Université de Toulouse **Année 2025 - 2026**

Master 1 - EDO et EDP **Semestre de printemps**

Sheet 1 on ODEs - Basic results and linear case

Exercice 1. *To prevent nonsense*

Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function

1. Assume that f admits a finite limit at $+\infty$. What can be deduced about f'?

2. Assume that f' admits a finite limit at $+\infty$. What can be deduced about f?

3. Assume that f and f' admits a finite limit at $+\infty$. What is the limit of f'?

Exercice 2. *Grönwall's lemma – Application to uniqueness*

1. Let a, b be real numbers such that $a < b$, and let $t_0 \in (a, b)$. Consider a function $\varphi \in C^0([a, b], \mathbb{R})$ such that there exists $C \in \mathbb{R}$ et $D \geq 0$ such that

$$\forall t \in [t_0, b], \quad \varphi(t) \leq C + D \int_{t_0}^{t} \varphi(s)\,\mathrm{d}s,$$

and

$$\forall t \in [a, t_0], \quad \varphi(t) \leq C + D \int_{t}^{t_0} \varphi(s)\,\mathrm{d}s,$$

Prove that

$$\forall\ t \in [a, b], \quad \varphi(t) \leq C \exp(D|t - t_0|).$$

2. Application: let (J_1, y_1) and (J_2, y_2) be two solutions of the following Cauchy problem

$$(\clubsuit) \quad y' = \cos(y), \quad y(0) = a \in \mathbb{R}.$$

a. Show that $y_1 = y_2$ on $J_1 \cap J_2$.

b. Deduce that if $(\clubsuit)$ admits a maximal solution, it is necessarily unique.

Exercice 3. *Generalized Grönwall's lemma*

Let I be a non-empty interval, $t \in I \mapsto a(t) \in \mathbb{R}$ be a continuous and non-negative function, and $b : I \to \mathbb{R}$ be a non-decreasing function. Let $x : I \to \mathbb{R}$ be a continuous function that satisfies, for some $s \in I$, the following inequality :

$$\forall t \in I, t \geq s, \quad x(t) \leq b(t) + \int_{s}^{t} a(\tau)x(\tau)\,\mathrm{d}\tau.$$

Show that

$$\forall t \in I, t \geq s, \quad x(t) \leq b(t) \exp\left(\int_{s}^{t} a(\tau)\,\mathrm{d}\tau\right).$$

Exercice 4. *There and back again*

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let a, b be real numbers such that $a < b$, and $u : [a, b] \mapsto \mathbb{R}$ be continuous and differentiable on (a, b). Show that if $u' = f(u)$ on (a, b) and $u(a) = u(b)$, then u is constant.

As an application, we consider the ordinary differential equation $y' = 2\sqrt{|y|}$.

2. Find all the maximal solutions of the equation.

3. Are there twice continuously differentiable solutions?

Exercice 5. *Convergence toward an equilibrium*

Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function.

1. Let $\ell \in \mathbb{R}^d$ and $y_\ell : t \in \mathbb{R} \mapsto \ell$. Show that $(\mathbb{R}, y_\ell)$ is a solution of the ordinary differential equation $y' = f(y)$ if and only if $f(\ell) = 0$.

2. Let (J, y) be a solution of the equation $y' = f(y)$ such that $\sup J = +\infty$. Assume that there exists $\ell \in \mathbb{R}^d$ such that $y(t) \to \ell$ as $t \to +\infty$. Prove that $f(\ell) = 0$.

Hint: consider the function $h : t \in J \mapsto f(\ell) \cdot y(t) \in \mathbb{R}$.

Exercice 6. *Linear Cauchy-Lipschitz theorem*

The aim of this exercise is to prove the following result:

Theorem 1. *Let $I \subset \mathbb{R}$ be an open interval, $A : I \to \mathcal{M}_d(\mathbb{R})$ and $b : I \to \mathbb{R}^d$ be two continuous functions. For all $t_0 \in I$ and $y_0 \in \mathbb{R}^d$, the Cauchy problem $y' = Ay + b$, $y(t_0) = y_0$ admits a unique global solution.*

1. Let $(\mathcal{E}, \|\cdot\|)$ be a Banach space, and $f : \mathcal{E} \mapsto \mathcal{E}$ be a function satisfying that there exists $p \in \mathbb{N}^*$ and $k \in [0, 1)$ such that

$$\forall (x, y) \in \mathcal{E} \times \mathcal{E}, \quad \|f^p(x) - f^p(y)\| \leq k\|x - y\|.$$

Prove that f admits a unique fixed point in $\mathcal{E}$.

