Sheet 0: Sobolev spaces and elliptic problems in one dimension

Exercise 1. Let $\Omega = (0,1)$.

1. Prove that the following continuous embeddings hold
$$W^{1,1}(\Omega) \hookrightarrow C^0(\bar{\Omega}) \quad \text{and} \quad W^{1,p}(\Omega) \hookrightarrow C^{0,1-1/p}(\bar{\Omega}) \quad \text{when } p \in (1,\infty],$$
with the convention $1/\infty = 0$.

2. Prove that for all $1 \leq p < \infty$, the space $W_0^{1,p}(\Omega)$ is given by
$$W_0^{1,p}(\Omega) = \big\{ u \in W^{1,p}(\Omega) : u(0) = u(1) = 0 \big\}.$$

Exercise 2. Let $0 < \alpha < 1$ and $p > 1$ be positive real numbers. Show that there exists a positive constant $C_{\alpha,p} > 0$ such that for all $u \in C_0^\infty(\mathbb{R})$,
$$\left(\iint_{\mathbb{R}\times\mathbb{R}} \left(\frac{|u(x) - u(y)|}{|x-y|^\alpha} \right)^p \frac{\mathrm{d}x\mathrm{d}y}{|x-y|} \right)^{1/p} \leq C_{\alpha,p} \|u\|_{L^p(\mathbb{R})}^{1-\alpha} \|\nabla u\|_{L^p(\mathbb{R})}^{\alpha}.$$

Hint: Consider the two regions $\{|x-y| > R\}$ and $\{|x-y| \leq R\}$, where $R > 0$ is to be chosen.

Exercise 3. Let $\Omega = (0,1)$. Establish the following Poincaré inequality
$$\forall f \in H_0^1(\Omega), \quad \|f\|_{L^2(\Omega)} \leq \frac{1}{\pi} \|f'\|_{L^2(\Omega)},$$
and prove that the constant $1/\pi$ is optimal. *Hint*: Use Fourier series.

Exercise 4. Let $\Omega = (0,1)$. We consider $f \in L^2(\Omega)$ and $\alpha \in L^\infty(\Omega)$ satisfying $0 < \alpha_{\min} \leq \alpha(x)$ a.e. in Ω.

1. By using the Riesz representation theorem, prove that there exists a unique $u \in H_0^1(\Omega)$ satisfying
$$-(\alpha u')' + u = f \quad \text{in } \mathscr{D}'(\Omega).$$
When in addition $\alpha \in C^\infty(\Omega)$, check that $u \in H^2(\Omega)$ and that the above equality holds in $L^2(\Omega)$. What about the case $\alpha = 1$?

2. We consider moreover $\beta \in C^1(\Omega)$ a function satisfying $\beta' \leq 2$ on Ω. By using the Lax-Milgram theorem, prove that there exists a unique $u \in H_0^1(\Omega)$ satisfying
$$-(\alpha u')' + \beta u' + u = f \quad \text{in } \mathscr{D}'(\Omega).$$

Exercise 5. Let $\Omega = (-1, 1)$. We consider $f \in C^0(\bar{\Omega})$ and g a function defined on $\partial\Omega$. Solve explicitly in $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ the following elliptic problem with Neumann boundary conditions

$$\begin{cases} -u'' = f & \text{on } \Omega, \\ \frac{\partial u}{\partial n} = g & \text{on } \partial\Omega. \end{cases}$$

Can the Lax-Milgram theorem be used to study the above boundary value equation ?

Exercise 6. Let $\Omega = (0, 1)$. We consider $f \in L^2(\Omega)$ and $\phi : \mathbb{R} \to \mathbb{R}_+$ a strictly convex C^1 function. The purpose is to prove with a variational method that there exists a unique function $u \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying

$$-u'' + \phi'(u) = f \quad \text{in } L^2(\Omega). \tag{1}$$

1. Preliminaries: Let H be a real Hilbert space and $J : H \to \mathbb{R}$ be a continuous convex functional. We assume that J is coercive, that is, $J(x) \to +\infty$ when $\|x\| \to +\infty$. Prove then that there exists $x_\star$ in H such that $J(x_\star) = \inf_{x \in H} J(x)$.

2. In this question, we prove that there exists a unique $u \in H_0^1(\Omega)$ such that

$$\forall v \in H_0^1(\Omega), \quad \int_0^1 \left(u'(x)v'(x) + \phi'(u(x))v(x) - f(x)v(x)\right) \mathrm{d}x = 0. \tag{2}$$

   To that end, we introduce the functional $J : H_0^1(\Omega) \to \mathbb{R}$ defined for all $v \in H_0^1(\Omega)$ by

$$J(v) = \int_0^1 \left(\frac{1}{2}|v'(x)|^2 + \phi(v(x)) - f(x)v(x)\right) \mathrm{d}x.$$

   a) Check that the functional J is well-defined, strictly convex and coercive.
   b) Prove that the functional J is differentiable on $H_0^1(\Omega)$ and give the expression of its derivative.
   c) Deduce from the preliminary question that the variational problem (2) admits a unique solution $u \in H_0^1(\Omega)$.

3. Prove that the unique function $u \in H_0^1(\Omega)$ satisfying (2) belongs to $H^2(\Omega)$ and is also the unique function that satisfies (1).

4. When the function f is moreover continuous on $[0, 1]$, check that $u \in C^2(\bar{\Omega})$ is a strong solution of (1), that is

$$\forall x \in [0, 1], \quad -u''(x) + \phi'(u(x)) = f(x).$$

Exercise 7. Let $\Omega = (0, 1)$. Prove that there exists a unique function $u \in C^\infty(\bar{\Omega})$ satisfying

$$\begin{cases} -u'' + u = \cos(u), \\ u(0) = u(1) = 0. \end{cases}$$

Hint: Use the Banach-Picard fixed point theorem on the space $L^2(\Omega)$.

Sheet 1: Weak formulation of elliptic equations

Exercise 1 (Ellipticity). For each of the following linear differential operator L, give the symbol, the principal symbol of L, and discuss the ellipticity and uniform ellipticity.

1. $Lu(x) = -\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x)\frac{\partial u}{\partial x_i} + c(x)u, \quad x \in \Omega \subset \mathbb{R}^d,$
2. $Lf(x,v) = v \cdot \nabla_x f + F(x) \cdot \nabla_v f, \quad x, v \in \mathbb{R}^d,\ F\colon \mathbb{R}^d \to \mathbb{R}^d,$
3. $Lu(t,x) = \partial_t u - \Delta u, \quad t > 0,\ x \in \mathbb{R}^d,$
4. $Lu(t,x) = \partial_t u - i\Delta u, \quad t > 0,\ x \in \mathbb{R}^d.$

Exercise 2 (Faber-Krahn inequality). Let Ω be an open bounded subset of $\mathbb{R}^d$ with $d \geq 3$ and $V \in L^\infty(\Omega)$ such that $V \geq 0$. We consider the problem

$$\begin{cases} -\Delta u = Vu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

1. Give the definition of a weak solution to (1).
2. Can you apply the Lax-Milgram theorem here?
3. Let $r > \frac{d}{2}$. Show that there is a constant $c_d > 0$ depending on d only such that, if (1) has a non-trivial weak solution, then
$$|\Omega|^{\frac{2}{d}-\frac{1}{r}} \|V\|_{L^r(\Omega)} \geq c_d.$$
Hint: Use the following Sobolev inequality
$$\|u\|_{L^{2^*}(\Omega)} \leq M_d \|\nabla u\|_{L^2(\Omega)}, \qquad \frac{1}{2^*} = \frac{1}{2} - \frac{1}{d},$$
which holds for all $u \in H^1_0(\Omega)$, where M_d depends on d only.
4. What do you obtain in the particular case $V = \lambda = \text{cst}$?

