-Freeness as “Robustness”

- **Inspired by:** Let $\rho = |\psi\rangle\langle\psi|$ be a ground state of a QL Hamiltonian:

$$H = \sum_k H_k, \quad H_k = H_{N_k} \otimes I_{\bar{N}_k}$$

**Def:** If all $|\psi\rangle$ are eigenvectors of minimal energy for both the global and neighborhood Hamiltonians, namely:

$$\langle\psi|H|\psi\rangle = \min \sigma(H) \implies \langle\psi|H_k|\psi\rangle = \min \sigma(H_k), \quad \forall k.$$  

such an $H$ is said Frustration-Free (FF).

- If the global ground state is unique, we can obtain it by simultaneously “cooling” the system on each neighborhood, and it does not change if we scale the neighborhood terms:

$$H = \sum_k \alpha_k H_k, \quad \alpha_1, \ldots, \alpha_k \in \mathbb{R},$$  

No fine tuning!

- Same robustness holds for a FF generator and its equilibria.

**Key Property:**

Summing neighborhood terms in FF generators does not add equilibria.
Asymptotic State Stabilization

\[ \rho \in \mathcal{D}(\mathcal{H}) := \{ \rho = \rho^\dagger > 0, \text{trace}(\rho) = 1 \} \]

**Task:** Prepare a target state irrespective of the initial one.

**When is it possible with FF dynamics?**

**Define:** \( \rho_d \) is **Frustration-Free Stabilizable [FFS]** if it is

1) **Invariant:** \( \mathcal{L}(\rho_d) = 0 \)

2) **Attracting:** \( \forall \rho \in \mathcal{D}(\mathcal{H}), \lim_{t \to +\infty} e^{\mathcal{L}t}(\rho) = \rho_d \)
   for some quasi-local FF dynamics.

**Relevance:** Basic task of QIP; Cooling to ground state; *Entanglement generation and preservation*; One-way computing; Metropolis-type sampling.
General Fact in Dissipative Design:

Making a state invariant is the hard part; After that, making everything else converge to it is (relatively) easy.

Invariance-ensuring generators are a zero-measure set. In there, stabilizing ones are generic.
[T. et al, IEEE TAC 2012]
[T., Viola, QIC 2014]
Characterizing Invariance: Schmidt Span

• When is a state invariant for a FF generator?

**FF hypothesis:** we have an equilibrium *if and only if*

\[ \mathcal{L}_{\mathcal{N}_k} \otimes \mathcal{I}_{\overline{\mathcal{N}}_k} (\rho_d) = 0, \ \forall k \]

Consider one neighborhood and its complement:

\[ \mathcal{N}_k \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \cdots \quad \overline{\mathcal{N}}_k \]

• Write the **operator Schmidt decomposition** with respect to the partition \( \mathcal{H}_{\mathcal{N}_k} \otimes \mathcal{H}_{\overline{\mathcal{N}}_k} \):

\[ \rho_d = \sum_j A_j \otimes B_j \]

• Define the **Schmidt Span**:

\[ \Sigma_k(\rho_d) = \text{span}\{A_j\} \]

• **Lemma:** \( \rho_d \) is invariant if and only if

\[ \Sigma_k(\rho_d) \subseteq \ker(\mathcal{L}_{\mathcal{N}_k}), \ \forall k \]

• This implies invariance of the reduced state:

\[ \rho_{\mathcal{N}_k} = \text{trace}_{\overline{\mathcal{N}}_k}(\rho_d) \]
Invariance is characterized!
Now we have a good idea of what the stabilizing QL generators have to do!

I. Locally preserve the *Schmidt spans*;
II. Perturb and destabilize everything else;

However....
Towards Stabilization: Distorted Algebras

- \( \mathcal{L}_{\mathcal{N}_k} \) is the generator of a CPTP semigroup. The structure of the fixed points is well known [Ng, Blume-Kohut, Viola; Wolf], they form a distorted algebra:

\[
\ker(\mathcal{L}_{\mathcal{N}_k}) = \left( \bigoplus_{\ell} \mathcal{B}(\mathcal{H}_\ell^A) \otimes \tau_\ell \right) \oplus \mathbb{O}
\]

- **Why is this important?** We (may) need to enlarge the set of invariant operators with respect to (just) the Schmidt span (~no pancake theorem).