We fix t_0 in I and $y_0 \in \mathbb{R}^d$. Let $a, b \in I$ be such that $a < t_0 < b$, and define $E = C^0([a, b]; \mathbb{R}^d)$. Recall that E, endowed with the sup-norm $\|z\|_\infty = \sup_{t\in[a,b]} \|z(t)\|$ is a Banach space. We define the map

$$T : z \in E \mapsto \left(t \in [a, b] \mapsto y_0 + \int_{t_0}^{t} (A(s)z(s) + b(s))\, \mathrm{d}s \right).$$

2. Explain briefly why $T(E) \subset E$.

3. Set $M = \sup_{t\in[a,b]} \|A(t)\|$. Show that, for all $u, v \in E$, $p \in \mathbb{N}$ and $t \in [a, b]$,

$$\|T^p(u)(t) - T^p(v)(t)\| \leq \frac{M^p|t - t_0|^p}{p!}\|u - v\|_\infty.$$

4. Deduce that there exists an unique $y \in E$ such that $T(y) = y$.

5. Let (I_1, y_1) and (I_2, y_2) be two solutions of the Cauchy problem $y' = Ay + b$, $y(t_0) = y_0$. Show that $y_1 = y_2$ on $I_1 \cap I_2$. Deduce the uniqueness of the maximal solution of this Cauchy problem.

6. Show that the Cauchy problem $y' = Ay + b$, $y(t_0) = y_0$ admits a global solution. Conclude.

7. Let us set

$$S_h = \left\{ y \in C^1(I, \mathbb{R}^d) : \forall t \in I,\ y'(t) = A(t)y(t) \right\}.$$

Show that S_h is a vector subspace of $C^1(I, \mathbb{R}^d)$ of dimension d.

Exercice 7. *Wronskian*

Let $A : I \to \mathcal{M}_d(\mathbb{R})$ be a continuous function, and let $(I, y_1), \dots, (I, y_d)$ be d global solutions of the equation $y' = Ay$. We define the *Wronskian* of the functions $y_1, \dots, y_d$, as the function

$$W : t \in I \mapsto \det \begin{bmatrix} y_1(t) & \dots & y_d(t) \end{bmatrix} \in \mathbb{R}.$$

1. Show that W is a global solution of $W' = \mathrm{Tr}(A)W$.

2. Deduce $W(t)$ as a function of $W(t_0)$ for $t, t_0 \in I$.

From now on, we suppose that $(y_1, \dots, y_d)$ is a basis of the space

$$\{y \in C^1(I, \mathbb{R}^d) : \forall t \in I,\ y'(t) = A(t)y(t)\}.$$

Let $b : I \to \mathbb{R}^d$ be a continuous function.

3. Prove that there exists d continuous functions $\mu_1, \dots, \mu_d : I \mapsto \mathbb{R}$ satisfying that for all $t \in I$, $\mu_1(t)y_1(t) + \dots + \mu_d(t)y_d(t) = b(t)$.

4. Deduce that there exit d functions $\rho_1, \dots \rho_d \in C^1(I, \mathbb{R})$ such that the function

$$y : t \in I \mapsto \rho_1(t)y_1(t) + \dots + \rho_d(t)y_d(t),$$

satisfies $y' = Ay + b$ on I.

5. Application: for $\omega > 0$, find a particular solution of the equation $y'' + \omega^2 y = e^{-t}$.

Exercice 8. *Isolated zeros*

Let $B, C : I \to \mathcal{M}_d(\mathbb{R})$ be continuous functions. Let (I, y) be a nontrivial solution of the second order equation $y'' + By' + Cy = 0$. Let $t_0 \in I$ be such that $y(t_0) = 0$. Show that there exists $\varepsilon > 0$ such $y(t) \neq 0$ for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon) \setminus \{t_0\}$.

Exercice 9. *Zeros again*

Let $\alpha : \mathbb{R} \to (0, \infty)$ be a continuous function. Prove that any function belonging to the space $\mathcal{S} = \{y \in C^2(\mathbb{R}) : y'' + \alpha y = 0\}$ cancels at least once.

Exercice 10. *Trapped on a sphere*

Let $A \in \mathcal{M}_d(\mathbb{R})$. To each solution $t \in \mathbb{R} \mapsto x(t)$ of the equation $x' = Ax$, we associate the corresponding trajectory $\{x(t) : t \in \mathbb{R}\}$. Show that every trajectory is contained in a sphere centered at 0 if and only if A is skew-symmetric.

Exercice 11. *Bounded solutions*

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function, and $a > 0$. Show that the equation $y''(t) - a^2 y(t) = f(t)$ admits a unique bounded solution.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and integrable function. Consider the equation $y''(t) + f(t)y(t) = 0$

  a. Show that if y is a bounded solution, then $\lim_{t \to \pm\infty} y'(t) = 0$.

  b. Let y_1, y_2 be two solutions, and define

$$W : t \in \mathbb{R} \mapsto \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}.$$

  Show that W is constant.

c. Deduce that the equation admits an unbounded solution.