Exercise 3 (Dirichlet problem). Let Ω be an open bounded subset of $\mathbb{R}^d$, $f \in L^2(\Omega)$ and $F \in L^2(\Omega)^d$. Show that the following elliptic problem with Dirichlet boundary condition

$$\begin{cases} -\Delta u = f - \operatorname{div} F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique weak solution $u \in H^1_0(\Omega)$.

Exercise 4 (Neumann problem). Let Ω be an open bounded subset of $\mathbb{R}^d$ with smooth boundary, the exterior unit normal being denoted by n, and $f \in L^2(\Omega)$. Show that, for all $\mu > 0$, the elliptic problem with Neumann boundary condition

$$\begin{cases} -\Delta u + \mu u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

has a unique weak solution $u \in H^1(\Omega)$. In the case $\mu = 0$, give a necessary condition on $\int_\Omega f$ to the existence of a weak solution to (2).

Exercise 5 (Fourier condition). Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with smooth boundary, $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$ and $\lambda > 0$. We consider the following elliptic problem with Fourier boundary condition

$$\left\{\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ \lambda u + \frac{\partial u}{\partial n} &= g \quad \text{on } \partial\Omega. \end{aligned}\right. \tag{3}$$

1. Give the variational formulation of the problem (3).
2. Prove that there exists a positive constant $C_\Omega > 0$ only depending on Ω such that for all $u \in H^1(\Omega)$,
$$\|u\|^2_{L^2(\Omega)} \leq C_\Omega\big(\|\nabla u\|^2_{L^2(\Omega)} + \lambda\|\gamma_0 u\|^2_{L^2(\partial\Omega)}\big),$$
where γ_0 denotes the trace operator $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$.
3. Prove that (3) has a unique weak solution.
4. * Is this weak solution a strong solution ?

Exercise 6 (System of equations). Let Ω be an open bounded subset of $\mathbb{R}^d$, $f, g \in L^2(\Omega)$ and A, B, C, D be four matrices in $\mathcal{M}_d(\mathbb{R})$. We analyse the following system of equations with Dirichlet boundary counditions

$$\left\{\begin{aligned} -\operatorname{div}(A\nabla u) - \operatorname{div}(B\nabla v) &= f \quad \text{in } \Omega, \\ -\operatorname{div}(C\nabla u) - \operatorname{div}(D\nabla v) &= g \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \\ v &= 0, \quad \text{on } \partial\Omega. \end{aligned}\right.$$

We assume that the *Legendre-Hadamard ellipticity condition* holds:

$$\exists\theta > 0, \forall\xi \in \mathbb{R}^d, \forall\eta \in \mathbb{R}^2, \quad \sum_{i,j=1}^{2} (\mathcal{A}_{ij}\xi \cdot \xi)\eta_i\eta_j \geq \theta|\xi|^2|\eta|^2 \quad \text{where} \quad \mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

1. Check that A and D are uniformly elliptic.
2. Show that the functional $a(U, V) := \int_\Omega \sum_{i,j=1}^2 \mathcal{A}_{ij}\nabla U_i \cdot \nabla V_j \, \mathrm{d}x$ is continuous and coercive on $H^1_0(\Omega) \times H^1_0(\Omega)$. *Hint*: Use Bessel-Parseval theorem after extending U and V by 0 outside Ω.
3. Conclude.

Exercise 7 (Resolution by minimization). Let $\Omega \subset \mathbb{R}^3$ be open, bounded with smooth boundary. The purpose is to prove that the following elliptic problem has a non-trivial weak solution

$$\left\{\begin{aligned} -\Delta u &= u^3 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right.$$

1. Prove that there exists a solution to the following minimization problem
$$\inf\big\{\|\nabla v\|_{L^2(\Omega)} : v \in H^1_0(\Omega),\ \|v\|_{L^4(\Omega)} = 1\big\}. \tag{4}$$
Recall: Since $d = 3$ here, the continuous embedding $H^1_0(\Omega) \hookrightarrow L^q(\Omega)$ holds for all $1 \leq q \leq 6$, and is moreover compact when $1 \leq q < 6$.
2. Prove that if the function $v \in H^1_0(\Omega)$ solves (4), there exists a positive constant $\lambda > 0$ such that $-\Delta v = \lambda v^3$ weakly in Ω.
3. Conclude.

Sheet 2: Elliptic regularity and maximum principle

Exercise 1 (Control of the L^∞ norm). Let Ω be an open bounded subset of $\mathbb{R}^d$ of class C^2. Let $A \in C^1(\overline{\Omega}, S_d(\mathbb{R}))$ satisfying the following ellipticity condition

$$\exists \alpha > 0, \forall (x,\xi) \in \Omega \times \mathbb{R}^d, \quad A(x)\xi \cdot \xi \geq \alpha |\xi|^2. \tag{1}$$

Let $f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$ be the weak solution of the following Dirichlet problem

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

1. In this question, we assume that $d \leq 3$. Show that there exists a constant $C \geq 0$ depending only on Ω and d such that

$$\|u\|_{L^\infty(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}). \tag{2}$$

2. We assume that $\Omega = B(0,R)$ where $R > 0$.
   (a) Compute Δv when $v(x) = \psi(|x|)$ is a radial function.
   (b) By considering the function $u(x) = \ln|\ln|x||$ and the case $A(x) = \mathrm{I}_d$, discuss the validity of the estimate (2) when $d \geq 4$.

Note: One can prove (this is a bit technical) that when $d \geq 4$ and $f \in L^p(\Omega)$, where $p > d/2$, there exists a positive constant $C > 0$ only depending on d, Ω and p such that the following estimate, somehow analogous to (2), holds

$$\|u\|_{L^\infty(\Omega)} \leq C(\|f\|_{L^p(\Omega)} + \|u\|_{L^2(\Omega)}).$$

Exercise 2 (Hölder regularity). The purpose is to show a gains of derivatives in Hölder spaces for the solution u to the Laplace equation $-\Delta u = f$, where $f \in C(\mathbb{R}^3)$ is a function with compact support. Let $G(x) = \frac{1}{4\pi}\frac{1}{|x|}$ be the Green function of the Laplacian in dimension 3. Let us recall that the function $u = G * f$ is a weak solution of the Poisson equation $-\Delta u = f$ in $\mathbb{R}^3$. We assume that $f \in C^\alpha(\mathbb{R}^3)$ for a given $\alpha \in (0,1)$, and we set

$$[f]_{\dot{C}^\alpha(\mathbb{R}^3)} = \sup_{x \neq z \in \mathbb{R}^3} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < +\infty.$$

Let K be a compact of $\mathbb{R}^3$. We want to prove that $u, \nabla u \in C^\alpha(K)$ and that there exists a positive constant $c_1 > 0$ only depending on K, d, α and on the support of f such that

$$[u]_{\dot{C}^\alpha(K)} + [\nabla u]_{\dot{C}^\alpha(K)} \leq c_1 [f]_{\dot{C}^\alpha(\mathbb{R}^3)}. \tag{3}$$

1. Show that $u \in C^\alpha(K)$ and that the estimate (3) holds for u.

2. By introducing a cut-off function ω_ε of the forme $\omega_\varepsilon(x) = \theta(\varepsilon^{-1}|x|)$ and considering the approximation $u_\varepsilon = (G\omega_\varepsilon) * f$, prove that $\nabla u \in C^\alpha(K)$ and that the estimate (3) holds for the function ∇u.

Note: By using similar techniques, one can prove that for all $\delta \in (0,\alpha)$, we have $\nabla^2 u \in C^\delta(K)$ and also that there exists a positive constant $c_2 > 0$ depending only on K, d, α, δ and the support of the function f such that

$$[\nabla^2 u]_{\dot{C}^\delta(K)} \leq c_2 [f]_{\dot{C}^\alpha(\mathbb{R}^3)}.$$

Exercise 3 (A non-linear equation). Let Ω be a bounded subset of $\mathbb{R}^d$ and $b : \mathbb{R}^d \to \mathbb{R}$ be a 1-Lipschitz function. Prove that the equation

$$\begin{cases} -\Delta u + u = b(\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique weak solution $u \in H_0^1(\Omega)$. Assuming moreover that $b \in C^\infty(\mathbb{R}^d)$, check that this solution u belongs to $C^\infty(\Omega)$.

