TD 1: First elliptic equations

Exercise 1. Let $\Omega = (0,1)$. Establish the following Poincaré inequality

$$\forall f \in H_0^1(\Omega), \quad \|f\|_{L^2(\Omega)} \leq \frac{1}{\pi}\|f'\|_{L^2(\Omega)},$$

and prove that the constant $1/\pi$ is optimal.
Hint: Use Fourier series.

Exercise 2. Let $\Omega = (0,1)$. The purpose of this exercice is to prove with a variational method that given a function $f \in L^2(\Omega)$, there exists a unique function $u \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying

$$-u'' + \sinh(u) = f \quad \text{in } L^2(\Omega). \tag{1}$$

1. Preliminaries: Let H be a real Hilbert space and $J : H \to \mathbb{R}$ be a continuous convex functional. We assume that J is coercive, that is, $J(x) \to +\infty$ when $\|x\| \to +\infty$. Prove then that there exists $x_\star$ in H such that $J(x_\star) = \inf_{x \in H} J(x)$.

2. In this question, we prove that there exists a unique $u \in H_0^1(\Omega)$ such that

$$\forall v \in H_0^1(\Omega), \quad \int_0^1 (u'(x)v'(x) + \sinh(u(x))v(x) - f(x)v(x))\,\mathrm{d}x = 0. \tag{2}$$

   To that end, we introduce the functional $J : H_0^1(\Omega) \to \mathbb{R}$ defined for all $v \in H_0^1(\Omega)$ by

$$J(v) = \int_0^1 \left(\frac{1}{2}|v'(x)|^2 + \cosh(v(x)) - f(x)v(x)\right)\mathrm{d}x.$$

   a) Check that the functional J is well-defined, strictly convex and coercive.
   b) Prove that the functional J is differentiable on $H_0^1(\Omega)$ and give the expression of its derivative.
   c) Deduce from the preliminary question that the variational problem (2) admits a unique solution $u \in H_0^1(\Omega)$.

3. Prove that the unique function $u \in H_0^1(\Omega)$ satisfying (2) belongs to $H^2(\Omega)$ and is also the unique function that satisfies (1).

4. When the function f is continuous on $[0,1]$, check that $u \in C^2(\bar{\Omega})$ is a strong solution of (1), in the sense that

$$\forall x \in [0,1], \quad -u''(x) + \sinh(u(x)) = f(x).$$

Exercise 3. Let $\Omega = (0,1)$. We aim at proving that there exists a unique $u \in H_0^1(\Omega) \cap H^2(\Omega)$ satisfying

$$\begin{cases} -u'' + u = \cos(u), \\ u(0) = u(1) = 0. \end{cases}$$

1. Given $v \in L^2(\Omega)$, check that the following problem

$$\begin{cases} -u'' + u = \cos(v), \\ u(0) = u(1) = 0, \end{cases}$$

admits a unique solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$.
Hint: Use Riesz' representation theorem in $H_0^1(\Omega)$.

2. Conclude by using the Banach-Picard fixed-point theorem on the space $L^2(\Omega)$.

Exercise 4. Let ρ be a compactly supported C^∞ function on $\mathbb{R}^3$. We are looking for a function $u \in C^2(\mathbb{R}^3)$ satisfying

$$-\Delta u = \rho, \tag{3}$$

under the following decreasing conditions at infinity

$$x \mapsto |x|u(x) \text{ is bounded}, \quad x \mapsto |x|^2 \nabla u(x) \text{ is bounded}. \tag{4}$$

1. Check that the function $x \mapsto 1/|x|$ is of class C^2 on $\mathbb{R}^3 \setminus \{0\}$ and compute its Laplacian.

2. Let Ω be a smooth open subset of $\mathbb{R}^3$. We denote by $n(x)$ the unit normal vector exiting at $x \in \partial\Omega$ and $\mathrm{d}\sigma$ the measure surface on $\partial\Omega$. We consider two functions u, v of class C^2 on $\overline{\Omega}$. By using Stokes' formula, prove Green's formula for the Laplacian:

$$\int_\Omega (v\Delta u - u\Delta v)\,\mathrm{d}x = \int_{\partial\Omega} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) \mathrm{d}\sigma(x).$$

3. For $0 < \alpha < \beta$, we define the following sphere and annulus

$$S_\alpha = \left\{x \in \mathbb{R}^3 : |x| = \alpha\right\} \quad \text{and} \quad A_{\alpha,\beta} = \left\{x \in \mathbb{R}^3 : \alpha \le |x| \le \beta\right\}.$$

Let $0 < \varepsilon < R$. We consider $u \in C^2(\mathbb{R}^3)$ satisfying (4). For all $x \in \mathbb{R}^3$, prove the following identity

$$\frac{1}{\varepsilon^2} \int_{S_\varepsilon} u(x+y)\,\mathrm{d}\sigma(y) = \int_{A_{\varepsilon,R}} \frac{(-\Delta u)(x+y)}{|y|}\,\mathrm{d}y + \mathcal{O}\left(\frac{1}{R}\right) + \mathcal{O}(\varepsilon).$$

4. Prove that the unique solution of (3) satisfying (4) is given for all $x \in \mathbb{R}^3$ by

$$u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|}\,\mathrm{d}y.$$

5. Let $p \in [1, 3)$. Check that there exists a constant C_p independent of ρ, such that

$$\|\nabla u\|_{L^\infty(\mathbb{R}^3)} \le C_p \|\rho\|_{L^p(\mathbb{R}^3)}^{p/3} \|\rho\|_{L^\infty(\mathbb{R}^3)}^{1-p/3}.$$

Hint: Consider the domains $\{|x-y| \le r\}$ and $\{|x-y| > r\}$, and optimize with respect to r.

6. Prove the following formula:

$$\|\nabla u\|_{L^2(\mathbb{R}^3)}^2 = \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y.$$

TD 2: Weak formulation of elliptic equations

Exercise 1 (Ellipticity). For each of the following linear differential operator L, give the symbol, the principal symbol of L, and discuss the ellipticity and uniform ellipticity.

1. $Lu(x) = -\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u, \quad x \in \Omega \subset \mathbb{R}^d,$
2. $Lf(x,v) = v \cdot \nabla_x f + F(x) \cdot \nabla_v f, \quad x, v \in \mathbb{R}^d,\ F\colon \mathbb{R}^d \to \mathbb{R}^d,$
3. $Lu(t,x) = \partial_t u - \Delta u, \quad t > 0,\ x \in \mathbb{R}^d,$
4. $Lu(t,x) = \partial_t u - i\Delta u, \quad t > 0,\ x \in \mathbb{R}^d.$

Exercise 2 (Faber-Krahn inequality). Let Ω be an open bounded subset of $\mathbb{R}^d$ with $d \geq 3$ and $V \in L^\infty(\Omega)$ such that $V \geq 0$. We consider the problem

$$\begin{cases} -\Delta u = Vu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

1. Give the definition of a weak solution to (1).
2. Can you apply the Lax-Milgram theorem here?
3. Let $r > \frac{d}{2}$. Show that there is a constant $c_d > 0$ depending on d only such that, if (1) has a non-trivial weak solution, then
$$|\Omega|^{\frac{2}{d}-\frac{1}{r}} \|V\|_{L^r(\Omega)} \geq c_d.$$
Hint: Use the following Sobolev inequality
$$\|u\|_{L^{2^*}(\Omega)} \leq M_d \|\nabla u\|_{L^2(\Omega)}, \qquad \frac{1}{2^*} = \frac{1}{2} - \frac{1}{d},$$
which holds for all $u \in H_0^1(\Omega)$, where M_d depends on d only.
4. What do you obtain in the particular case $V = \lambda = \text{cst}$?

