Thresholds for entanglement criteria in quantum information theory

Joint results with Nicolae Lupa, Ion Nechita and David Reeb

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- Entanglement via positive maps approach
- Reduction Criterion (RED)
- Absolute Reduction Criterion (ARED)
- Approximations of (A)SEP
- Thresholds for RED and ARED
- Comparing entanglement criteria via thresholds

Introduction: Entanglement versus separability

Entanglement=inseparability

Separable state:

$$\rho = \sum_i p_i e_i e_i^* \otimes f_i f_i^*, p_i \geq 0, \sum_i p_i = 1, e_i \in \mathbb{C}^n, f_i \in \mathbb{C}^k$$

Goal: efficient methods to characterize entangled states

PPT criterion:[1] if a state is separable, then the partial transpose respect to one of the subsystems is positive-semidef.

$$SEP := \{\rho_{AB} : \rho_{AB} - sep\} \subset PPT = \{\rho_{AB} / \rho^{\Gamma} \geq 0\}$$

Tool to detect entanglement: if the partial transpose is not positive-semidefinite, then the state is entangled

Question: exists other postive maps φ such that $(id \otimes \varphi)(\rho) \ngeq 0$, for some ρ entangled state

[1] A. Peres, Separability criterion for density matrices, PRL 77,1996

Separability criteria based on positive maps approach

Mathematical formulation[2]:

$$\rho \in \mathcal{M}_n \otimes \mathcal{M}_k \text{ separable iff}$$
$$\rho^{\varphi} := [\mathrm{id} \otimes \varphi](\rho) \geq 0, \forall \varphi \geq 0, \varphi : \mathcal{M}_k \to \mathcal{M}_m, \text{ all positive integers } m \in \mathbb{N}$$

$$SEP :\subset \{\rho : \rho^{\varphi} \geq 0\}$$

- transposition map $\varphi \Rightarrow$ Positive Partial Transposition (PPT)
- reduction map: $\varphi(X) := \mathrm{I} \cdot \mathrm{Tr}X - X \Rightarrow$

Reduction Criterion (RC)[3]:

$$\rho_{AB} - sep \Rightarrow \rho_A \otimes I_B - \rho_{AB} \geq 0 \quad , I_A \otimes \rho_B - \rho_{AB} \geq 0 \qquad (1)$$

$\rho_A = [id \otimes Tr](\rho_{AB})$ partial trace over the second subsystem

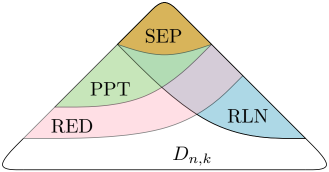
[2] Horodecki M, Separability of mixed states: necessary and sufficient conditions, Phys. Lett A, 1996

[3] Horodecki and all.'99, Cerf and all. '99

Separability via Reduction Criterion

- $SEP \subset RED := \{\rho : \rho^{red} := \rho_A \otimes I_B - \rho_{AB} \geq 0\}$
- ρ_{AB} rank one entangled state, then $\rho^{red} \ngeq 0$
- $\rho^{\Gamma} \geq 0 \Rightarrow \rho^{red} \geq 0, (PPT \subset RED)$
- If $dimB = 2$, then PPT=RC
- RC connected to entanglement distillation: all states that violate RC are distillable.

$$SEP \subset PPT \subset RED$$



Absolutely Separable States

Knill's question[4]: *given an self-adjoint positive semi-definite operator ρ, which are the conditions on the spectrum of ρ such that ρ is separable respect to any decomposition?*

ASEP: states that remain separable under any unitary transformation

$$ASEP_{n,k} = \bigcap_{U \in \mathcal{U}_{nk}} U\mathrm{SEP}_{n,k} U^*$$

Goal= conditions on the spectrum such that to be separable!

$$APPT_{n,k} := \{\rho \in \mathcal{D}_{n,k} / \forall U \in \mathcal{U}_{nk} : (U\rho U^*)^\Gamma \geq 0\} = \bigcap_{U \in \mathcal{U}_{nk}} U\mathrm{PPT}_{n,k} U^*$$

$$ARED_{n,k} := \{\rho \in \mathcal{D}_{n,k} / \forall U \in \mathcal{U}_{nk} : (U\rho U^*)^{red} \geq 0\} = \bigcap_{U \in \mathcal{U}_{nk}} U\mathrm{RED}_{n,k} U^*$$