Exercise 4 (Weak maximum principle). Let Ω be a bounded open subset of $\mathbb{R}^d$ with smooth boundary and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying $\Delta u \leq 0$ on Ω. Proof by hand that

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

Hint: Assume first that $\Delta u < 0$.

Exercise 5 (Weak maximum principle for weak solutions). Let $\Omega \subset \mathbb{R}^d$ be a bounded open set.

1. Let $G \in C^1(\mathbb{R})$ a function with bounded derivative satisfying $G(0) = 0$.
   a) Check that for all $u \in H^1(\Omega)$, we have $G \circ u \in L^2(\Omega)$.
   b) Prove that $G \circ u \in H^1(\Omega)$ and that for all $1 \leq j \leq n$,

$$\partial_{x_j}(G \circ u) = (G' \circ u)\partial_{x_j} u.$$

2. We consider the following operator $L = -\operatorname{div}(A(x)\nabla u)$, where $A \in L^\infty(\Omega, M_d(\mathbb{R}))$ satisfies the following ellipticity assumption

$$\exists \alpha > 0, \forall (x,\xi) \in \Omega \times \mathbb{R}^d, \quad A(x)\xi \cdot \xi \geq \alpha |\xi|^2.$$

   We want to prove that if $u \in H_0^1(\Omega)$ is a weak solution of the equation $Lu \leq 0$, then $u \leq 0$ a.e. in Ω.
   (a) Prove that there exists a non-negative function $G \in C^1(\mathbb{R})$ with bounded derivative such that $G' > 0$ on $(0, +\infty)$ and $G' = 0$ on $(-\infty, 0]$.
   (b) By considering $\langle Lu, G \circ u \rangle_{L^2(\Omega)}$, prove that

$$\int_\Omega |\nabla u(x)|^2 (G' \circ u)(x) \, \mathrm{d}x \leq 0.$$

   (c) Conclude.

Exercise 6 (Estimates on the gradient). Let Ω be an open bounded subset of $\mathbb{R}^d$. Let A be a symmetric definite positive $d \times d$ matrix and $f \in \mathrm{Lip}(\bar{\Omega})$. We will establish gradient estimates for solutions u to the equation $Lu = f$ with Dirichlet homogeneous boundary condition, where L is the elliptic operator $Lu = -\operatorname{div}(A\nabla u)$, under the assumption that there exists a function $\psi \in \mathrm{Lip}(\Omega) \cap C^2(\Omega)$ such that $L\psi \geq f$ in Ω and $\psi = 0$ on $\partial\Omega$. For simplicity, we will consider the case where the function f is constant.

1. Let $\omega \subset \Omega$ and $u, v \in C^2(\omega) \cap C(\bar{\omega})$ satisfying $Lu \leq Lv$ in ω. Show that

$$\sup_{\overline{\omega}}(u - v) \leq \sup_{\partial\omega}(u - v).$$

2. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying $Lu = f$ in Ω.
   (a) Prove that

$$\sup\left\{\frac{|u(x) - u(y)|}{|x - y|} : x, y \in \Omega, x \neq y\right\} = \sup\left\{\frac{|u(x) - u(y)|}{|x - y|} : x \in \Omega, y \in \partial\Omega\right\}.$$

   Hint: given $x_1, x_2 \in \Omega$ with $\tau = x_2 - x_1 \neq 0$, compare u and $u_\tau \colon x \mapsto u(x + \tau)$ in $\omega = \Omega \cap (-\tau + \Omega)$.

   (b) We assume furthermore that $u = 0$ on $\partial\Omega$. Show that $\mathrm{Lip}(u) \leq \mathrm{Lip}(\psi)$.

3. A ψ as above is called a *barrier function*. Construct a barrier function in the case $\Omega = B(0, 1)$. *Hint*: consider $\psi(x) = -\gamma|x|^2/2 + C$ for some given constants $\gamma > 0$ and $C \in \mathbb{R}$.

Exercise 7. (Localization) Let Ω be an open subset of $\mathbb{R}^d$. We consider $A \in L^\infty_{\mathrm{loc}}(\Omega, M_d(\mathbb{R}))$, $b \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^d)$ and $c \in L^\infty_{\mathrm{loc}}(\Omega)$ and L the operator defined by

$$Lu = -\operatorname{div}(A(x)\nabla u) + b \cdot \nabla u + cu.$$

Assume that A satisfies the following ellipticity assumption

$$\exists \alpha > 0, \forall (x, \xi) \in \Omega \times \mathbb{R}^d, \quad A(x)\xi \cdot \xi \geq \alpha|\xi|^2.$$

Let Ω' and Ω'' be open subsets of Ω satisfying $\overline{\Omega'} \subset \Omega''$ and $\overline{\Omega''} \subset \Omega$. Prove that there exists a positive constant $C > 0$ such that for all $u \in H^1_{\mathrm{loc}}(\Omega)$,

$$\|\nabla u\|_{L^2(\Omega')} \leq C\big(\|Lu\|_{H^{-1}(\Omega'')} + \|u\|_{L^2(\Omega'')}\big).$$

Sheet 3: Heat equation

Exercise 1 (Heat kernel). Let $d \geq 1$ and $E_d \in \mathscr{S}'(\mathbb{R}_t \times \mathbb{R}^d_x)$ be the tempered distribution defined by

$$E_d(t,x) = \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}} \mathbf{1}_{]0,+\infty[}(t).$$

Prove that E_d is a fundamental solution of the heat operator, that is, satisfies

$$\Big(\partial_t - \frac{1}{2}\Delta\Big) E_d = \delta_{(t,x)=(0,0)} \quad \text{in } \mathscr{S}'(\mathbb{R}_t \times \mathbb{R}^d_x).$$

Check that E_d is unique under the condition $\operatorname{Supp} E_d \subset \mathbb{R}_+ \times \mathbb{R}^d$.

Exercise 2 (Heat equation on $\mathbb{R}^d$). Let $u_0 \in L^2(\mathbb{R}^d)$. We consider the homogeneous heat equation posed on the whole space $\mathbb{R}^d$:

$$\left\{ \begin{array}{rcll} \partial_t u - \frac{1}{2}\Delta u &=& 0 & \text{on } (0,+\infty) \times \mathbb{R}^d, \\ u(0,\cdot) &=& u_0 & \text{on } \mathbb{R}^d. \end{array} \right. \tag{1}$$

1. (Regularity) Compute explicitly the solution of the equation (1). What is its regularity ?
2. (Energy estimate) Show that for all $t \geq 0$,
$$\|u(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} + \int_0^t \|\nabla u(s,\cdot)\|^2_{L^2(\mathbb{R}^d)} \, \mathrm{d}s = \|u_0\|^2_{L^2(\mathbb{R}^d)}.$$
3. (Maximum principle) Show that if $u_0 \in L^\infty(\mathbb{R}^d)$, then $u(t,\cdot) \in L^\infty(\mathbb{R}^d)$ for all $t \geq 0$ and
$$\sup_{t\geq 0} \|u(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}.$$
4. (Infinite speed of propagation) Prove that if $u_0 \geq 0$ is a function not identically equal to zero and non-negative, then $u > 0$ in $\mathbb{R}_+ \times \mathbb{R}^d$.

Exercise 3 (Spectral theory). Let Ω be a bounded open subset of $\mathbb{R}^d$.