Exercise 3 (Dirichlet problem). Let Ω be an open bounded subset of $\mathbb{R}^d$, $f \in L^2(\Omega)$ and $F \in L^2(\Omega)^d$. Show that the following elliptic problem with Dirichlet boundary condition

$$\begin{cases} -\Delta u = f - \operatorname{div} F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique weak solution $u \in H_0^1(\Omega)$.

Exercise 4 (Neumann problem). Let Ω be an open bounded subset of $\mathbb{R}^d$ with smooth boundary, the exterior unit normal being denoted by n, and $f \in L^2(\Omega)$. Show that, for all $\mu > 0$, the elliptic problem with Neumann boundary condition

$$\begin{cases} -\Delta u + \mu u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

has a unique weak solution $u \in H^1(\Omega)$. In the case $\mu = 0$, give a necessary condition on $\int_\Omega f$ to the existence of a weak solution to (2).

Exercise 5 (Fourier condition). Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with smooth boundary, $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$ and $\lambda > 0$. We consider the following elliptic problem with Fourier boundary condition

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \lambda u + \dfrac{\partial u}{\partial n} = g & \text{on } \partial\Omega. \end{cases} \tag{3}$$

1. Give the variational formulation of the problem (3).
2. Prove that there exists a positive constant $C_\Omega > 0$ only depending on Ω such that for all $u \in H^1(\Omega)$,
$$\|u\|^2_{L^2(\Omega)} \le C_\Omega\big(\|\nabla u\|^2_{L^2(\Omega)} + \lambda\|\gamma_0 u\|^2_{L^2(\partial\Omega)}\big),$$
where γ_0 denotes the trace operator $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$.
3. Prove that (3) has a unique weak solution.
4. * Is this weak solution a strong solution ?

Exercise 6 (The method of continuity).

1. Solve the equation $u - \Delta u = f$ on $\mathbb{T}^d$ and show that it defines a map $L^2(\mathbb{T}^d) \to H^2(\mathbb{T}^d)$.
2. Let X, Y be some Banach spaces. Let $(T_t)_{t\in[0,1]}$ be a *continuous* path of continuous linear operators from X to Y satisfying
$$\exists C \ge 0, \forall u \in X, \forall t \in [0,1], \quad \|u\|_X \le C\|T_t u\|_Y. \tag{4}$$
Prove that T_0 is surjective if and only if T_1 is surjective as well.
3. Let $A \in C^1(\mathbb{T}^d, M_d(\mathbb{R}))$. We assume that the following ellipticity condition holds
$$\exists \alpha \in (0,1), \forall x \in \mathbb{T}^d, \forall \xi \in \mathbb{R}^d, \quad A(x)\xi\cdot\xi \ge \alpha|\xi|^2.$$
We define the path $(T_t)_{t\in[0,1]}$ of operators $H^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$ by the formula
$$T_t u = u - \operatorname{div}(A^{(t)}(x)\nabla u), \quad A^{(t)} = tA + (1-t)I_d.$$
   (a) Show that $t \mapsto T_t$ is continuous.
   (b) Check that (4) is satisfied.
   (c) Conclude.

Exercise 7 (Resolution by minimization). Let $\Omega \subset \mathbb{R}^3$ be open, bounded with smooth boundary. The purpose is to prove that the following elliptic problem has a non-trivial weak solution

$$\begin{cases} -\Delta u = u^3 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

1. Prove that there exists a solution to the following minimization problem
$$\inf\big\{\|\nabla v\|_{L^2(\Omega)} : v \in H^1_0(\Omega),\ \|v\|_{L^4(\Omega)} = 1\big\}. \tag{5}$$
Recall: Since $d = 3$ here, the continuous embedding $H^1_0(\Omega) \hookrightarrow L^q(\Omega)$ holds for all $1 \le q \le 6$, and is moreover compact when $1 \le q < 6$.
2. Prove that if the function $v \in H^1_0(\Omega)$ solves (5), there exists a positive constant $\lambda > 0$ such that $-\Delta v = \lambda v^3$ weakly in Ω.
3. Conclude.

TD 3: Elliptic regularity and maximum principles

Exercise 1 (Control of the L^∞ norm). Let Ω be an open bounded subset of $\mathbb{R}^d$ of class C^2. Let $A \in C^1(\overline{\Omega}, S_d(\mathbb{R}))$ satisfying the following ellipticity condition

$$\exists \alpha > 0, \forall (x,\xi) \in \Omega \times \mathbb{R}^d, \quad A(x)\xi \cdot \xi \geq \alpha |\xi|^2. \tag{1}$$

Let $f \in L^2(\Omega)$ and $u \in H^1_0(\Omega)$ be the weak solution of the following Dirichlet problem

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

1. In this question, we assume that $d \leq 3$. Show that there exists a constant $C \geq 0$ depending only on Ω and d such that
$$\|u\|_{L^\infty(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}). \tag{2}$$
2. We assume that $\Omega = B(0,R)$ where $R > 0$.
   (a) Compute Δv when $v(x) = \psi(|x|)$ is a radial function.
   (b) By considering the function $u(x) = \ln|\ln|x||$ and the case $A(x) = \mathrm{I}_d$, discuss the validity of the estimate (2) when $d \geq 4$.

Remark: One can prove (this is a bit technical) that when $d \geq 4$ and $f \in L^p(\Omega)$, where $p > d/2$, there exists a positive constant $C > 0$ only depending on d, Ω and p such that the following estimate, somehow analogous to (2), holds

$$\|u\|_{L^\infty(\Omega)} \leq C(\|f\|_{L^p(\Omega)} + \|u\|_{L^2(\Omega)}).$$

Exercise 2 (Hölder regularity). The purpose of this exercise is to show a gain of derivatives in Hölder spaces for the solution u to the Poisson equation $-\Delta u = \rho$, where $\rho \in C^0(\mathbb{R}^3)$ is a function with compact support. Let G be the Green function of the Laplacian in dimension 3. Let us recall that the function

$$u(x) = (G * \rho)(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|}\,\mathrm{d}y,$$

is solution of the Poisson equation in $\mathbb{R}^3$. We assume that $\rho \in C^\alpha(\mathbb{R}^3)$ for a given $\alpha \in (0,1)$, and we set

$$[\rho]_{\dot{C}^\alpha(\mathbb{R}^3)} = \sup_{x \neq z \in \mathbb{R}^3} \frac{|\rho(x) - \rho(y)|}{|x-y|^\alpha} < +\infty.$$

Let K be a compact of $\mathbb{R}^3$. We want to prove that $u, \nabla u \in C^\alpha(K)$ and that there exists a positive constant $c > 0$ only depending on K, d, α and on the support of ρ such that

$$[u]_{\dot{C}^\alpha(K)} + [\nabla u]_{\dot{C}^\alpha(K)} \leq c[\rho]_{\dot{C}^\alpha(\mathbb{R}^3)}. \tag{3}$$

1. Show that $u \in C^\alpha(K)$ and that the estimate (3) holds for u.
2. By introducing a cut-off function ω_ε of the form $\omega_\varepsilon(x) = \theta(\varepsilon^{-1}|x|)$ and considering the approximation $u_\varepsilon = (G\omega_\varepsilon) * \rho$, prove that $\nabla u \in C^\alpha(K)$ and that the estimate (3) holds for the function ∇u.