[4] Open Problems in Quantum Information Theory, http://www.imaph.tu-bs.de/qi/problems

Absolutely PPT states

- necessary and sufficient conditions [5] on the spectrum under which the absolute PPT property holds
- the condition is to check the positivity of an exponential number of Hermitian matrices (the number of LMI is bounded above by $e^{2p\ln p(1+o(1))}$, $p = \min(n, k)$)
- if $\rho \in \mathcal{H}_{2n} = \mathcal{H}_2 \otimes \mathcal{H}_n$, then $\rho \in APPT$ iff

$$\lambda_1 \leq \lambda_{2n-1} + 2\sqrt{\lambda_{2n}\lambda_{2n-2}}$$

.

- if $\rho \in \mathcal{H}_{3n} = \mathcal{H}_3 \otimes \mathcal{H}_n$, then $\rho \in APPT$ iff

$$\begin{pmatrix} 2\lambda_{3n} & \lambda_{3n-1}-\lambda_1 & \lambda_{3n-2}-\lambda_2 \\ \lambda_{3n-1}-\lambda_1 & 2\lambda_{3n-3} & \lambda_{3n-4}-\lambda_3 \\ \lambda_{3n-2}-\lambda_2 & \lambda_{3n-4}-\lambda_3 & 2\lambda_{3n-5} \end{pmatrix} \geq 0,$$

$$\begin{pmatrix} 2\lambda_{3n} & \lambda_{3n-1}-\lambda_1 & \lambda_{3n-3}-\lambda_2 \\ \lambda_{3n-1}-\lambda_1 & 2\lambda_{3n-2} & \lambda_{3n-4}-\lambda_3 \\ \lambda_{3n-3}-\lambda_2 & \lambda_{3n-4}-\lambda_3 & 2\lambda_{3n-5} \end{pmatrix} \geq 0. \qquad (2)$$

[5] Hildebrand, Positive partial transpose from spectra, PRA, 2007

Absolutely Reduced pure states

$$\mathrm{ARED}_{n,k} := \{\rho \in D_{n,k} \,|\, \forall U \in \mathcal{U}_{nk} : (U\rho U^*)^{red} \geq 0\} \qquad (3)$$

Reduction of pure state[6]:
Given a vector $\psi \in \mathbb{C}^n \otimes \mathbb{C}^k$ with Schmidt coefficients $\{x_i\}_{i=1}^r$, the eigenvalues of the reduced matrix $(\psi\psi^*)^{red}$ are

$$\mathrm{spec}\left((\psi\psi^*)^{red}\right) = (\underbrace{x_1, \dots, x_1}_{k-1 \text{ times}}, \eta_1, x_2, \dots, \eta_{r-1}, \underbrace{x_r, \dots, x_r}_{k-1 \text{ times}}, \underbrace{0, \dots, 0}_{(n-r)k \text{ times}}, \eta_r)$$

where $x_i \geq \eta_i \geq x_{i+1}$ for $i \in [r-1]$ and $\eta_r = -\sum_{i=1}^{r-1} \eta_i \leq 0$.
The set $\{\eta_i\}_{i=1}^r \setminus \{x_i\}_{i=1}^r$ is the set of solutions $\eta \in \mathbb{R} \setminus \{x_i\}_{i=1}^r$ of the equation

$$\sum_{i=1}^{r} \frac{x_i}{x_i - \eta} = 1$$

[6] M.A. Jivulescu, N. Lupa, I. Nechita, D. Reeb, Positive reduction from spectra, Linear Algebra and its Applications, 2014

Characterizing Absolutely Reduced states

$$\mathrm{ARED}_{n,k} = \{\rho \in D_{n,k} : \forall x \in \Delta_{\min(n,k)}, \langle \lambda_\rho^{\downarrow}, \hat{x}^{\uparrow} \rangle \geq 0\}, \qquad (4)$$

where $\lambda_\rho^{\downarrow}$ is the vect. of the eigenvalues of ρ and $\hat{x}^{\uparrow}$ is the vector of Schmidt coefficients.

$$\hat{x} := (\underbrace{x_1, \ldots, x_1}_{k-1 \text{ times}}, \eta_1, \underbrace{x_2, \ldots, x_2}_{k-1 \text{ times}}, \ldots, \eta_{r-1}, \underbrace{x_r, \ldots, x_r}_{k-1 \text{ times}}, \underbrace{0, \ldots, 0}_{(n-r)k \text{ times}}, \eta_r),$$

η_i are the solutions of the equation $F_x(\lambda) := \sum_{i=1}^{q} \frac{m_i x_i}{x_i - \lambda} - 1 = 0$.