1. Explain why the operator $\Delta^{-1} : L^2(\Omega) \to H^1_0(\Omega)$ is a continuous isomorphism.
2. Let $\iota : H^1_0(\Omega) \to L^2(\Omega)$ be the canonical injection. Check that the operator $T = -\Delta^{-1} \circ \iota : H^1_0(\Omega) \to H^1_0(\Omega)$ is non-negative, selfadjoint, one to one and compact.
3. Deduce that the spectrum of the Laplacian operator $-\Delta$ with Dirichlet boundary condition is a sequence $(\lambda_n)_{n\geq 0}$ of positive real numbers which is increasing and diverges to $+\infty$, and also that there exists a Hilbert basis $(e_n)_{n\geq 0}$ of $H^1_0(\Omega)$ composed of eigenfunctions of $-\Delta$ and such that
$$\forall n \geq 0, \quad -\Delta e_n = \lambda_n e_n.$$
4. Compute explicitly those eigenvalues and those eigenfunctions when $d = 1$ and $\Omega = (0,1)$.

Exercise 4 (Heat equation on bounded domains). Let Ω be a bounded open subset of $\mathbb{R}^d$ with regular boundary, $T > 0$ be a final time, $u_0 \in L^2(\Omega)$ be an initial datum and $f \in L^2((0,T), L^2(\Omega))$ be a source term. We aim at proving that there exists a unique solution $u \in L^2((0,T), H_0^1(\Omega)) \cap C^0([0,T], L^2(\Omega))$ to the following heat equation with Dirichlet boundary conditions

$$\begin{cases} \partial_t u - \Delta u = f & \text{a.e. in } (0,T)\times\Omega, \\ u = 0 & \text{a.e. on } (0,T)\times\partial\Omega, \\ u(0,\cdot) = u_0 & \text{a.e. in } \Omega. \end{cases} \tag{2}$$

We will also check that this solution satisfies the following energy estimate for all $0 \le t \le T$,

$$\|u(t,\cdot)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u(s,\cdot)\|^2_{L^2(\Omega)}\,\mathrm{d}s \le C\left(\|u_0\|^2_{L^2(\Omega)} + \int_0^t \|f(s,\cdot)\|^2_{L^2(\Omega)}\,\mathrm{d}s\right), \tag{3}$$

where $C > 0$ is a positive constant only depending on Ω. In the following, we consider $(e_n)_{n\ge 0}$ a Hilbert basis of $L^2(\Omega)$ composed of eigenfunctions of the operator $-\Delta$. Moreover, we set λ_n the eigenvalue associated with the eigenfunction e_n.

1. We first prove that there exists a unique $u \in L^2((0,T), H_0^1(\Omega)) \cap C^0([0,T], L^2(\Omega))$ satisfying

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\langle u(t,\cdot), v\rangle_{L^2(\Omega)} + \displaystyle\int_\Omega \nabla u(t,\cdot)\cdot\nabla v = \langle f(t,\cdot), v\rangle_{L^2(\Omega)} & \forall v \in H_0^1(\Omega), \forall t \in (0,T), \\ u(0,\cdot) = u_0. \end{cases}$$

   a) Define properly this variational formulation.
   b) Give the expansion in the Hilbert basis $(e_n)_{n\ge 0}$ of such a solution.
   c) Prove that this expansion converges in $L^2((0,T), H_0^1(\Omega))$ and also in $C^0([0,T], L^2(\Omega))$.
   d) Conclude.

2. We now want to prove that this weak solution u is a strong solution, that is, is solution of the problem (2).

   a) Check that the boundary condition and the initial value condition hold.
   b) * Prove that $\partial_t u - \Delta u = f$ a.e. in $(0,T)\times\Omega$.

3. When $f = 0$, check that

$$\forall t \ge 0, \quad \|u(t,\cdot) - \langle u_0, e_0\rangle_{L^2(\Omega)} e^{-\lambda_0 t} e_0\|_{L^2(\Omega)} \le e^{-\lambda_1 t}\|u_0\|_{L^2(\Omega)}.$$

Exercise 5 (Maximum principle). Let Ω be a bounded open subset of $\mathbb{R}^d$ with smooth boundary, $T > 0$ be a final time, $u_0 \in H_0^1(\Omega)$ be an initial datum and $f \in L^2((0,T), L^2(\Omega))$ be a a source term. We consider $u \in L^2((0,T), H_0^1(\Omega)) \cap C^0([0,T], L^2(\Omega))$ the unique solution of the problem (2). Prove that when $f \ge 0$ a.e. in $(0,T)\times\Omega$ and $u_0 \ge 0$ a.e. in Ω, then $u \ge 0$ a.e. on $(0,T)\times\Omega$. *Hint*: Admit that $\partial_t u \in L^2((0,T), L^2(\Omega))$ and $u \in L^2((0,T), H^2(\Omega)) \cap C^0([0,T], H_0^1(\Omega))$.

Application *: Assume now that $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and $f \in L^\infty([0,+\infty)\times\Omega)$. Show that

$$\sup_{t\ge 0}\|u(t,\cdot)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} + \frac{\mathrm{diam}(\Omega)^2}{2d}\sup_{t\ge 0}\|f(t,\cdot)\|_{L^\infty(\Omega)}.$$

SHEET 4: EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR REACTION-DIFFUSION EQUATIONS

EXERCISE 1. We consider the following reaction-diffusion equation:

$$(1) \qquad \begin{cases} \partial_t u - \Delta u = u^2 & \text{in } (0,+\infty)\times\mathbb{R}, \\ u(0,\cdot) = u_0 & \text{in } \mathbb{R}. \end{cases}$$

1. Establish *a priori* energy estimates for any smooth solution of the equation (1).
2. Assume that $u_0 \in H^1(\mathbb{R})$. We aim at proving, by using an iterative method, that there exist $T > 0$ and a solution $u \in C^0([0,T], H^1(\mathbb{R}))$ of the equation (1). We therefore consider the sequence $(u^n)_{n\geq 0}$ recursively defined by $u^0 = u_0$ and

$$(2) \qquad \begin{cases} \partial_t u^{n+1} - \Delta u^{n+1} = (u^n)^2 & \text{in } (0,+\infty)\times\mathbb{R}, \\ u^{n+1}(0,\cdot) = u_0 & \text{in } \mathbb{R}. \end{cases}$$

   (a) Discuss the well-posedness of the sequence $(u^n)_{n\geq 0}$.

   (b) (Bound in H^1) Prove that there exists a positive time $T_1 > 0$ and a positive constant $c_1 > 0$ such that for all $n \geq 0$ and $0 \leq t \leq T_1$,
$$\|u^n(t,\cdot)\|_{H^1(\mathbb{R})} \leq c_1.$$

   (c) (Convergence in H^1) Prove that there exists another positive time $0 < T_2 < T_1$ and another positive constant $c_2 > 0$ satisfying that for all $n \geq 0$ and $0 \leq t \leq T_2$,
$$\|u^{n+1}(t,\cdot) - u^n(t,\cdot)\|_{H^1(\mathbb{R})} \leq \frac{c_2}{2^n}.$$

   (d) Conclude.

3. Is this solution unique ?

EXERCISE 2. Let $u_0 \in H^1(\mathbb{R})$ be a smooth initial value. We consider $T > 0$ the positive time and $u \in C^0([0,T], H^1(\mathbb{R}))$ the solution of the equation (1), both given by the previous exercise. By using a bootstrap argument, prove that the function u is smooth, precisely $u \in C^\infty(]0,T[\times\mathbb{R})$.

EXERCISE 3. By adapting the strategy used in the first exercise, investigate the existence of solutions for the following reaction-diffusion equation:

$$(3) \qquad \begin{cases} \partial_t u - \Delta u = \arctan(u) & \text{in } (0,+\infty)\times\mathbb{R}^d, \\ u(0,\cdot) = u_0 & \text{in } \mathbb{R}^d, \end{cases}$$

with initial datum $u_0 \in L^2(\mathbb{R}^d)$.
Hint: The function arctan is globally Lipschitz continuous, only L^2 estimates are required.