Remark: By using similar techniques, one can prove that for all $\delta \in (0, \alpha)$, we have $\nabla^2 u \in C^\delta(K)$ and also that there exists a positive constant $c' > 0$ depending only on K, d, α, δ and the support of the function ρ such that

$$[\nabla^2 u]_{\dot{C}^\delta(K)} \leq c'[\rho]_{\dot{C}^\alpha(\mathbb{R}^3)}.$$

Exercise 3 (Weak maximum principle). Let Ω be a bounded open subset of $\mathbb{R}^d$ with smooth boundary and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying $\Delta u \leq 0$ on Ω. Proof by hand that

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

Hint: Assume first that $\Delta u < 0$.

Exercise 4 (Weak maximum principle for weak solutions). Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. We consider the following operator $L = -\operatorname{div}(A(x)\nabla \cdot)$, where $A \in L^\infty(\Omega, M_d(\mathbb{R}))$ satisfies the following ellipticity assumption

$$\exists \alpha > 0, \forall (x, \xi) \in \Omega \times \mathbb{R}^d, \quad A(x)\xi \cdot \xi \geq \alpha |\xi|^2.$$

We want to prove that if $u \in H_0^1(\Omega)$ is a weak solution of the equation $Lu \leq 0$, then $u \leq 0$ a.e. in the set Ω.

1. Prove that there exists a non-negative function $G \in C^1(\mathbb{R})$ with bounded derivative such that $G' > 0$ on $(0, +\infty)$ and $G' = 0$ on $(-\infty, 0]$.
2. Check that we have
$$\int_\Omega |\nabla u(x)|^2 (G' \circ u)(x)\, \mathrm{d}x \leq 0.$$
3. Conclude.

Exercise 5 (No solution). Let $L > 0$. We aim at proving that when $L \gg 1$ is large enough, there is no smooth solution u satisfying $-u'' = e^u$ in $(0, L)$ and $u(0) = u(L) = 0$. We assume by contradiction that such a solution $u \in C^0[0, L] \cap C^2(0, L)$ exists.

1. Given $\varepsilon > 0$, we consider the function $w = u + \varepsilon$. Give the equation satisfied by this new function w.
2. We consider the family of functions $(v_\lambda)_{\lambda \geq 0}$ defined on $[0, L]$ by $v_\lambda(x) = \lambda \sin(\pi x / L)$. Give the equations satisfied by these functions.
3. Prove that when $L \gg 1$ is large enough, the function w is a sub-solution of the equation established in the above question. Check moreover that when $0 < \lambda \ll 1$ is sufficiently small, then $v_\lambda < w$ on $[0, L]$.
4. Let us now start increasing λ until the graphs of v_λ and w touch at some point
$$\lambda_0 = \sup\big\{\lambda \geq 0 : \forall x \in [0, L],\ v_\lambda(x) < w(x)\big\}.$$
By considering the function $p = v_{\lambda_0} - w$ and using the weak maximum principle, obtain a contradiction.

TD 4: Heat equation

Exercise 1 (Heat kernel). Let $d \geq 1$ and $E_d \in \mathscr{S}'(\mathbb{R}_t \times \mathbb{R}^d_x)$ be the tempered distribution defined by

$$E_d(t,x) = \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}} \mathbb{1}_{]0,+\infty[}(t).$$

Prove that E_d is a fundamental solution of the heat operator, that is, satisfies

$$\Big(\partial_t - \frac{1}{2}\Delta\Big) E_d = \delta_{(t,x)=(0,0)} \quad \text{in } \mathscr{S}'(\mathbb{R}_t \times \mathbb{R}^d_x).$$

Check that E_d is unique under the condition $\operatorname{Supp} E_d \subset \mathbb{R}_+ \times \mathbb{R}^d$.

Exercise 2 (Heat equation on $\mathbb{R}^d$). Let $u_0 \in L^2(\mathbb{R}^d)$. We consider the homogeneous heat equation posed on the whole space $\mathbb{R}^d$:

$$\text{(1)} \qquad \begin{cases} \partial_t u - \frac{1}{2}\Delta u = 0 & \text{on } (0,+\infty) \times \mathbb{R}^d, \\ u(0,\cdot) = u_0 & \text{on } \mathbb{R}^d. \end{cases}$$

1. (Regularity) Compute explicitly the solution of the equation (1). What is its regularity ?
2. (Energy estimate) Show that for all $t \geq 0$,
$$\|u(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} + \int_0^t \|\nabla u(s,\cdot)\|^2_{L^2(\mathbb{R}^d)} \, \mathrm{d}s = \|u_0\|^2_{L^2(\mathbb{R}^d)}.$$
3. (Maximum principle) Show that if $u_0 \in L^\infty(\mathbb{R}^d)$, then $u(t,\cdot) \in L^\infty(\mathbb{R}^d)$ for all $t \geq 0$ and
$$\sup_{t \geq 0} \|u(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}.$$
4. (Infinite speed of propagation) Prove that if $u_0 \geq 0$ is a function being not identically equal to zero and non-negative, then $u > 0$ in $\mathbb{R}_+ \times \mathbb{R}^d$.

Exercise 3 (Spectral theory). Let Ω be a bounded open subset of $\mathbb{R}^d$.

1. Prove that there exists a continuous operator $T \in \mathcal{L}(L^2(\Omega), H^1_0(\Omega))$ satisfying $\langle f, v \rangle_{L^2(\Omega)} = \langle Tf, v \rangle_{H^1_0(\Omega)}$ for all $f \in L^2(\Omega)$ and $v \in H^1_0(\Omega)$.
2. Let $\iota : H^1_0(\Omega) \to L^2(\Omega)$ be the canonical injection. Check that the operator $T \circ \iota : H^1_0(\Omega) \to H^1_0(\Omega)$ is non-negative, selfadjoint, one to one and compact.
3. Deduce that the spectrum of the Laplacian operator $-\Delta$ with Dirichlet boundary condition is a sequence $(\lambda_n)_{n \geq 0}$ of positive real numbers which is increasing and diverges to $+\infty$, and also that there exists a Hilbert basis $(e_n)_{n \geq 0}$ of $H^1_0(\Omega)$ composed of eigenfunctions of $-\Delta$ and such that $-\Delta e_n = \lambda_n e_n$ for all $n \geq 0$.
4. Compute explicitly those eigenvalues and those eigenfunctions when $d = 1$ and $\Omega = (0,1)$.