- Let \( \rho \) be a maximum rank fixed state for \( \mathcal{L}_{\mathcal{N}_k} \). Given the Schmidt span, we can construct the **minimal distorted algebra** \( \mathcal{A}_k \) so that \( \Sigma_k(\rho_d) \subseteq \mathcal{A}_k \), by making it closed with respect to:
  
  (i) Linear combinations and adjoint;
  (ii) Modified product:
  
  \[
  X \times_\rho Y = X \rho^{-1}Y
  \]
  
  with: \( \rho = \rho_{\mathcal{N}_k} \).

**Lemma:** \( \rho_d \) is invariant if and only if \( \mathcal{A}_k \subseteq \ker(\mathcal{L}_{\mathcal{N}_k}), \quad \forall k \)

- As we hoped for, for generic states, the condition turns out to be not only necessary, but also sufficient....
Main Result: Full-rank States

• For each neighborhood, we can construct the enlarged distorted algebra:
  \[ \mathcal{A}_k^g = \mathcal{A}_k \otimes \mathcal{B}(\mathcal{N}_k) \]

**Theorem:** Assume \( \rho_d \) is full rank. Then it is FFS if and only if

\[ \cap_k \mathcal{A}_k^g = \text{span}(\rho_d) \]

• **Proof idea:** **Necessity** follows from Lemmas. Proving **sufficiency**, we consider an explicit choice of generators:
  \[ \mathcal{L}_{\mathcal{N}_k}(\rho) = \mathcal{E}_{\mathcal{N}_k}(\rho) - \rho; \]

with **CPTP non-orthogonal projections onto the minimal distorted algebras** (dual of conditional expectations):

\[ \mathcal{E}_{\mathcal{N}_k}(\rho) \in \mathcal{A}_k; \quad \mathcal{E}^2_{\mathcal{N}_k}(\rho) = \mathcal{E}_{\mathcal{N}_k}(\rho). \]

**Key technical point:** proving the dynamics is frustration free. Then the shared equilibrium is unique, and there cannot be any other one.
Key Result

- Assume that for all $k$, $\mathcal{L}_k = \mathcal{L}_{N_k}(j) \otimes I_{\tilde{N}_k}$:
  \[
  \text{alg}(\mathcal{L}_k) \subseteq \text{alg}(\mathcal{L})
  \]

- **Note:** This is true if there is no Hamiltonian;

- Then we have the following chain of equality/inclusions (with full rank states):
  \[
  \ker(\mathcal{L}) = \rho^{\frac{1}{2}} \ker(\mathcal{L}^\dagger) \rho^{\frac{1}{2}} = \rho^{\frac{1}{2}} \text{alg}\{\mathcal{L}\}' \rho^{\frac{1}{2}}
  \]
  \[
  \ker(\mathcal{L}_k) = \rho^{\frac{1}{2}} \ker(\mathcal{L}_k^\dagger) \rho^{\frac{1}{2}} = \rho^{\frac{1}{2}} \text{alg}\{\mathcal{L}_k\}' \rho^{\frac{1}{2}},
  \]

- This proves that the chosen generator is FF (does not have Hamiltonian).
Main Result: Comments and Extensions

• **What is this useful for?**
  Allows for checking if a target state is *in principle* stabilizable under given (and strict) locality constraints, with frustration-free dynamics. The checking procedure can be automated.

• **If full quasi-local control/simulation is available,** we give a recipe for stabilization of desired state, where possible. More constraints can be included later, e.g. via suitable numerical methods. Our result gives a preliminary check.

• **It can be seen as a way to construct quantum “sampler”**
  
  [Kastoryano,Brandao, 2014] - *a way to obtain a density we do not have.* Complements to other work by Temme, Cubitt, Wolf, and co-workers where focus is on studying the scalability/speed, when convergence is already guaranteed.

• **For general states,** the same necessary condition holds. However, we do not have a full proof for sufficiency.
  
  *An additional condition is used, but we conjecture is not needed.*

• Full and simpler characterization for *pure states.*
Specialization for Pure States

- For each *neighborhood* compute the reduced states;

  
  \[
  \mathcal{N}_1 \quad \mathcal{N}_2 \quad \mathcal{N}_3 \quad \Rightarrow \quad \rho_{\mathcal{N}_1}, \rho_{\mathcal{N}_2}, \rho_{\mathcal{N}_3}
  \]

- **Beingρ_d pure, it can be shown that:** \( A_k = \Sigma_k(\rho) = \mathcal{B}(\text{supp}(\rho_{\mathcal{N}_k})) \)

- Instead of intersecting distorted algebras, I can just look at heir supports.