Exercise 4. Let $T > 0$ and $u_0 \in L^2(\mathbb{R}^d)$. We consider the following initial value problem:

$$\left\{ \begin{aligned} \partial_t u - \Delta u &= \sqrt{1+u^2}, &&\text{in } (0,T] \times \mathbb{R}^d, \\ u(0,\cdot) &= u_0 &&\text{in } \mathbb{R}^d, \end{aligned} \right. \tag{4}$$

We say that a continuous function $u \in C^0([0,T], L^2(\mathbb{R}^d))$ is a *mild* solution of the initial value problem (4) when it satisfies the following integral equation for all $0 \le t \le T$:

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \sqrt{1 + u(s)^2}\, \mathrm{d}s,$$

where, for all $v \in L^2(\mathbb{R}^d)$, $e^{t\Delta} v$ denotes the solution of the heat equation posed on $\mathbb{R}^d$ with initial datum v.

1. We consider the function $F : C^0([0,T], L^2(\mathbb{R}^d)) \to C^0([0,T], L^2(\mathbb{R}^d))$ defined by

$$(Fu)(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \sqrt{1 + u(s)^2}\, \mathrm{d}s.$$

   By using a fixed-point theorem on the function F, prove that the equation (4) admits a unique mild solution $u \in C^0([0,T], L^2(\mathbb{R}^d))$.

2. Check that the function $u_0 \in L^2(\mathbb{R}^d) \mapsto u \in C^0([0,T], L^2(\mathbb{R}^d))$ is Lipschitz continuous.

Exercise 5. Study the existence of mild solutions for the equation (3) and make the link with the solution constructed by iterative method in Exercise 3.

Exercise 6. Let $\Omega = (0,1)$, $t_0 > 0$ and $u_0 \in H_0^1(\Omega)$. We aim at proving that there exist a positive time $t^* > 0$ and a unique function $u \in C^0([t_0, t^*[, H_0^1(\Omega))$ solution of the following integral equation for all $t_0 \le t < t^*$:

$$u(t) = e^{(t-t_0)\Delta} u_0 + \int_{t_0}^t e^{(t-s)\Delta} \sinh(u(s))\, \mathrm{d}s. \tag{5}$$

Let us recall that there exists a Hilbert basis $(e_n)_{n\ge 0}$ of the space $H_0^1(\Omega)$ composed of eigenvalues of the operator $-\Delta$. In the above integral equation, the operator $e^{t\Delta} \in \mathcal{L}(H_0^1(\Omega))$ is defined by

$$e^{t\Delta} = \sum_{n=0}^{+\infty} e^{-t\lambda_n} \langle \cdot, e_n \rangle_{H_0^1} e_n,$$

with $\lambda_n > 0$ the eigenvalue associated with the eigenfunction e_n.

1. By using a fixed-point theorem, prove that there exists a positive time $t_1 > t_0$ such that the equation (5) has a solution in the space $C^0([t_0, t_1], H_0^1(\Omega))$.

2. Explain how this solution can be extended to the interval $[t_0, t_1 + \delta]$ with $\delta > 0$. Deduce, proceeding by contradiction, that if $[t_0, t^*[$ stands for the maximal interval of existence of the solution u and if $t^* < +\infty$, then

$$\lim_{t \nearrow t^*} \|u(t)\|_{H_0^1(\Omega)} = +\infty.$$

3. Investigate the uniqueness of such a solution.

4. Of which equation is the function u a mild solution ?

Sheet 5: Maximum principles and stability of steady states

Exercise 1. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $T > 0$ be a final time and $Q_T = (0,T) \times \Omega$. We consider the following differential operator

$$L = -\sum_{i,j=1}^{d} a^{i,j}(t,x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{n} b^i(t,x)\partial_{x_i} + c(t,x), \quad (t,x) \in Q_T,$$

the coefficients $a^{i,j}, b^i$ and c being bounded on Q_T, with moreover $a^{i,j} = a^{j,i}$. We assume that the operator $\partial_t + L$ is uniformly parabolic, that is,

$$\exists \theta > 0, \forall (t,x) \in Q_T, \forall \xi \in \mathbb{R}^d, \quad \sum_{i,j=1}^{d} a^{i,j}(t,x)\xi_i\xi_j \geq \theta|\xi|^2.$$

State as many maximum principles as you can for the parabolic operator $\partial_t + L$.

Exercise 2. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $T > 0$ be a positive time and $Q_T = (0,T) \times \Omega$. We also consider $f \in C^\infty(\mathbb{R})$ a smooth function. Let $u, v \in C^2(Q_T) \cap C^0(\bar{Q}_T)$ be two functions satisfying

$$\begin{cases} \partial_t v - \Delta v - f(v) \leq \partial_t u - \Delta u - f(u) & \text{in } Q_T, \\ v \leq u & \text{on } \partial Q_T. \end{cases}$$

Prove that $v \leq u$ on Q_T.

Application: Consider $u \in C^2(Q_T) \cap C^0(\bar{Q}_T)$ a solution of the equation

$$\begin{cases} \partial_t u - \Delta u = u(1-u)(u-a) & \text{in } Q_T, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,\cdot) = u_0 & \text{in } \Omega, \end{cases}$$

where $0 < a < 1$ is a positive constant and u_0 is a smooth initial datum satisfying $0 \leq u_0 \leq 1$ in Ω. Prove that the function u is bounded as follows

$$\forall (t,x) \in Q_T, \quad 0 \leq u(t,x) \leq 1.$$

Can you be more precise when assuming $0 \leq u_0 < a$ in Ω ?

Exercise 3. Let $L > 0$. Prove that there exists a critical length $L_c > 0$ such that the equation

$$\begin{cases} q'' + q(1-q) = 0 & x \in (0,L), \\ q(0) = q(L) = 0, \end{cases} \tag{1}$$

has a non-trivial non-negative solution if and only if $L > L_c$. Why is this exercise in this sheet ?
Hint: The function $H(q_1,q_2) = q_1^2/2 + q_2^2/2 - q_1^3/3$ is a Lyapunov function for this equation.

Exercise 4. Let $L > 0$ be a length, $u_0 \in L^2(0, L)$ be an initial datum satisfying $u_0 > 0$ and u be the solution of the Fisher-KPP equation

$$(2) \qquad \begin{cases} \partial_t u - \partial_{xx} u = u(1-u), & t > 0, \; x \in (0, L), \\ u(t, 0) = u(t, L) = 0, & t > 0, \\ u(0, x) = u_0(x), & x \in (0, L), \end{cases}$$

We aim at proving that when $0 < L < \pi$, then

$$\forall x \in [0, L], \quad u(t, x) \underset{t \to +\infty}{\to} 0.$$

1. Find a subsolution $\underline{u}$ of the equation (2).
2. We consider $\overline{u}$ the solution of the equation

$$\begin{cases} \partial_t \overline{u} - \partial_{xx} \overline{u} = \overline{u} & t > 0, \; x \in (0, L), \\ \overline{u}(t, 0) = \overline{u}(t, L) = 0 & t > 0, \\ \overline{u}(0, x) = u_0(x) & x \in (0, L), \end{cases}$$

   Check that $\overline{u}$ is a supersolution of the equation (2).
3. Prove that

$$\forall x \in [0, L], \quad \overline{u}(t, x) \underset{t \to +\infty}{\to} 0.$$

   Hint: Use Fourier series.
4. Conclude.

Exercise 5. We still consider the Fisher-KPP equation (2). Assuming this time that $L > \pi$, we aim at proving that there exists a supersolution $\overline{u}$ of the equation (2) such that $u(t, x) \le \overline{u}(t, x)$ for all $t \ge 0$ and $x \in (0, L)$, and satisfying

$$\forall x \in [0, L], \quad \overline{u}(t, x) \underset{t \to +\infty}{\to} q(x),$$

where q is the non-trivial non-negative steady state given by Exercice 3.

1. Let $\overline{u}$ be the solution of the equation

$$\begin{cases} \partial_t \overline{u} - \partial_{xx} \overline{u} = \overline{u}(1 - \overline{u}), & t > 0, \; x \in (0, L), \\ \overline{u}(t, 0) = \overline{u}(t, L) = 0, & t > 0, \\ \overline{u}(0, x) = M, & x \in (0, L), \end{cases}$$

   with $M = \max(1, \sup_{(0,L)} u_0)$. Prove that $\overline{u}$ is a supersolution of the equation (2) which dominates the function u.
2. By comparing $\overline{u}(t + h, x)$ and $\overline{u}(t, x)$, prove that for all $x \in [0, L]$, the limit $w(x) = \lim_{t \to +\infty} \overline{u}(t, x)$ exists and satisfies the estimate $0 \le w(x) \le M$.
3. Admit that w is a solution of the equation (1). Deduce then that $w = q$ and conclude.