Exercise 4 (Heat equation on bounded domains). Let Ω be a bounded open subset of $\mathbb{R}^d$ with regular boundary, $T > 0$ be a final time, $u_0 \in L^2(\Omega)$ be an initial datum and $f \in L^2((0,T), L^2(\Omega))$ be a source term. We aim at proving that there exists a unique solution $u \in L^2((0,T), H_0^1(\Omega)) \cap C^0([0,T], L^2(\Omega))$ to the following heat equation with Dirichlet boundary conditions

$$\begin{cases} \partial_t u - \Delta u = f & \text{a.e. in } (0,T)\times\Omega, \\ u = 0 & \text{a.e. on } (0,T)\times\partial\Omega, \\ u(0,\cdot) = u_0 & \text{a.e. in } \Omega. \end{cases} \tag{2}$$

We will also check that this solution satisfies the following energy estimate for all $0 \le t \le T$,

$$\|u(t,\cdot)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u(s,\cdot)\|^2_{L^2(\Omega)}\,\mathrm{d}s \le C\left(\|u_0\|^2_{L^2(\Omega)} + \int_0^t \|f(s,\cdot)\|^2_{L^2(\Omega)}\,\mathrm{d}s\right), \tag{3}$$

where $C > 0$ is a positive constant only depending on Ω. In the following, we consider $(e_n)_{n\ge 0}$ a Hilbert basis of $L^2(\Omega)$ composed of eigenfunctions of the operator $-\Delta$. Moreover, we set λ_n the eigenvalue associated with the eigenfunction e_n.

1. We first prove that there exists a unique $u \in L^2((0,T), H_0^1(\Omega)) \cap C^0([0,T], L^2(\Omega))$ satisfying

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\langle u(t,\cdot), v\rangle_{L^2(\Omega)} + \displaystyle\int_\Omega \nabla u(t,\cdot)\cdot\nabla v = \langle f(t,\cdot), v\rangle_{L^2(\Omega)} & \forall v \in H_0^1(\Omega), \forall t \in (0,T), \\ u(0,\cdot) = u_0. \end{cases}$$

   a) Define properly this variational formulation.
   b) Give the formal expansion in the Hilbert basis $(e_n)_{n\ge 0}$ of such a solution.
   c) Prove that this expansion converges in $L^2((0,T), H_0^1(\Omega))$ and also in $C^0([0,T], L^2(\Omega))$.
   d) Conclude.

2. We now want to prove that this weak solution u is a strong solution, that is, is solution of the problem (2).

   a) Check that the boundary condition and the initial value condition hold.
   b) * Prove that $\partial_t u - \Delta u = f$ a.e. in $(0,T)\times\Omega$.

3. When $f = 0$, check that

$$\forall t \ge 0, \quad \|u(t,\cdot) - \langle u_0, e_0\rangle_{L^2(\Omega)} e^{-\lambda_0 t} e_0\|_{L^2(\Omega)} \le e^{-\lambda_1 t}\|u_0\|_{L^2(\Omega)}.$$

Exercise 5 (Maximum principle). Let Ω be a bounded open subset of $\mathbb{R}^d$ with smooth boundary, $T > 0$ be a final time, $u_0 \in H_0^1(\Omega)$ be an initial datum and $f \in L^2((0,T), L^2(\Omega))$ be a source term. We consider $u \in L^2((0,T), H_0^1(\Omega)) \cap C^0([0,T], L^2(\Omega))$ the unique solution of the problem (2). Prove that when $f \ge 0$ a.e. in $(0,T)\times\Omega$ and $u_0 \ge 0$ a.e. in Ω, then $u \ge 0$ a.e. on $(0,T)\times\Omega$.
Hint: Admit that $\partial_t u \in L^2((0,T), L^2(\Omega))$ and $u \in L^2((0,T), H^2(\Omega)) \cap C^0([0,T], H_0^1(\Omega))$.

Application *: Assume now that $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and $f \in L^\infty([0,+\infty)\times\Omega)$. Show that

$$\sup_{t\ge 0}\|u(t,\cdot)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} + \frac{\operatorname{diam}(\Omega)^2}{2d}\sup_{t\ge 0}\|f(t,\cdot)\|_{L^\infty(\Omega)}.$$

TD 5: Existence and uniqueness of solutions for reaction-diffusion equations

Exercise 1. We consider the following reaction-diffusion equation:

$$\begin{cases} \partial_t u - \Delta u = u^2 & \text{in } (0,+\infty)\times\mathbb{R}, \\ u(0,\cdot) = u_0 & \text{in } \mathbb{R}. \end{cases} \tag{1}$$

1. Establish *a priori* energy estimates for any smooth solution of the equation (1).
2. Assume that $u_0 \in H^1(\mathbb{R})$. We aim at proving, by using an iterative method, that there exist $T > 0$ and a solution $u \in C^0([0,T], H^1(\mathbb{R}))$ of the equation (1). We therefore consider the sequence $(u^n)_{n\geq 0}$ recursively defined by $u^0 = u_0$ and

$$\begin{cases} \partial_t u^{n+1} - \Delta u^{n+1} = (u^n)^2 & \text{in } (0,+\infty)\times\mathbb{R}, \\ u^{n+1}(0,\cdot) = u_0 & \text{in } \mathbb{R}. \end{cases} \tag{2}$$

   (a) Discuss the well-posedness of the sequence $(u^n)_{n\geq 0}$.

   (b) (Bound in H^1) Prove that there exists a positive time $T_1 > 0$ and a positive constant $c_1 > 0$ such that for all $n \geq 0$ and $0 \leq t \leq T_1$,
$$\|u^n(t,\cdot)\|_{H^1(\mathbb{R})} \leq c_1.$$

   (c) (Convergence in H^1) Prove that there exists another positive time $0 < T_2 < T_1$ and another positive constant $c_2 > 0$ satisfying that for all $n \geq 0$ and $0 \leq t \leq T_2$,
$$\|u^{n+1}(t,\cdot) - u^n(t,\cdot)\|_{H^1(\mathbb{R})} \leq \frac{c_2}{2^n}.$$

   (d) Conclude.

3. Is this solution unique ?

Exercise 2. Let $u_0 \in H^1(\mathbb{R})$ be a smooth initial datum. We consider $T > 0$ the positive time and $u \in C^0([0,T], H^1(\mathbb{R}))$ the solution of the equation (1), both given by the previous exercise. By using a bootstrap argument, prove that the function u is smooth, precisely $u \in C^\infty(]0,T[\times\mathbb{R})$.