- necessary and sufficient condition on the spectrum as family of linear inequalities in terms of the spectrum of reduced of a pure state
- given $\rho \in M_2(\mathbb{C}) \otimes M_k(\mathbb{C})$, then $\rho \in \mathrm{ARED}_{2,k}$ if and only if

$$\lambda_1 \leq \lambda_{k+1} + 2\sqrt{(\lambda_2 + \cdots + \lambda_k)(\lambda_{k+2} + \cdots + \lambda_{2k})}.$$

Approximations of SEP

$$\mathrm{SEPBALL}_{n,k} = \left\{ \rho \in D_{n,k} \,|\, \mathrm{Tr}(\rho^2) \leq \frac{1}{nk-1} \right\}$$

- largest Euclidian ball[7] inside $D_{n,k}$, centered at $\frac{I}{nk}$
- contains states on the boundary of $D_{n,k}$
- all states within SEPBALL are separable
- depends only on the spectrum, i.e. $\mathrm{SEPBALL}_{n,k} \subset \mathrm{ASEP}$
- it is smaller than other sets

$$\mathrm{GER}_{n,k} = \left\{ \lambda \in \Delta_{nk} \ : \ \sum_{i=1}^{r-1} \lambda_i^{\downarrow} \leq 2\lambda_{nk}^{\downarrow} + \sum_{i=1}^{r-1} \lambda_{nk-i}^{\downarrow} \right\}$$

- the defining equation[8] represents the sufficient condition provided by Gershgorin's theorem for all Hildebrand APPT matrix inequalities to be satisfied
- lower approximation of APPT, since $GER_{n,k} \subset APPT_{n,k}$
- provides easily-checkable sufficient condition to be APPT, much simpler that Hildebrand's conditions

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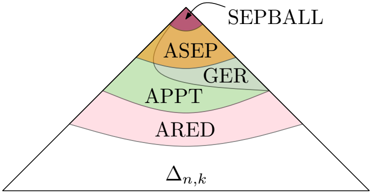
[7] Gurvitz L., PRA 2002

[8] M.A. Jivulescu, N. Lupa, I. Nechita, D. Reeb, Positive reduction from spectra, Linear Algebra and its Applications, 2014

Approximations of (A)SEP



$$\mathrm{LS}_p := \{\lambda \in \Delta_{nk} \ : \ \lambda_1^{\downarrow} \leq \lambda_{nk-p+1}^{\downarrow} + \lambda_{nk-p+2}^{\downarrow} + \cdots + \lambda_{nk}^{\downarrow}\}$$

- LS_P -the sets of eigenvalue vectors for which the largest eigenvalue is less or equal than the sum of the p smallest (arbitrary $p \in [nk]$)
- For $n, k \geq 3$, $\mathrm{APPT} \subseteq \mathrm{LS}_3 \subseteq \mathrm{LS}_k \subseteq \mathrm{ARED}_{n,k} \subseteq \mathrm{LS}_{2k-1}$.

Threshold concept

Threshold=the value c of the parameter, giving an scaling of the environment, **value at which a sharp phase transition of the system occurs!**

Mathematical characterisation[9]:

Consider a random bipartite quantum state $\rho_{AB} \in M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$, obtained by partial tracing over $\mathbb{C}^s$ a uniformly distributed, pure state $x \in \mathbb{C}^n \otimes \mathbb{C}^k \otimes \mathbb{C}^s$.
When one (or both)of the system dimensions n and k are large, a threshold phenomenon occurs:
if $s \sim cnk$, then there is a *threshold value* c_0 such that

1. for all $c < c_0$, as dimension nk grows, $P(\rho_{AB}$ satisfies the entangled criterion$) = 0$;
2. for all $c > c_0$, as dimension nk grows, $P(\rho_{AB}$ satisfies the entangled criterion$) = 1$;

[9] Aubrun, G. *Partial transposition of random states and non-centered semicircular distributions.* Random Matrices: Theory Appl. (2012)

Reduction Criterion: RMT approach

$W = XX^* \in M_d(\mathbb{C})$ -Wishart matrix of parmeters d and s.
Wishart matrices – physically reasonable models for random density matrices on a tensor product space.