Remark: One can also prove that there exists a subsolution $\underline{u}$ converging pointwise to q and bounding the function u from below. As a consequence, $u(t, x) \to_{t \to +\infty} q(x)$ for all $x \in [0, L]$.

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Département de Mathématiques **M1 - EDP**

Sheet 6: Lotka-Volterra system

Exercise 1. Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a function of class C^1. Assume that the ODE

$$x'(t) = f(x(t)), \tag{1}$$

admits a Lyapunov function $V : \mathbb{R}^d \to \mathbb{R}$ of class C^1. Let us recall that by definition, V satisfies

$$\forall x \in \mathbb{R}^d, \quad \mathrm{d}V(x) \cdot f(x) \leq 0.$$

1. Let (x, I) be a solution of the equation (1). Check that the function $t \mapsto V(x(t))$ is non-increasing on I.
2. Let $x_0 \in \mathbb{R}^d$ be an equilibrium state of the equation (1), that is, satisfying $f(x_0) = 0$. We assume moreover that x_0 is a strict local minimum of the function V:

$$\exists r_1 > 0, \forall r \in]0, r_1[, \quad \alpha_r = \min_{|x-x_0|=r} V(x) > V(x_0).$$

   (a) Let $0 < r < r_1$. Prove that the set $U_r = \{x \in \mathbb{R}^d : V(x) < \alpha_r\} \cap B(x_0, r)$ is open and contains x_0, and that any trajectory starting from U_r stays in $B(x_0, r)$.

   (b) Prove that x_0 is a stable stationary point.

3. Keeping the same hypothesis as in the previous question, we assume moreover that

$$\forall x \in \mathbb{R}^d \setminus \{x_0\}, \quad \mathrm{d}V(x) \cdot f(x) < 0.$$

   We aim at proving that the point x_0 is asymptotically stable.

   (a) Let $0 < r < r_1$. Check that the flow $\phi_t(x)$ of (1) is well-defined on $\mathbb{R}_+ \times U_r$.

   (b) Let $x \in U_r$. We define the ω-limit set $\omega(x)$ of the point x as

$$\omega(x) = \big\{y \in \mathbb{R}^d : \exists (t_n)_{n\geq 0} \in \mathbb{R}^{\mathbb{N}},\ t_n \underset{n\to+\infty}{\to} +\infty,\ \phi_{t_n}(x) \underset{n\to+\infty}{\to} y\big\}.$$

   Check that for all $y \in \omega(x)$, the function $t \mapsto V(\phi_t(y))$ is constant on $\mathbb{R}_+$.

   (c) Check that $\omega(x) \subset \{y \in \mathbb{R}^d : \mathrm{d}V(y) \cdot f(y) = 0\}$ for all $x \in U_r$.

   (d) Prove that $\omega(x) = \{x_0\}$ for all $x \in U_r$, and conclude.

Exercise 2. Let $a, b, c, d > 0$ be positive real numbers and $x_0, y_0 \geq 0$ be non-negative real numbers. We consider the Lotka-Volterra system

$$\begin{cases} x'(t) &= x(a - by), \\ y'(t) &= y(-c + dx), \end{cases} \tag{2}$$

with initial conditions $x(0) = x_0$ and $y(0) = y_0$.

1. Give an interpretation of this system in terms of sharks and sardines.
2. Prove that there exists a unique maximal solution defined on an open interval I of $\mathbb{R}$.
3. When $x_0 > 0$, check that $x(t) > 0$ for all $t \in I$. Similarly, prove that $y(t) > 0$ for all $t \in I$ under the assumption $y_0 > 0$.
4. We now assume that $x_0 > 0$ and $y_0 > 0$. By considering the function $H : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$H(x, y) = dx - c \ln x + by - a \ln y, \quad (x, y) \in (\mathbb{R}_+^*)^2,$$

prove that the maximal solution of the system (2) is bounded. What to conclude?
5. What are the stationary points of the system (2) ? Study their stability.
6. Draw the phase portrait of this system.
7. How to model the influence of fishing?

Exercise 3. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. We also consider $a, b, c, d, \lambda > 0$ some positive real numbers and $u_0, v_0 \in L^2(\mathbb{R}^d)$ be smooth initial data satisfying $0 < u_0 < c_1$ and $0 < v_0 < c_2$, with $c_1, c_2 > 0$. We consider (u, v) the solution of the following Lotka-Volterra system

$$\begin{cases} \partial_t u - \lambda \Delta u &= u(a - bv) & \text{in } (0, +\infty) \times \Omega, \\ \partial_t v - \lambda \Delta v &= v(-c + du) & \text{in } (0, +\infty) \times \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} &= 0 & \text{on } (0, T) \times \partial\Omega, \\ u(0, \cdot) &= u_0 & \text{in } \Omega, \\ v(0, \cdot) &= v_0 & \text{in } \Omega. \end{cases}$$

We aim at proving that (u, v) tends to a spatially uniform state as $t \to +\infty$. To that end, we consider the energy s of the system without diffusion

$$s(t, x) = du(t, x) - c \ln u(t, x) + bv(t, x) - a \ln v(t, x), \quad (t, x) \in (0, +\infty) \times \Omega.$$

1. What is the equation satisfied by the energy s ?
2. We define the total energy S of the system at time t by

$$S(t) = \int_\Omega s(t, x) \, \mathrm{d}x.$$

Check that S is non-increasing.
3. Conclude.
4. How to interpret this result ?
5. By the way, does such a solution (u, v) exist?

Sheet 7: Travelling waves

In all this sheet, we consider the one-dimensional Fisher-KPP equation posed on the whole space

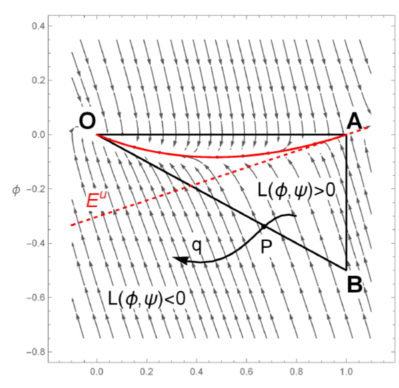
$$\partial_t u - \partial_{xx} u = u(1-u), \quad t>0,\ x\in\mathbb{R}. \tag{1}$$

Exercise 1. First, we aim at proving that there are traveling waves solutions of the equation (1), that is, solutions of the form $u(t,x)=\phi(x-ct)$ for some function $\phi:\mathbb{R}\to[0,1]$ and $c\in\mathbb{R}$. Precisely, we are interested in traveling wavefronts, that is, satisfying $\lim_{+\infty}\phi=0$ and $\lim_{-\infty}\phi=1$.

1. Check that a traveling wave is solution of the equation (1) if and only if the wave profile ϕ satisfies the following ordinary equation,

   $$\phi''(z)+c\phi'(z)+\phi(z)(1-\phi(z))=0, \quad z\in\mathbb{R}, \tag{2}$$

   where $z=x-ct$ denotes the co-moving frame.
2. Write this equation as a two-dimensional system of first order equations.
3. Study the stationary points of this system.
4. Explain why such a traveling wave does not exist when $0<c<2$.
5. Assuming that $c\geq 2$, the purpose is to prove the existence of such a traveling wave with velocity c. We denote the origin of the phase space by O, the point $(1,0)$ by A and the point $(1,-b)$ by B as represented in the following draw (with $c=3$), where $b>0$ is to be chosen.



   a) What is E^u in the above picture ?

   b) Check that $b>0$ can be chosen so that no orbit can leave the triangle OAB.
   Hint: For the side OB, introduce the function $L(\phi,\psi)=b\phi+\psi$.

   c) * By using the Poincaré-Bendixon theorem, prove that the equation (2) has a unique solution ϕ satisfying $\lim_{+\infty}\phi=0$ and $\lim_{-\infty}\phi=1$.