Exercise 3. By adapting the strategy used in the first exercise, investigate the existence of solutions for the following reaction-diffusion equation:

$$\begin{cases} \partial_t u - \Delta u = \arctan(u) & \text{in } (0,+\infty)\times\mathbb{R}^d, \\ u(0,\cdot) = u_0 & \text{in } \mathbb{R}^d, \end{cases} \tag{3}$$

with initial datum $u_0 \in L^2(\mathbb{R}^d)$. Assuming then that $d = 1$ and $\langle x\rangle u_0, \langle x\rangle \partial_x u_0 \in L^2(\mathbb{R})$, prove pointwise estimates for the function u.
Hint: The function arctan *is globally Lipschitz continuous, only* L^2 *estimates are required.*

Exercise 4. Let $T > 0$ and $u_0 \in L^2(\mathbb{R}^d)$. We consider the following initial value problem:

$$(4) \qquad \begin{cases} \partial_t u - \Delta u = \sqrt{1+u^2} - 1, & \text{in } (0,T] \times \mathbb{R}^d, \\ u(0,\cdot) = u_0 & \text{in } \mathbb{R}^d, \end{cases}$$

We say that a continuous function $u \in C^0([0,T], L^2(\mathbb{R}^d))$ is a *mild* solution of the initial value problem (4) when it satisfies the following integral equation for all $0 \leq t \leq T$:

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}(\sqrt{1+u(s)^2} - 1)\,\mathrm{d}s,$$

where, for all $v \in L^2(\mathbb{R}^d)$, $e^{t\Delta}v$ denotes the solution of the heat equation posed on $\mathbb{R}^d$ with initial datum v.

1. We consider the function $F : C^0([0,T], L^2(\mathbb{R}^d)) \to C^0([0,T], L^2(\mathbb{R}^d))$ defined by

$$(Fu)(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}(\sqrt{1+u(s)^2} - 1)\,\mathrm{d}s.$$

   By using a fixed-point theorem on the function F, prove that the equation (4) admits a unique mild solution $u \in C^0([0,T], L^2(\mathbb{R}^d))$.
2. Check that the function $u_0 \in L^2(\mathbb{R}^d) \mapsto u \in C^0([0,T], L^2(\mathbb{R}^d))$ is Lipschitz continuous.

Exercise 5. Study the existence of mild solutions for the equation (3) and make the link with the solution constructed by iterative method in Exercise 3.

Exercise 6. Let $\Omega = (0,1)$, $t_0 > 0$ and $u_0 \in H_0^1(\Omega)$. We aim at proving that there exist a positive time $t^* > 0$ and a unique function $u \in C^0([t_0, t^*[, H_0^1(\Omega))$ solution of the following integral equation for all $t_0 \leq t < t^*$:

$$(5) \qquad u(t) = e^{(t-t_0)\Delta}u_0 + \int_{t_0}^t e^{(t-s)\Delta}\sinh(u(s))\,\mathrm{d}s.$$

Let us recall that there exists a Hilbert basis $(e_n)_{n\geq 0}$ of the space $H_0^1(\Omega)$ composed of eigenvalues of the operator $-\Delta$. In the above integral equation, the operator $e^{t\Delta} \in \mathcal{L}(H_0^1(\Omega))$ is defined by

$$e^{t\Delta} = \sum_{n=0}^{+\infty} e^{-t\lambda_n}\langle \cdot, e_n\rangle_{H_0^1} e_n,$$

with $\lambda_n > 0$ the eigenvalue associated with the eigenfunction e_n.

1. By using a fixed-point theorem, prove that there exists a positive time $t_1 > t_0$ such that the equation (5) has a solution in the space $C^0([t_0, t_1], H_0^1(\Omega))$.
2. Explain how this solution can be extended to the interval $[t_0, t_1 + \delta]$ with $\delta > 0$. Deduce, proceeding by contradiction, that if $[t_0, t^*[$ stands for the maximal interval of existence of the solution u and if $t^* < +\infty$, then

$$\lim_{t \nearrow t^*} \|u(t)\|_{H_0^1(\Omega)} = +\infty.$$

3. Investigate the uniqueness of such a solution.
4. Of which equation is the function u a mild solution ?

TD 6: Maximum principles and stability of steady states

Exercise 1. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $T > 0$ be a final time and $Q_T = (0, T] \times \Omega$. We consider the following differential operator

$$L = -\sum_{i,j=1}^{d} a^{i,j}(t,x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{n} b^i(t,x)\partial_{x_i} + c(t,x), \quad (t,x) \in Q_T,$$

the coefficients $a^{i,j}, b^i$ and c being bounded on $\overline{Q_T}$, with moreover $a^{i,j} = a^{j,i}$. We assume that the operator $\partial_t + L$ is uniformly parabolic, that is,

$$\exists \theta > 0, \forall (t,x) \in Q_T, \forall \xi \in \mathbb{R}^d, \quad \sum_{i,j=1}^{d} a^{i,j}(t,x)\xi_i\xi_j \geq \theta|\xi|^2.$$

State as many maximum principles as you can for the parabolic operator $\partial_t + L$.

Exercise 2. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $T > 0$ be a positive time and $Q_T = (0, T] \times \Omega$. We also consider $f \in C^1(\mathbb{R})$ a smooth function. Let $u, v \in C^2(Q_T) \cap C^0(\overline{Q_T})$ be two functions satisfying

$$\begin{cases} \partial_t v - \Delta v - f(v) \leq \partial_t u - \Delta u - f(u) & \text{in } Q_T, \\ v \leq u & \text{on } \Gamma_T. \end{cases}$$

Prove that $v \leq u$ on Q_T.

Application: Consider $u \in C^2(Q_T) \cap C^0(\overline{Q_T})$ a solution of the equation

$$\begin{cases} \partial_t u - \Delta u = u(1-u)(u-a) & \text{in } Q_T, \\ u = 0 & \text{on } (0,T] \times \partial\Omega, \\ u(0,\cdot) = u_0 & \text{in } \Omega, \end{cases}$$

where $0 < a < 1$ is a positive constant and u_0 is a smooth initial datum satisfying $0 \leq u_0 \leq 1$ in Ω. Prove that the function u is bounded as follows

$$\forall (t,x) \in Q_T, \quad 0 \leq u(t,x) \leq 1.$$

Can you be more precise when assuming $0 \leq u_0 < a$ in Ω ?

Exercise 3. Let $L > 0$. Prove that there exists a critical length $L_c > 0$ such that the equation

$$\begin{cases} q'' + q(1-q) = 0 & x \in (0, L), \\ q(0) = q(L) = 0, \end{cases} \tag{1}$$

has a non-trivial non-negative solution if and only if $L > L_c$.
Hint: The function $H(q_1, q_2) = q_1^2/2 + q_2^2/2 - q_1^3/3$ is a Lyapunov function for this equation.

Exercise 4. Let $0 < L < \pi$ be a length, $u_0 \in L^2(0, L)$ be an initial datum such that $0 \leq u_0 \leq 1$ a.e. and u be the solution of the Fisher-KPP equation

$$(2) \qquad \begin{cases} \partial_t u - \partial_{xx} u = u(1-u), & t > 0,\ x \in (0, L), \\ u(t, 0) = u(t, L) = 0, & t > 0, \\ u(0, x) = u_0(x), & x \in (0, L), \end{cases}$$

1. Prove the following estimate

$$\forall t \geq 0, \quad \|u(t)\|_{L^2(0,L)} \leq e^{(1-\pi^2/L^2)t} \|u_0\|_{L^2(0,L)},$$

   and deduce that $u(t) \to 0$ in $L^2(0, L)$ as $t \to +\infty$.