- For each neighborhood calculate the *support* of the reduced state times the identity on the rest:

  \[
  \mathcal{H}_{\mathcal{N}_k} = \text{supp}(\rho_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k})
  \]

- **Theorem** [T.-Viola, 2012]:

  \[
  \mathcal{H}_0 := \bigcap_k \mathcal{H}_{\mathcal{N}_k} = \text{supp}(\rho)
  \]

  *if and only if* \( \rho \) *is FFS;*

**IDEA:** the support is “where the probability is”;
Locally I only see the reduced state, and I try to prepare it.
**Equivalent characterization:** \( \rho = |\psi\rangle \langle \psi| \) is FFS if and only if it is the unique ground state of a Frustration-Free QL Hamiltonian, that is:

- There exists a QL Hamiltonian for which \( |\psi\rangle \) is the unique ground state and

\[
H = \sum_k H_k, \quad H_k = H_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k}
\]

such that \( \langle \psi | H_k | \psi \rangle = \min \sigma(H_k), \quad \forall k \).

**Proof:** It suffices to choose \( H_k = \Pi_{\mathcal{N}_k}^\perp \otimes I_{\bar{\mathcal{N}}_k}, \Pi_{\mathcal{N}_k}^\perp \) projects on \( \text{supp}(\rho_{\mathcal{N}_k})^\perp \).

- We retrieve the FF Hamiltonian - the analogy with FF generators fully works!

- Interesting connection to physically-relevant cases, and previous work by Verstraete, Perez-Garcia, Cirac, Wolf, B. Kraus, Zoller and co-workers.

**Differences:**

In their setting, the proper locality notion is induced by the target state itself. In our setting, the locality is fixed a priori. We also prove necessity of the condition.
Applications

Generating entanglement from quasi-local dissipation.
Is Frustration-Free Enough for Pure States?

• Which states are FFS? Using our test, it turns out that...

  • All product states are FFS.
  
  • **GHZ states (maximally entangled) and W states are not FFS**
    
    Unless we have neighborhoods that cover the whole network/nonlocal interactions;
    
    \[ \rho_{\text{GHZ}} = |\Psi\rangle\langle\Psi|, \quad |\Psi\rangle \equiv |\Psi_{\text{GHZ}}\rangle = (|0000\rangle + |1111\rangle)/\sqrt{2}. \]
    
  • **Any graph state is FFS with respect to the locality induced by the graph**;
    
    To each node is assigned a neighborhood, which contain all the nodes connected by edges.
    
    \[ U_G|00\ldots0\rangle = |\varphi_{\text{graph},0}\rangle \]
    
  • **Generic (injective) MPS/PEPS are FFS** for some locality definition...
    
    Neighborhood size may be big! [see work by Peres-Garcia, Wolf, Cirac and co-workers]
    
  • Some **Dicke states** *that are not graph* can be stabilized!
    
    E.g. on linear graph with NN interaction:
    
    \[ \frac{1}{\sqrt{6}}(|1100⟩ + |1010⟩ + |0110⟩ + |0101⟩ + |0011⟩ + |1001⟩) \]
Is Frustration-Free Enough for Mixed States?

• Which states are FFS? Using our test, it turns out that...
  
  • There are non-entangled states that are not FFS!
    \[ \rho_{\text{sep}} = \frac{1}{2}(00 \otimes n + 11 \otimes n). \]
  
  • Product graph states are FFS, with locality induced by the graph.
    \[ U_G : \text{prepares the graph basis.} \]
    \[ \rho_G = U_G \left( \bigotimes_{j=1}^{n} \rho_j \right) U_G^\dagger, \]

  • Commuting Gibbs states are FFS, with locality generated by the Hamiltonian (NNN).
    \[ \rho_{\beta} = \frac{e^{-\beta H}}{\text{Tr}(e^{-\beta H})} \]
    with:
    \[ H = \sum_k H_k, \quad H_k = H_{N_k} \otimes I_{\bar{N}_k}, \quad [H_k, H_j] = 0, \quad \forall j, k \]

  • Some non-commuting Gibbs states are FFS!
    e.g. zero-temperature states as certain Dicke states, and their mixtures with e.g. GHZ states!
Summary and Outlook

- Locality constraints are key for state preparation.
- We obtain a way to check if a target state is “compatible” with given constraints.
- If it is, we provide intuition on what the stabilizing dynamics should do, as well as one that works.
- We show that there are new (non commuting) states that are genuinely FFS.
- It is possible to relax invariance constraints for preparation of GHZ and W. Two steps: first initialization and then conditional stabilization.