Exercice 12. *Interlacing of zeros*

1. Let $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ be two continuous functions, and y_1, $y_2 \in C^2(\mathbb{R})$ be linearly independant solutions of the equation $y'' + \alpha y' + \beta y = 0$.

a. Determine the Wronskian $W : t \in \mathbb{R} \mapsto \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$.

b. Let $t_a < t_b$ be two consecutive zeros of y_1. Show that there exits $t \in [t_a, t_b]$ such that $y_2(t) = 0$.

2. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous functions such that $f(t) \leq g(t)$ for all $t \in \mathbb{R}$. Let $y_f \in C^2(\mathbb{R})$ be a nontrivial solution of the equation $y'' + fy = 0$, and $y_g \in C^2(\mathbb{R})$ be a solution of the equation $y'' + gy = 0$.

a. We set $W : t \in \mathbb{R} \mapsto \begin{vmatrix} y_f(t) & y_g(t) \\ y_f'(t) & y_g'(t) \end{vmatrix}$. Determine W'.

b. Deduce that between two consecutive zeros of y_f, there is a zero of y_g.

3. Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a continuous function and $y \in C^2(\mathbb{R})$ be a nontrivial solution of $y'' + \gamma y = 0$. Consider some $\omega > 0$.

a. Assume that $\gamma(t) \geq \omega^2$ for all $t \in \mathbb{R}$. Let a, b be two real numbers such that $\omega(b - a) \geq \pi$. Show that there exists $t \in [a, b]$ such that $y(t) = 0$.

b. Assume $\gamma(t) \leq \omega^2$ for all $t \in \mathbb{R}$. Let t_1 and t_2 be such that $t_1 \neq t_2$ and $y(t_1) = y(t_2) = 0$. Show that $\omega|t_2 - t_1| \geq \pi$.

Hint: consider the function $\sin(\omega(t - t_1) + \rho\pi)$, *with* $\rho \in (0, 1)$.

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Sheet 2 on ODEs - Existence of solutions and qualitative analysis

Exercice 1. *Quadratic growth*
Let $f : I \times \mathbb{R}^d \to \mathbb{R}^d$ be a C^1 function. Assume that there exist two continuous functions $c_1, c_2 : I \to [0, +\infty)$ such that for all $t \in I$ and all $y \in \mathbb{R}^d$,

$$|f(t, y) \cdot y| \leq c_1(t)\|y\|^2 + c_2(t).$$

Show that all the maximal solutions of the equation $y' = f(t, y)$ are global.

Exercice 2. *Periodic solution*
Let $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be a function being continuous and locally lipschitz with respect to the second variable. Assume there exists $T > 0$ such that for all $t \in \mathbb{R}$ and $y \in \mathbb{R}^d$,

$$f(t + T, y) = f(t, y).$$

Let (J, y) be a maximal solution of the equation $y' = f(t, y)$. Assume that there exists t_0 in J such that $t_0 + T \in J$ and $y(t_0 + T) = y(t_0)$. Show that $J = \mathbb{R}$ and that y is T-periodic.

Exercice 3. *Solution in a compact set*
Let $f : I \times \Omega \mapsto \mathbb{R}^d$ be continuous and locally lipschitz with respect to the second variable. We consider (J, y) a maximal solution of the equation $y' = f(t, y)$. The aim of this exercise is to give a direct proof of the following lemma.

Lemma 1. *Let $t_0 \in J$. Assume that there exists a compact set $K \subset \Omega$ such that $y(t)$ belongs to K for all $t \in J \cap [t_0, \infty)$. Then $\sup J = \sup I$.*

From now on, we assume that there exists a compact set $K \subset \Omega$ such that $y(t)$ belongs to K for all $t \in J \cap [t_0, \infty)$. We furthermore assume that $\sup J < \sup I$.

1. Show that y' is bounded on $[t_0, \sup J)$.
2. Prove that y is a lipschitz function on $[t_0, \sup J)$.
3. Deduce that there exists $y_\ell \in K$ such that $y(t) \to y_\ell$ as $t \to \sup J$ with $t < \sup J$.
4. Show that $y' \to f(\sup J, y_\ell)$ as $t \to \sup J$.
5. By considering the Cauchy problem $z' = f(t, z)$, $z(\sup J) = y_\ell$, obtain a contradiction.

Exercice 4. *Separation of variables*
Show that the Cauchy problem

$$(t^2 + 1)\, y' = (t + 1)\, y^2, \quad y(0) = 1,$$

admits a unique maximal solution (J, y). Compute y, and show that J is a bounded interval.