The spectral properties of $\rho^{red} \to$ *reduced matrix*

$$R = W^{red} := W_A \otimes \mathrm{I}_k - W_{AB},$$

where W_{AB} is a Wishart matrix of parameters nk and s, W_A is its partial trace with respect to the second subsystem B.
Issues:

- study the distribution of the eigenvalues of the random matrix $R = W^{red}$
- evaluating the probability that R is positive semidefinite

Moment formula for R

Theorem

The moments of the random matrix $R = W^{red} = W_A \otimes \mathrm{I}_k - W_{AB} \in M_{nk}(\mathbb{C})$ *are given by*[10]

$$\forall p \geq 1, \quad \mathbb{E}\mathrm{Tr}(R^p) = \sum_{\alpha \in \mathcal{S}_p,\, f \in \mathcal{F}_p} (-1)^{|f^{-1}(2)|} s^{\#\alpha} n^{\#(\gamma^{-1}\alpha)} k^{\mathbf{1}_{f\equiv 1} + \#(P_f^{-1}\alpha)}, \tag{5}$$

($\#$ -number of cycles of α, $f : \{1, \ldots, p\} \to \{1, 2\}$).

Examples:

$$\mathbb{E}\mathrm{Tr}(R) = nk(k-1)s$$

$$\mathbb{E}\mathrm{Tr}\left(R^2\right) = (k-2)\left[(ks)^2 n + ksn^2\right] + nks^2 + (nk)^2 s.$$

[10] M.A. Jivulescu, N. Lupa, I. Nechita, On the reduction criterion for random quantum states, Journal of Mathematical Physics, Volume: 55, Issue: 11, 2014

Moment formula

The proof is based on

- the development of R^p using the non-commutative binomial formula $R^p = \sum_{f \in \mathcal{F}_p} (-1)^{|f^{-1}(2)|} R_f$
- $f : \{1, \ldots, p\} \to \{1, 2\}$ encodes the choice of the term (choose the $f(i)$-th term) in each factor in the product

$$R^p = (W_A \otimes \mathrm{I}_k - W_{AB})(W_A \otimes \mathrm{I}_k - W_{AB}) \cdots (W_A \otimes \mathrm{I}_k - W_{AB}).$$

- R_f denotes the ordered product

$$R_f = R_{f(1)} R_{f(2)} \cdots R_{f(p)} = \overrightarrow{\prod_{1 \le i \le p}} R_{f(i)},$$

  for the two possible values of the factors $R_1 = W_A \otimes \mathrm{I}_k, R_2 = W_{AB}$.
- Wick graphical calculus to compute $\mathbb{E} \operatorname{Tr} R_f$;

Three asymptotics regimes

We aim to study the behavior of the combinatorial powers of n, k and s (dominant term) from the moment formula in three asymptotics regimes:

Balanced asymptotics: $(\exists)\quad c, t > 0$ such that

$$n \to \infty; k \to \infty, \quad k/n \to t; s \to \infty, \quad s/(nk) \to c.$$

Unbalanced asymptotics, first case: $(\exists)\quad c > 0$ such that

$$n - \text{ fixed}; k \to \infty; s \to \infty, \quad s/(nk) \to c.$$

Unbalanced asymptotics, second case: $(\exists)\quad c > 0$ such that

$$n \to \infty; k - \text{ fixed}; s \to \infty, \quad s/(nk) \to c.$$

Asymptotics regimes: balanced and unbalanced, first case

Balanced asymptotics: : the spectrum of the reduced Wishart matrix R becomes trivial when $n \to \infty$, in the sense that $R/(ks) \approx \mathrm{I}$.
Unbalanced asymptotics: first case: the spectrum of the reduced Wishart matrix R becomes trivial when $k \to \infty$, in the sense that $R/(ks) \approx \mathrm{I}$.