2. We now aim at proving that $u(t, x) \to 0$ as $t \to +\infty$ for all $x \in [0, L]$.
   (a) Find a subsolution $\underline{u}$ of the equation (2).
   (b) We consider $\overline{u}$ the solution of the equation

$$\begin{cases} \partial_t \overline{u} - \partial_{xx} \overline{u} = \overline{u} & t > 0,\ x \in (0, L), \\ \overline{u}(t, 0) = \overline{u}(t, L) = 0 & t > 0, \\ \overline{u}(0, x) = u_0(x) & x \in (0, L), \end{cases}$$

   Check that $\overline{u}$ is a supersolution of the equation (2).
   (c) Prove that $\overline{u}(t, x) \to 0$ for all $x \in [0, L]$ as $t \to +\infty$ and conclude.

Exercise 5. We still consider the Fisher-KPP equation (2). Assuming this time that $L > \pi$, we aim at proving that there exists a supersolution $\overline{u}$ of the equation (2) such that $u(t, x) \leq \overline{u}(t, x)$ for all $t \geq 0$ and $x \in (0, L)$, and satisfying $\overline{u}(t, x) \to_{t \to +\infty} q(x)$ for all $x \in [0, L]$, where q is the non-trivial non-negative steady state given by Exercice 3.

1. Let $\overline{u}$ be the solution of the equation

$$\begin{cases} \partial_t \overline{u} - \partial_{xx} \overline{u} = \overline{u}(1 - \overline{u}), & t > 0,\ x \in (0, L), \\ \overline{u}(t, 0) = \overline{u}(t, L) = 0, & t > 0, \\ \overline{u}(0, x) = M, & x \in (0, L), \end{cases}$$

   with $M = \max(1, \sup_{(0,L)} u_0)$. Prove that $\overline{u}$ is a supersolution of the equation (2) which dominates the function u.

2. By comparing $\overline{u}(t + h, x)$ and $\overline{u}(t, x)$, prove that for all $x \in [0, L]$, the limit $w(x) = \lim_{t \to +\infty} \overline{u}(t, x)$ exists and satisfies the estimate $0 \leq w(x) \leq M$.

3. Admit that w is a solution of the equation (1). Deduce then that $w = q$ and conclude.

Remark: One can also prove that there exists a subsolution $\underline{u}$ converging pointwise to q and bounding the function u from below. As a consequence, $u(t, x) \to_{t \to +\infty} q(x)$ for all $x \in [0, L]$.

TD 7: Travelling waves

Exercise 1. We aim at proving that there are traveling waves solutions for the Fisher-KPP equation

$$\partial_t u - \partial_{xx} u = u(1-u), \quad t>0,\ x \in \mathbb{R}, \tag{1}$$

i.e. solutions of the form $u(t,x) = \phi(x-ct)$ for some function $\phi : \mathbb{R} \to [0,1]$ and $c \in \mathbb{R}$. Precisely, we are interested in traveling wavefronts, i.e. satisfying $\lim_{+\infty} \phi = 0$ and $\lim_{-\infty} \phi = 1$.

1. Check that a traveling wave is solution of the equation (1) if and only if the wave profile ϕ satisfies the following ordinary equation,

$$\phi''(z) + c\phi'(z) + \phi(z)(1-\phi(z)) = 0, \quad z \in \mathbb{R},$$

where $z = x - ct$ denotes the co-moving frame.

2. Write this equation as a system of two first order ordinary equations.
3. Study the stationary points of this system.
4. Explain why such a traveling wave does not exist when $0 < c < 2$.
5. Admitting that such a travelling wave exists when $c \geq 2$, prove that the wave profile ϕ has the following asymptotics

$$\phi(z,c) = \frac{1}{1+e^{z/c}} + \frac{1}{c^2}\frac{e^{z/c}}{(1+e^{z/c})^2} \ln\left(\frac{4e^{z/c}}{(1+e^{z/c})^2}\right) + \mathcal{O}(c^{-4}).$$

Hint: Set $\varepsilon = 1/c^2$ *and* $\xi = z/c$*, and consider the expansion of* ϕ *in powers of* ε*, that is,* $\phi(\xi,\varepsilon) = \phi_0(\xi) + \varepsilon\phi_1(\xi) + \varepsilon^2\phi_2(\xi) + \cdots$

Exercise 2. We still consider the Fisher-KPP equation (1). The purpose is now to deal with the appearance of propagation speeds in the reality. Assume that the initial condition of the equation (1) is given by

$$u(0,x) = e^{-a|x|}, \quad x \in \mathbb{R},$$

where $a > 0$ is a positive constant.

1. By considering supersolutions of the form

$$\overline{u}(t,x) = e^{\pm s_a(x \pm c_a t)}, \quad t>0,\ x \geq 0,$$

where $c_a > 0$ and $s_a > 0$ are positive constants depending on a, establish an estimate of the form

$$\forall t \geq 0, \forall x \in \mathbb{R}, \quad |u(t,x)| \leq e^{-s_a(|x| - c_a t)}.$$

2. Deduce that

$$\forall c > a + \frac{1}{a}, \quad \lim_{t\to+\infty} \sup_{|x|\geq ct} |u(t,x)| = 0, \quad \text{when } 0 < a < 1,$$

$$\forall c > 2, \quad \lim_{t\to+\infty} \sup_{|x|\geq ct} |u(t,x)| = 0, \quad \text{when } a \geq 1.$$

3. Draw a picture, admitting that

$$\forall 0 < c < a + \frac{1}{a}, \quad \lim_{t\to+\infty} \sup_{|x|\leq ct} |1-u(t,x)| = 0, \quad \text{when } 0 < a < 1,$$

$$\forall 0 < c < 2, \quad \lim_{t\to+\infty} \sup_{|x|\leq ct} |1-u(t,x)| = 0, \quad \text{when } a \geq 1.$$

Remark: Those limits can be obtained by constructing adapted subsolutions.

4. Comment.

Exercise 3. Rabies may infect all warm-blooded animals, also birds, and also humans, and affects the central nervous system. Vaccines are available (but expensive); but no further cure is known. The spread seems to occur in waves, e.g. one coming from the Polish-Russian border; the spread velocity is approx. 30-60 km/year.

Let us consider two groups of foxes:

. Susceptible foxes (S), with no diffusion (as they are territorial),

. Infective foxes (I), with diffusion (loss of sense of territory), constant death rate.

The infection rate is assumed to be proportional to their densities, no reproduction or further spread:

$$\begin{cases} \partial_t S = -rIS, & t > 0,\ x \in \mathbb{R}, \\ \partial_t I = rIS - aI + \nu \partial^2_{xx} I, & t > 0,\ x \in \mathbb{R}. \end{cases}$$

The non-dimensionalised version of the above system is the following:

$$\begin{cases} \partial_t S = -IS, & t > 0,\ x \in \mathbb{R}, \\ \partial_t I = IS - mI + \partial^2_{xx} I, & t > 0,\ x \in \mathbb{R}, \end{cases}$$

with $m = a/(rS_0)$, S_0 being the initial (maximum) susceptible density. We look for a travelling wave solution of this system of the form

$$S(t,x) = S(x-ct) = S(z) \quad \text{and} \quad I(t,x) = I(x-ct) = I(z),$$

where $z = x - ct$, the wave fronts S and I satisfying $0 \leq S \leq 1$ and $0 \leq I \leq 1$.