Exercise 2. We keep the notations introduced in Exercice 1 and assume that $c \geq 2$. In this exercice, we aim at determining the profile of the wave front ϕ. We make the change of variable $\xi = z/c$, so that ϕ satisfies the following ordinary equation:

$$\varepsilon\phi''(\xi) + \phi'(\xi) + \phi(\xi)(1 - \phi(\xi)) = 0, \quad \xi \in \mathbb{R}, \tag{3}$$

with $\varepsilon = 1/c^2$. We can expand ϕ in powers of ε:

$$\phi(\xi, \varepsilon) = \phi_0(\xi) + \varepsilon\phi_1(\xi) + \varepsilon^2\phi_2(\xi) + \dots.$$

1. By substituting this expansion in (3) and splitting the different powers of ε, give the equations satisfied by the functions ϕ_0 and ϕ_1. We recall that $\lim_{+\infty} \phi = 0$ and $\lim_{-\infty} \phi = 1$.
2. Why can we choose $\phi(0) = 1/2$?
3. Solve the equations satisfied by ϕ_0 and ϕ_1, and deduce that

$$\phi(z, c) = \frac{1}{1 + e^{z/c}} + \frac{1}{c^2}\frac{e^{z/c}}{(1 + e^{z/c})^2} \ln\left(\frac{4e^{z/c}}{(1 + e^{z/c})^2}\right) + \mathcal{O}(c^{-4}).$$

Exercise 3. We keep the previous notations. The purpose is now to deal with the appearance of propagation speeds in the reality. Assume that the initial condition of the equation (1) is given by

$$u(0, x) = e^{-a|x|}, \quad x \in \mathbb{R},$$

where $a > 0$ is a positive constant.

1. By considering supersolutions of the form

$$\overline{u}(t, x) = e^{\pm s_a(x \pm c_a t)}, \qquad t > 0,\ x \geq 0,$$

   where $c_a > 0$ and $s_a > 0$ are positive constants depending on a, establish an estimate of the form

$$\forall t \geq 0, \forall x \in \mathbb{R}, \quad |u(t, x)| \leq e^{-s_a(|x| - c_a t)}.$$

   Hint: Consider the leading edge of the evolving wave where, since u is small, we can neglect u^2 in comparison with u.
2. Deduce that

$$\forall c > a + \frac{1}{a}, \quad \lim_{t \to +\infty} \sup_{|x| \geq ct} |u(t, x)| = 0, \quad \text{when } 0 < a < 1,$$
$$\forall c > 2, \qquad \lim_{t \to +\infty} \sup_{|x| \geq ct} |u(t, x)| = 0, \quad \text{when } a \geq 1.$$

3. Draw a picture, admitting that

$$\forall 0 < c < a + \frac{1}{a}, \quad \lim_{t \to +\infty} \sup_{|x| \leq ct} |1 - u(t, x)| = 0, \quad \text{when } 0 < a < 1,$$
$$\forall 0 < c < 2, \qquad \lim_{t \to +\infty} \sup_{|x| \leq ct} |1 - u(t, x)| = 0, \quad \text{when } a \geq 1.$$

   Remark: Those limits can be obtained by constructing adapted subsolutions.
4. Comment.

Sheet 8: Systems of reaction-diffusion equations

Exercise 1. Rabies may infect all warm-blooded animals, also birds, and also humans, and affects the central nervous system. Vaccines are available (but expensive); but no further cure is known. The spread seems to occur in waves, e.g. one coming from the Polish-Russian border; the spread velocity is approx. 30-60 km/year.

Let us consider two groups of foxes:

. Susceptible foxes (S), with no diffusion (as they are territorial),

. Infective foxes (I), with diffusion (loss of sense of territory), constant death rate.

The infection rate is assumed to be proportional to their densities, no reproduction or further spread:

$$\begin{cases} \partial_t S = -rIS, & t > 0,\ x \in \mathbb{R}, \\ \partial_t I = rIS - aI + \nu \partial^2_{xx} I, & t > 0,\ x \in \mathbb{R}. \end{cases}$$

The non-dimensionalised version of the above system is the following:

$$\begin{cases} \partial_t S = -IS, & t > 0,\ x \in \mathbb{R}, \\ \partial_t T = IS - mI + \partial^2_{xx} I, & t > 0,\ x \in \mathbb{R}, \end{cases}$$

with $m = a/(rS_0)$, S_0 being the initial (maximum) susceptible density. We look for a travelling wave solution of this system of the form

$$S(t,x) = S(x-ct) = S(z) \quad \text{and} \quad I(t,x) = I(x-ct) = I(z),$$

where $z = x - ct$, the wave fronts S and I satisfying $0 \leq S \leq 1$ and $0 \leq I \leq 1$.

1. Write the system of ODEs satisfied by the functions S and I.
2. Justify the following boundary conditions: $S(+\infty) = 1$, $I(+\infty) = 0$, $S'(-\infty) = 0$, $I(-\infty) = 0$.
3. Check that
$$S(-\infty) - m \ln S(-\infty) = 1.$$
Deduce the fraction of susceptibles which survives the "rabies wave" (draw a picture).
4. Draw the phase plane associated with the system satisfied by S and I.
5. Explain why $c = 2\sqrt{1-m}$ is the minimal wave speed.
6. Draw the shapes of the waves fronts S and I.

Exercise 2. We consider a simple predator-prey model with logistic growth of the prey

$$\begin{cases} \partial_t u = u(1-u-v) + \nu \partial^2_{xx} u, & t > 0,\ x \in \mathbb{R}, \\ \partial_t v = av(u-b) + \partial^2_{xx} v, & t > 0,\ x \in \mathbb{R}, \end{cases}$$

with $a > 0$, $0 < b < 1$ and $\nu \geq 0$ some constants.

1. Check that the spatially independent system admits three stationary points, namely $(0,0)$, $(1,0)$ and $(b, 1-b)$. Study their stability.

We look for constant shape travelling wavefront solutions moving to the left:

$$u(t,x) = U(z) \quad \text{and} \quad v(t,x) = V(z),$$

where z denotes the wave variable $z = x + ct$ and $c > 0$ is positive.

2. Write the system of ODEs satisfied by the wave fronts U and V.

As a simpler case, we assume that the prey is diffusing much slower than the predators (e.g. consider a system where animals eat some plants), thus $\nu = 0$ is assumed.

3. Transform the system obtained in Question 2 in a new system of three ODEs of order one. Check that its stationary points are $(0,0,0)$, $(1,0,0)$ and $(b, 1-b, 0)$.
4. Compute the Jacobian matrix $J(U, V, W)$ of this system.
5. By studying the stability of the point $(1,0,0)$, explain why the wave speed c should necessarily satisfy $c \geq \sqrt{4a(1-b)}$ for keeping the possibility of a travelling wavefront.
6. Check that the point $(0,0,0)$ is unstable.
7. Let p be the characteristic polynomial of the matrix $J(b, 1-b, 0)$. What can we say about the local maxima of p ? Draw the typical graph of p for various values of a.
8. Justify that we could find some solutions with the following boundary conditions:

$$U(-\infty) = 1, \quad V(-\infty) = 0, \quad U(+\infty) = b, \quad V(+\infty) = 1-b, \tag{1}$$

   and / or:

$$U(-\infty) = 0, \quad V(-\infty) = 0, \quad U(+\infty) = b, \quad V(+\infty) = 1-b.$$

9. By considering the boundary conditions (1), draw the possible shapes for the fronts U and V.

Exercise 3. We consider the Belousov-Zhabotinskii chemical reaction modeled by the following system:

$$\begin{cases} \partial_t u = Lrv + u(1-u-rv) + \partial^2_{ss} u, & t > 0,\ x \in \mathbb{R}, \\ \partial_t v = -Mv - buv + \partial^2_{ss} v, & t > 0,\ x \in \mathbb{R}, \end{cases}$$

where L and M are of order 10^{-4}, b is of order 1, r is something between 5 and 50.