Asymptotically, all random quantum states satisfy the reduction criterion ($c_{red} = 0$).

Three asymptotics regimes: unbalanced, second case

Theorem

The moments of the rescaled random matrix R converge to the following combinatorial quantity:

$$\forall p \geq 1, \quad \lim_{n\to\infty} \mathbb{E}\frac{1}{nk}\mathrm{Tr}\left(\frac{R}{n}\right)^p = \sum_{\alpha\in NC(p)} \prod_{b\in\alpha} c\left[(1-k)^{|b|} + k^2 - 1\right]. \tag{6}$$

Therefore, the empirical eigenvalue distribution μ_n of $\frac{R}{n}$ converges, in moments, to a compound free Poisson distribution $\mu_{k,c} = \pi_{\nu_{k,c}}$, where

$$\nu_{k,c} = c\delta_{1-k} + c(k^2-1)\delta_1.$$

Moreover, the above convergence holds in a strong sense: the extremal eigenvalues converge, almost surely, to the edges of the support of the limiting measure $\mu_{k,c}$.

Its support is positive if and only if $c > c_{red} := \frac{(1+\sqrt{k+1})^2}{k(k-1)}$.

Thresholds for RED in different asymptotics regimes

1. **Unbalanced asymptotics, second case** [11] : $(n \to \infty, k$ fixat, $s \sim cnk)$,

$$c_{red} = \frac{(\sqrt{k+1}+1)^2}{k(k-1)}$$

2. **Balanced asymptotics**[12]**:** $(n, k_n \to \infty, s \sim cn)$

$$c_{red} = 1;$$

3. **Unbalanced asymptotics, first case** [12]: $(k \to \infty, n, s$ fixed)

$$c_{red} = n.$$

[11] M.A. Jivulescu, N. Lupa, I. Nechita, On the reduction criterion for random quantum states, Journal of Mathematical Physics, 2014

[12] M.A. Jivulescu, N. Lupa, I. Nechita, Thresholds for reduction-related entanglement criteria in quantum information theory, Quantum Information and Computation, 2015:

Spectrum of Wishart matrices

Theorem

Let $\{\lambda_i\}, \lambda_i \geq 0$ *the eigenvalues of a Wishart matrix* (d, s)*.*
Then, when $d \to \infty$ *and* $s = s_d \sim cd$ *for some constant* $c > 0$*,*

1. *The empirical eigenvalue distribution* $\mu_d = \frac{1}{d}\sum_{i=1}^{d} \delta_{d^{-1}\lambda_i}$ *converges to Marčenko-Pastur distribution*

$$\pi_c = \max(1-c, 0)\delta_0 + \sqrt{4c - (x - 1 - c)^2}\, \mathbf{1}_{[(\sqrt{c}-1)^2, (\sqrt{c}+1)^2]}(x)dx;$$

2. *For any function* $j_d = o(d)$*, almost surely, as* $d \to \infty$*, the rescaled eigenvalues* $\tilde{\lambda}_i = d^{-1}\lambda_i$ *have the following limits*[13]

$$\tilde{\lambda}_d, \tilde{\lambda}_{d-1}, \ldots, \tilde{\lambda}_{d-j_d+1} \to a_c = \begin{cases} 0, & \text{if } c \leq 1, \\ (\sqrt{c}-1)^2, & \text{if } c > 1, \end{cases}$$

and

$$\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_{j_d} \to b_c = (\sqrt{c}+1)^2;$$

[13] Bai, Yin, Ann. Probab. 1993

Comparing entanglement criteria via thresholds

<table>
<tr><td></td><td>$n, k \to \infty$</td><td colspan="2">$m = \min(n,k)$ fixed,
$\max(n,k) \to \infty$</td></tr>
<tr><td>SEP[14]</td><td>$n^3 \lesssim s \lesssim n^3 \log^2 n$</td><td colspan="2">$mnk \lesssim s \lesssim mnk \log^2(nk)$</td></tr>
<tr><td rowspan="2">PPT[15]</td><td>$s \sim cnk$</td><td colspan="2">$s \sim cnk$</td></tr>
<tr><td>$c = 4$</td><td colspan="2">$c = 2 + 2\sqrt{1 - \frac{1}{m^2}}$</td></tr>
<tr><td rowspan="2">RLN[16]</td><td>$s \sim cnk$</td><td colspan="2">s fixed</td></tr>
<tr><td>$c = (8/3\pi)^2$</td><td colspan="2">$s = m^2$</td></tr>
<tr><td rowspan="2">RED</td><td>$s \sim cn$</td><td>$m = n$, s is fixed</td><td>$m = k$, $s = cnk$</td></tr>
<tr><td>$c = 1$</td><td>$s = n$</td><td>$c = \frac{(1+\sqrt{k+1})^2}{k(k-1)}$</td></tr>
</table>