➡ Next:
Relation to Encoders and Memories; Numerical approaches; When is FFS generic? More general constraints.

➡ Open problems: The above mentioned conjecture and...
   Better classification of FFS states; Scalable non-commuting Gibbs; Stabilization beyond Frustration-Free; Discrete-time models; Speed of convergence (when the system size grows - scalability).
A case study: GHZ States

• **GHZ states are never QLS for non trivial topology:**

\[
\rho_{\text{GHZ}} = |\Psi\rangle\langle\Psi|, \quad |\Psi\rangle \equiv |\Psi_{\text{GHZ}}\rangle = (|000\ldots0\rangle + |111\ldots1\rangle)/\sqrt{2}.
\]

By symmetry, \(\mathcal{H}_0\) **must contain** \(|000\ldots0\rangle, |111\ldots1\rangle\).

Hence the following orthogonal states **must remain** stable for the QL dynamics.

\[
|\Psi_{\text{GHZ}+}\rangle = (|000\ldots0\rangle + |111\ldots1\rangle)/\sqrt{2}; \\
|\Psi_{\text{GHZ}-}\rangle = (|000\ldots0\rangle - |111\ldots1\rangle)/\sqrt{2};
\]

We need to "select" the right one How?

\[
\sigma_x^{\otimes n}|\Psi_{\text{GHZ}+}\rangle = |\Psi_{\text{GHZ}+}\rangle \quad \sigma_x^{\otimes n}|\Psi_{\text{GHZ}-}\rangle = -|\Psi_{\text{GHZ}-}\rangle
\]

• **Trick:** First prepare the system in the +1-eigenspace of \(\sigma_x^{\otimes n}\) (e.g. \(|+\rangle^{\otimes n}\)).
Then we show there exists a QL \(\{\mathcal{E}_t\}_{t \geq 0}\) that prepares \(\mathcal{H}_0\) leaving the eigenspace invariant.

• By our Theorem, \(\rho_{\text{GHZ}}\) is **Conditionally QLS!** (scalable on the linear graph)
Conditional Preparation: Some Intuition

FFS Problem: unfeasible global stabilization task because I can only prepare (nec. cond.):

\[ \mathcal{H}_0 := \bigcap_k \mathcal{H}_{\mathcal{N}_k} \]

The necessity follows from:

\[ \forall t \geq 0 \; \mathcal{E}_t(\rho) = \rho \]

If we relax this assumptions, we can obtain scalable protocols!

\[ \forall t \geq T > 0 \; \mathcal{E}_t(\rho) = \rho \]

First I prepare a subspace that

1. is invariant for the QL sequence;
2. is attracted directly to \( \rho_d \)

**Problem**: finding such \( \mathcal{H}' \)!
Conditional Preparation: Definition & Result

- **Definition:** A state \( \rho = |\psi\rangle\langle\psi| \) is **Quasi-Local Stabilizable (QLS) conditional to** \( \mathcal{H}' \) if there exist a dynamical semigroup \( \{\mathcal{E}_t\}_{t \geq 0} \) such that
  \[
  \forall t \geq 0 \quad \mathcal{E}_t(\rho) = \rho \quad \lim_{t \to \infty} \|\rho_t - \rho\| = 0
  \]
  for every \( \rho_0 \) with support on \( \mathcal{H}' \).

- **Lemma:** It is not restrictive to take \( \mathcal{H}' \) invariant.

- **Theorem:** If \( \mathcal{H}' \)
  1. contains \( |\Psi\rangle \);
  2. is orthogonal to \( \mathcal{H}_0 \ominus \{ |\Psi\rangle \} \);
  3. is invariant for \( \{\mathcal{E}_t\}_{t \geq 0} \) that stabilizes \( \mathcal{H}_0 \);

Then \( \rho = |\psi\rangle\langle\psi| \) is QLS conditional to \( \mathcal{H}' \).