1. Check that the spatially homogeneous stationary states are $(0,0)$ and $(1,0)$.

Due to $L \ll 1$ and $M \ll 1$, we neglect the corresponding terms, which yields a model for the leading edge of travelling waves in the Belousov-Zhabotinskii reaction:

$$\begin{cases} \partial_t u = u(1-u-rv) + \partial^2_{ss} u, & t > 0,\ x \in \mathbb{R}, \\ \partial_t v = -buv + \partial^2_{ss} v, & t > 0,\ x \in \mathbb{R}. \end{cases}$$

We search for travelling wavefront solutions $u(t,x) = U(x+ct)$ and $v(t,x) = V(x+ct)$ for this new system, moving to the left and satisfying the boundary conditions

$$U(-\infty) = 0, \quad V(-\infty) = 1, \quad U(+\infty) = 1, \quad V(+\infty) = 0.$$

2. * By using what was stated previously for the Fisher-KPP equation and the comparison theorem, show that necessarily, the wave speed satisfies $c \leq 2$.

Remark: The best known result is *a priori* $((r^2 + \frac{2b}{2})^{1/2} - r)(2(b+2r))^{-1/2} \leq c \leq 2$.

Sheet 9: Reviews

Exercise 1 (Faber-Krahn inequality). Let Ω be an open bounded subset of $\mathbb{R}^d$, with $d \geq 3$, and $V \in L^\infty(\Omega)$ such that $V \geq 0$. We consider the problem

$$\begin{cases} -\Delta u = Vu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

1. Give the definition of a weak solution to the equation (1).
2. Can you apply the Lax-Milgram theorem here?
3. Let $r > \frac{d}{2}$. Show that there is a constant $c_d > 0$ depending on d only such that, if (1) has a non-trivial weak solution, then
$$|\Omega|^{\frac{2}{d}-\frac{1}{r}} \|V\|_{L^r(\Omega)} \geq c_d.$$
Hint: Use the following Sobolev inequality
$$\|u\|_{L^{2^*}(\Omega)} \leq M_d \|\nabla u\|_{L^2(\Omega)}, \quad \frac{1}{2^*} = \frac{1}{2} - \frac{1}{d},$$
which holds for all $u \in H_0^1(\Omega)$, where M_d depends on d only.
4. What do you obtain in the particular case $V = \lambda = \text{cst}$?

Exercise 2 (Estimates on the gradient). Let Ω be an open bounded subset of $\mathbb{R}^d$. Let A be a symmetric definite positive $d \times d$ matrix and $f \in \text{Lip}(\bar{\Omega})$. We will establish gradient estimates for solutions u to the equation $Lu = f$ with Dirichlet homogeneous boundary condition, where L is the elliptic operator $Lu = -\operatorname{div}(A\nabla u)$, under the assumption that there exists a function $\psi \in \text{Lip}(\Omega) \cap C^2(\Omega)$ such that $L\psi \geq f$ in Ω and $\psi = 0$ on $\partial\Omega$. For simplicity, we will consider the case where the function f is constant.

1. Let $\omega \subset \Omega$ and $u, v \in C^2(\omega) \cap C(\bar{\omega})$ satisfying $Lu \leq Lv$ in ω. Show that
$$\sup_{\overline{\omega}}(u - v) \leq \sup_{\partial\omega}(u - v).$$
2. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying $Lu = f$ in Ω.
   (a) Prove that
   $$\sup\left\{\frac{|u(x) - u(y)|}{|x - y|} : x, y \in \Omega, x \neq y\right\} \leq \sup\left\{\frac{|u(x) - u(y)|}{|x - y|} : x \in \Omega, y \in \partial\Omega\right\}.$$
   Hint: given $x_1, x_2 \in \Omega$ with $\tau = x_2 - x_1 \neq 0$, compare u and $u_\tau \colon x \mapsto u(x + \tau)$ in $\omega = \Omega \cap (-\tau + \Omega)$.
   (b) We assume furthermore that $u = 0$ on $\partial\Omega$. Show that $\text{Lip}(u) \leq \text{Lip}(\psi)$.
3. A ψ as above is called a *barrier function*. Construct a barrier function in the case $\Omega = B(0, 1)$.
Hint: consider $\psi(x) = -\gamma|x|^2/2 + C$ for some given constants $\gamma > 0$ and $C \in \mathbb{R}$.

Exercise 3 (The method of continuity).

1. Solve the equation $u - \Delta u = f$ on $\mathbb{T}^d$ and show that it defines a map $L^2(\mathbb{T}^d) \to H^2(\mathbb{T}^d)$.
2. Let X, Y be some Banach spaces. Let $(T_t)_{t\in[0,1]}$ be a *continuous* path of linear operators from X to Y satisfying

$$\exists C \geq 0, \forall u \in X, \forall t \in [0,1], \quad \|u\|_X \leq C\|T_t u\|_Y. \tag{2}$$

   Prove that T_0 is surjective if and only if T_1 is surjective as well.
3. Let $(a_{i,j})_{1\leq i,j\leq d}$ be a family of maps of class C^1 on $\mathbb{T}^d$. We assume that the following ellipticity condition holds

$$\exists \alpha > 0, \forall x \in \mathbb{T}^d, \forall \xi \in \mathbb{R}^d, \quad a_{i,j}(x)\xi_i\xi_j \geq \alpha|\xi|^2.$$

   We define the path $(T_t)_{t\in[0,1]}$ of operators $H^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$ by the formula

$$T_t u = u - \partial_i(a_{ij}^{(t)}(x)\partial_j u), \quad a_{ij}^{(t)} = t a_{ij} + (1-t)\delta_{ij}.$$

   (a) Show that $t \mapsto T_t$ is continuous.
   (b) Check that (2) is satisfied.
   (c) What to conclude ?

Exercise 4 (The heat equation). Let $u_0 : \mathbb{R} \to \mathbb{R}$ be a continuous, piecewise C^1 and 2π-periodic function. Prove that there exists a unique function $u \in C^0([0,+\infty) \times \mathbb{R}) \cap C^\infty((0,+\infty) \times \mathbb{R})$ satisfying

$$\begin{cases} \partial_t u(t,x) = \partial_x^2 u(t,x), & (t,x) \in (0,+\infty) \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

the function $u(t,\cdot)$ being moreover 2π-periodic for all $t \geq 0$.

Exercise 5 (A reaction-diffusion equation). We consider the following reaction-diffusion equation:

$$\begin{cases} \partial_t u - \Delta u \;=\; u^3 & \text{in } (0,+\infty) \times \mathbb{R}, \\ u(0,\cdot) \;=\; u_0 & \text{in } \mathbb{R}, \end{cases} \tag{3}$$

with initial datum $u_0 \in H^1(\mathbb{R})$.

1. Establish *a priori* energy estimates for the equation (3).
2. By using an iterative method, prove that there exists a positive time $T > 0$ and a unique solution $u \in C^0([0,T], H^1(\mathbb{R}))$ of the equation (3). Check that $u \in C^\infty((0,T) \times \mathbb{R})$.
3. Assuming moreover that the initial datum u_0 is fast decaying, establish pointwise estimates for the solution u.

Exercise 6 (The Fisher-KPP equation with Allee effect). We consider the one-dimensional Fisher-KPP equation with Allee effect posed on the whole space

$$\partial_t u - \partial_{xx} u = u(1-u)(u-a), \quad t > 0,\ x \in \mathbb{R}, \tag{4}$$

where $0 < a < 1/2$ is a parameter. Study the existence of traveling wave solutions for this equation, that is, solutions of the form

$$u(t,x) = \phi(x - ct), \quad t > 0,\ x \in \mathbb{R},$$

with $c > 0$.