1. Write the system of ODEs satisfied by the functions S and I.
2. Justify the following boundary conditions: $S(+\infty) = 1$, $I(+\infty) = 0$, $S'(-\infty) = 0$, $I(-\infty) = 0$.
3. Check that

$$S(-\infty) - m \ln S(-\infty) = 1.$$

   Deduce the fraction of susceptibles which survive the "rabies wave" (draw a picture).

4. Draw the phase plane associated with the system satisfied by S and I.
5. Explain why $c = 2\sqrt{1-m}$ is the minimal wave speed.
6. Draw the shapes of the wave fronts S and I.

TD 8: Pseudo-differential operators

Exercise 1.

1. Let $L = \sum_{|\alpha|\le m} a_\alpha(x)\partial_x^\alpha$ be a differential operator of order $m \ge 0$ with smooth and fast decaying coefficients $a_\alpha \in C^\infty(\mathbb{R}^d)$. Prove that for all $u \in \mathscr{S}(\mathbb{R}^d)$,
$$(Lu)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x,\xi)\widehat{u}(\xi)\,\mathrm{d}\xi, \quad x \in \mathbb{R}^d,$$
where the symbol a is defined by
$$a(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)(i\xi)^\alpha, \quad (x,\xi) \in \mathbb{R}^{2d}.$$

2. For all $u_0 \in L^2(\mathbb{R}^d)$ and $t \ge 0$, we set $e^{t\Delta}u_0$ as the mild solution at time t of the heat equation
$$\begin{cases} \partial_t u - \Delta u = 0 & \text{on } (0,+\infty)\times\mathbb{R}^d, \\ u(0,\cdot) = u_0 & \text{on } \mathbb{R}^d. \end{cases}$$
Prove that for all $t \ge 0$, the evolution operator $e^{t\Delta}$ is a pseudo-differential operator and give the expression of its symbol.

3. Let $m \in \mathbb{R}$ and $A \in \mathrm{Op}(S^m)$. Prove that there exists a unique $a \in S^m$ such that $\mathrm{Op}(a) = A$.

Exercise 2. Let $a \in S^m$ be a symbol of order $m \in \mathbb{R}$.

1. We denote by $[\mathrm{Op}(a), \partial_{x_j}]$ the commutator between the operator $\mathrm{Op}(a)$ and the partial derivative ∂_{x_j} with respect to x_j. Prove that $[\mathrm{Op}(a), \partial_{x_j}]$ is also a pseudo-differential operator and compute its symbol as a function of a.
2. Same question with $[\mathrm{Op}(a), x_j]$, where x_j stands for the multiplication by x_j.

Exercise 3.

1. Let $m \in \mathbb{R}$ and $a \in S^m$. Prove that for all $s \in \mathbb{R}$, there exists a positive constant $c_s > 0$ such that
$$\forall u \in \mathscr{S}(\mathbb{R}^d), \quad \|\mathrm{Op}(a)u\|_{H^s} \le c_s\|u\|_{H^{s+m}}.$$
Hint: Any operator in $\mathrm{Op}(S^0)$ *is bounded in* $L^2(\mathbb{R}^d)$.

2. Let $m_1, m_2 \in \mathbb{R}$ and $a_1 \in S^{m_1}$, $a_2 \in S^{m_2}$. Check that
$$[\mathrm{Op}(a_1), \mathrm{Op}(a_2)] - \mathrm{Op}\left(\frac{1}{i}\{a_1,a_2\}\right) \in \mathrm{Op}(S^{m_1+m_2-2}),$$
where $\{a_1,a_2\}$ stands for the following Poisson bracket
$$\{a_1,a_2\} = \nabla_\xi a_1 \cdot \nabla_x a_2 - \nabla_x a_1 \cdot \nabla_\xi a_2.$$

Exercise 4. Let $m \in \mathbb{R}$ and $a \in S^m$.

1. Assume that there exists $b \in S^{-m}$ such that $\operatorname{Op}(a)\operatorname{Op}(b) - I \in \operatorname{Op}(S^{-\infty})$. Prove that there exist $R > 0$ and $c > 0$ such that

$$\forall (x,\xi) \in \mathbb{R}^{2d}, \quad |\xi| \geq R \Rightarrow |a(x,\xi)| \geq c\langle\xi\rangle^m. \tag{1}$$

Hint: Begin by checking that $ab - 1 \in S^{-1}$.

2. Let us now assume that the symbol a satisfies the condition (1). We aim at proving that there exists a symbol $b \in S^{-m}$ such that $\operatorname{Op}(a)\operatorname{Op}(b) - I \in \operatorname{Op}(S^{-\infty})$. The operator $\operatorname{Op}(b)$ is called a *parametrix* of the operator $\operatorname{Op}(a)$. To that end, we will construct a sequence of symbols $(b_j)_j$ such that $b_j \in S^{-m-j}$ and

$$\forall n \geq 0, \quad a \sharp (b_0 + \cdots + b_n) - 1 \in S^{-n-1}.$$

(a) Let $F \in C^\infty(\mathbb{C})$ such that $F(z) = 1/z$ when $|z| \geq c$. We set

$$b_0(x,\xi) = \frac{1}{\langle\xi\rangle^m} F(a(x,\xi)\langle\xi\rangle^{-m}), \quad (x,\xi) \in \mathbb{R}^{2d}.$$

Prove that $b_0 \in S^{-m}$ and that $a \sharp b_0 - 1 \in S^{-1}$.

(b) Construct then the other symbols b_j and conclude by using Borel's summation lemma.

(c) Check that we also have $\operatorname{Op}(b)\operatorname{Op}(a) - I \in \operatorname{Op}(S^{-\infty})$.

(d) *Application:* Prove that for all $s, t \in \mathbb{R}$, there exist some positive constants $a_s, b_{s,t} > 0$ such that

$$\forall u \in \mathscr{S}(\mathbb{R}^d), \quad \|u\|_{H^{s+m}} \leq a_s \|\operatorname{Op}(a)u\|_{H^s} + b_{s,t}\|u\|_{H^t}. \tag{2}$$

Exercise 5. Let $m \in \mathbb{R}$ and $a \in S^m$ be a symbol satisfying that there exist $c, R > 0$ such that

$$\forall (x,\xi) \in \mathbb{R}^{2d}, \quad |\xi| \geq R \Rightarrow \operatorname{Re} a(x,\xi) \geq c\langle\xi\rangle^m.$$

1. Prove that there exists $r \in S^{m-1}$ such that

$$\forall u \in \mathscr{S}(\mathbb{R}^d), \quad \operatorname{Re}\langle \operatorname{Op}(a)u, u\rangle_{L^2} = \langle \operatorname{Op}(\operatorname{Re} a)u, u\rangle_{L^2} + \langle \operatorname{Op}(r)u, u\rangle.$$

2. Prove that for all $\tilde{r} \in S^{m-1}$, there exists a positive constant $c > 0$ such that

$$\forall u \in \mathscr{S}(\mathbb{R}^d), \quad |\langle \operatorname{Op}(\tilde{r})u, u\rangle_{L^2}| \leq c\|u\|^2_{H^{(m-1)/2}}.$$

3. Prove that there exists $b \in S^{m/2}$ which is elliptic in the sense that (1) holds with $m/2$, and such that $\operatorname{Op}(\operatorname{Re} a) - \operatorname{Op}(b)^*\operatorname{Op}(b) \in \operatorname{Op}(S^{m-1})$.