Conclusion: RLN is weaker than PPT (asymptotically);

[14] Aubrun, G., Szarek, S.J., and Ye, D. *Entanglement thresholds for random induced states.* Comm. Pure Appl. Math. (2014).

[15] Aubrun, G. *Partial transposition of random states and non-centered semicircular distributions.* Random Matrices: Theory Appl. (2012).

[16] Aubrun, G. and Nechita, I. *Realigning random states,* J.M.P. (2012).

Thresholds for ARED in different asymptotics regimes[17]

1. **Unbalanced asymptotics, first case** : $(k \to \infty, n - \textit{fixed}, s \sim ck)$

$$c_{ared} = n - 2$$

2. **Balanced asymptotics:** $(n, k_n \to \infty, s \sim cnk)$

$$c_{ared} = 1;$$

3. **Unbalanced asymptotics, second case** : ($n \to \infty$, k fixed, $s \sim cnk$),

$$c_{ared} = (1 + \frac{2}{k} + \frac{2}{k}\sqrt{k+1})^2$$

[17] M.A. Jivulescu, N. Lupa, I. Nechita, Thresholds for reduction-related entanglement criteria in quantum information theory, QIC, 2015;

Comparing entanglement criteria via thresholds

	$n, k \to \infty$	$m = \min(n, k)$ fixed $\max(n, k) \to \infty$	
APPT[18]	$s \sim c \min(n, k)^2 nk$	$s \sim cnk$	
	$c = 4$	$c = \left(m + \sqrt{m^2 - 1}\right)^2$	
GER	$s \sim c \min(n, k)^2 nk$	$s \sim cnk$	
	$c = 4$	$c = \left(m + \sqrt{m^2 - 1}\right)^2$	
ARED	$s \sim cnk$	$m = n$, $s \sim ck$	$m = k$, $s = cnk$
	$c = 1$	$c = n - 2$	$c = \left(\frac{(2+k+2)\sqrt{k+1}}{k}\right)^2$

- the thresholds for GER and APPT are the same, strenghting the claim that GER is a good approximation of APPT

[18] Collins, B., Nechita, I., and Ye, D. *The absolute positive partial transpose property for random induced states.* Random Matrices: Theory Appl. 01, 1250002 (2012).

Thresholds for related sets

<table>
<tr><td></td><td colspan="3">$d = nk \to \infty$</td></tr>
<tr><td rowspan="2">SEPBALL</td><td colspan="3">$s \sim cd^2$</td></tr>
<tr><td colspan="3">$c = 1$</td></tr>
<tr><td rowspan="3">$\mathrm{LS_p}$</td><td colspan="3">$s \sim cd$</td></tr>
<tr><td>$p \geq 2$ fixed</td><td>$1 \ll p = o(d)$</td><td>$p = \lfloor td \rfloor$</td></tr>
<tr><td>$c = \left(1 + \frac{2}{\sqrt{p}-1}\right)^2$</td><td>$c = 1$</td><td>$c = 1 - t$</td></tr>
</table>

- both sets depend on the product $d = nk$, so it is sufficent to consider one asymptotic regim $d \to \infty$
- SEPBALL case: the size of the enviroment s_d scales like the square of the total system $d = nk$

Perspectives

- description of ARLN and to find thresholds for ARLN
- integrate into the theory the recent results about thresholds for k-extendibility criterion[19]
- validate/invalidate the conjecture that ASEP=APPT[20]

[19] Lancien C. k-extendibility of high-dimensional bipartite quantum states, arxiv: 1504. 06459

[20] Arunachalam, S., Johnston, N., and Russo, V. *Is absolute separability determined by the partial transpose?* Quantum Inform. Comput. 2015

Thank you for your attention!

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