4. Check that there exist $c_0, c_1 > 0$ such that

$$\forall u \in \mathscr{S}(\mathbb{R}^d), \quad \operatorname{Re}\langle \operatorname{Op}(a)u, u\rangle_{L^2} + c_1\|u\|^2_{H^{(m-1)/2}} \geq c_0\|u\|^2_{H^{m/2}}.$$

Hint: Use the estimate (2) with the operator $\operatorname{Op}(b)$.

5. Prove finally that for all $s \in \mathbb{R}$, there exist some positive constants $a_s, b_s > 0$ such that

$$\forall u \in \mathscr{S}(\mathbb{R}^d), \quad \operatorname{Re}\langle \operatorname{Op}(a)u, u\rangle_{L^2} + a_s\|u\|^2_{H^s} \geq b_s\|u\|^2_{H^{m/2}}.$$

Hint: When $s < (m-1)/2$, use Young's inequality with the exponents $p = 2(m-2s)/(m-2s-1)$ and $q = 2(m-2s)$.

TD 9: Pseudo-differential operators II

Exercise 1. Let $K : \mathbb{R}^{2d} \to \mathbb{C}$ be a continuous function. Assume that there exists $A > 0$ such that

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |K(x,y)|\,\mathrm{d}y \leq A, \quad \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} |K(x,y)|\,\mathrm{d}x \leq A.$$

For all $u \in C_0^\infty(\mathbb{R}^d)$, we set

$$(Pu)(x) = \int_{\mathbb{R}^d} K(x,y)u(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^d.$$

1. Check that Pu is well-defined and belongs to $L^\infty(\mathbb{R}^d)$.
2. We will prove Schur's lemma, stating that P can be uniquely extended to a bounded operator in $L^2(\mathbb{R}^d)$ satisfying $\|P\|_{\mathcal{L}(L^2)} \leq A$.
   a) By using Cauchy-Schwarz' inequality, check that for all $u \in C_0^0(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

   $$|(Pu)(x)|^2 \leq A \int_{\mathbb{R}^d} |K(x,y)||u(y)|^2\,\mathrm{d}y.$$

   b) Conclude.

Exercise 2. The purpose of this exercise is to prove Calderón-Vaillancourt's theorem: any pseudo-differential operator $\mathrm{Op}(a)$, with $a \in S^0$, is bounded in $L^2(\mathbb{R}^d)$.

1. We first assume that $a \in S^{-(d+1)}$.
   a) Check that $\mathrm{Op}(a)$ can be written

   $$\mathrm{Op}(a)u(x) = \int_{\mathbb{R}^d} K(x,y)u(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^d.$$

   where K is a kernel to be precised.
   b) Prove that the function $(x,y) \in \mathbb{R}^{2d} \mapsto (1 + |x-y|^{d+1})K(x,y)$ is bounded.
   c) Prove the theorem by using Exercise 1.
2. Prove with an induction that for all $k \in \{0,\dots,d\}$, the theorem is true when $a \in S^{k-(d+1)}$. *Hint: Consider the operator* $\mathrm{Op}(a)^*\,\mathrm{Op}(a)$.
3. The previous question implies in particular that the theorem holds when $a \in S^{-1}$. We now assume that $a \in S^0$.
   a) Prove that if $M > 0$ is large enough, there exist symbols $c \in S^0$ and $r \in S^{-1}$ such that

   $$\mathrm{Op}(c)^*\,\mathrm{Op}(c) = M\,\mathrm{Id} - \mathrm{Op}(a)^*\,\mathrm{Op}(a) + \mathrm{Op}(r).$$

   b) Conclude.

Exercise 3. Let $m \in \mathbb{R} \cup \{-\infty\}$ and $a \in S^m$.

1. Recall the expression of the kernel K of the operator $\operatorname{Op}(a)$.
2. Prove that when $m = -\infty$, K belongs to $C^\infty(\mathbb{R}^{2d})$.
3. Let $x, y \in \mathbb{R}^d$ such that $x \neq y$. We consider $\varphi, \psi \in C_0^\infty(\mathbb{R}^d)$ satisfying

   a) $\varphi = 1$ is a neighborhood of x,

   b) $\psi = 1$ is a neighborhood of y,

   c) $\operatorname{supp} \varphi \cap \operatorname{supp} \psi = \emptyset$.

   Show that $M_\varphi \operatorname{Op}(a) M_\psi$ belongs to $\operatorname{Op}(S^{-\infty})$, where M_φ and M_ψ denote the multiplication by φ and ψ respectively.
4. Compute the kernel of the operator $M_\varphi \operatorname{Op}(a) M_\psi$ as a function of K.
5. Prove that K is C^∞ in a neighborhood of (x, y).

Exercise 4.

1. Let $a \in C^\infty(\mathbb{R}^{2d})$ and $\chi \in C^\infty(\mathbb{R}^d)$ satisfying

   $$\chi(\xi) \neq 0 \Longleftrightarrow 1/2 < |\xi| < 2.$$

   For all $\lambda \geq 1$, we set $a_\lambda(x, \xi) = \chi(\xi) a(x, \lambda\xi)$. Prove that the following conditions are equivalent:

   a) $a \in S^m$,

   b) $\forall (\alpha, \beta) \in \mathbb{N}^{2d}, \exists C_{\alpha,\beta} > 0, \forall \lambda \geq 1,\ \|\partial_\xi^\alpha \partial_x^\beta a_\lambda\|_{L^\infty} \leq C\lambda^m$.
2. Let $f \in C^k(\mathbb{R}^d)$ satisfying that f and $\partial^\alpha f$ are bounded for all $\alpha \in \mathbb{N}^d$ such that $|\alpha| = k$.

   a) Prove that there exists a positive constant $c > 0$ independent on f such that for all $\beta \in \mathbb{N}^d$ satisfying $0 \leq |\beta| \leq k$,

   $$\|\partial^\beta f\|_{L^\infty} \leq c\bigg(\|f\|_{L^\infty} + \sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^\infty}\bigg).$$

   b) Prove that for all $\beta \in \mathbb{N}^d$ satisfying $0 \leq |\beta| \leq k$,

   $$\|\partial^\beta f\|_{L^\infty} \leq c\|f\|_{L^\infty}^{1-|\beta|/k}\bigg(\sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^\infty}\bigg)^{|\beta|/k}.$$

   Hint: Consider the function $g : x \in \mathbb{R}^d \mapsto f(\lambda x)$ for a well-chosen $\lambda > 0$.
3. Let $a \in S^m$. Assume that there exists $\mu > 0$ and $c > 0$ such that

   $$\forall (x, \xi) \in \mathbb{R}^{2d}, \quad |a(x, \xi)| \leq c\langle\xi\rangle^\mu.$$

   Prove that $a \in S^{\mu+\varepsilon}$ for all $\varepsilon > 0$.
4. Let A be a nilpotent pseudo-differential operator, *i.e.* satisfying $A^k = 0$ for some $k \geq 1$.

   a) Prove that $A \in \operatorname{Op}(S^{-\infty})$.

   b) Give a non-trivial example when $k = 2$.