Exercice 5. *Global solution*

Let $y_0 \in \mathbb{R}$. Show that there exists a unique $y \in C^2(\mathbb{R})$ such that

$$y'' + |y| = \cos(t) + \sin\left((y')^2\right), \quad y(0) = y_0.$$

Exercice 6. *Bounded solutions*

For $x_0 \in \mathbb{R}$, we consider the Cauchy problem

$$(\star) \qquad y' = y\,\sin(y)^2, \quad y(0) = x_0.$$

1. Show that $(\star)$ admits a unique maximal solution (J, y).

2. Show that $J = \mathbb{R}$ and y is constant if and only if $x_0 = k\pi$ for some $k \in \mathbb{Z}$.

3. Show that for all $x_0 \in \mathbb{R}$, y is bounded. Deduce that $J = \mathbb{R}$.

4. Show that y is monotone.

5. Show that y admits limits in $\pm\infty$. What are these limits ?

6. For $x_0 \in \mathbb{R}$, we denote by $(\mathbb{R}, y_+)$ the solution of the equation $(\star)$, and by $(\mathbb{R}, y_-)$ the solution of the Cauchy problem $y' = y\sin(y)^2$, $y(0) = -x_0$. Prove that $y_- = -y_+$.

Exercice 7. *Draw a picture*

Let $f : (t, y) \in \mathbb{R} \times \mathbb{R} \mapsto t^2 + y^2$. We consider the Cauchy problem

$$(\star) \qquad y' = f(t, y), \quad y(0) = 0.$$

1. Show that $(\star)$ admits a unique maximal solution (J, y).

2. Set $J = (-\alpha, \beta)$, with $\alpha, \beta > 0$. We define $\tilde{J} = (-\beta, \alpha)$ and $\tilde{y} : t \in \tilde{J} \mapsto -y(-t)$.

   a. Show that $\tilde{y}$ is a well-defined C^1 function on $\tilde{J}$.

   b. Show that $(\tilde{J}, \tilde{y})$ is solution of $(\star)$. Deduce that $y = \tilde{y}$ on $J \cap \tilde{J}$.

   c. Deduce that $\alpha = \beta$ and y is odd.

3. Study the monotonicity and the convexity of y on J.

4. Show that $y(t) = \frac{t^3}{3} + o(t^3)$ as $t \to 0$.

5. Assume that $\sup J = +\infty$. Show there exists $c \in \mathbb{R}$ such that for all $t \geq 1$, $\arctan(y(t)) \geq t + c$. What to conclude?

6. What is $\lim_{t \to \sup J} y(t)$?

7. Draw y on J.

Exercice 8. *Around the origin*

Consider the following system of equations

$$\begin{cases} x' = -x + \sin y, \\ y' = \dfrac{x}{\mathrm{ch} y} - \dfrac{y}{2}\exp(x^2 + y^2). \end{cases}$$

1. Let $(J, (x, y))$ be a maximal solution of the equation. Show that $\sup J = +\infty$.

2. Prove that there exists $\rho > 0$ such that for any $R \geq \rho$, if $\|(x(t_0), y(t_0))\| < R$ for some $t_0 \in J$, then for any $t \geq t_0$, we have $\|(x(t), y(t))\| < R$.

Exercice 9. *Matrix-valued ODE*

Let $A \in M_n(\mathbb{R})$ with $n \geq 1$. Consider the following matrix-valued Cauchy problem:

$$X' = -XAX, \quad X(0) = I_n.$$

1. Justify that we can apply the Cauchy-Lipschitz theorem to this equation. From now on, we denote by $((a, b), X)$ the maximal solution of the Cauchy problem.

2. Show that, in a neighborhood of 0, we have $X(t) = (I_n + tA)^{-1}$. Deduce that $|a| \geq \|A\|^{-1}$ and $b \geq \|A\|^{-1}$, where $\|\cdot\|$ is a sub-multiplicative norm on $M_n(\mathbb{R})$.

Exercice 10. *Global and periodic solutions*

Consider the following ordinary differential equation

$$\begin{cases} y'' = 1 - 3y^2, \\ y(0) = y'(0) = 0. \end{cases} \tag{1}$$

The goal of this exercise is to show that this equation has a unique maximal solution, that is global and periodic.

1. Show that (1) has a unique maximal solution. Denote by I the interval on which this solution is defined.

2. Show that $(y')^2 = 2y(1 - y^2)$. Deduce that for all $t \in I$, we have $0 \leq y(t) \leq 1$. Deduce then that $I = \mathbb{R}$.

3. Show that y is an even function.

4. Prove the existence of a time $T > 0$ such that $y'(t) > 0$ for all $t \in (0, T)$. Define

$$T_{\max} = \sup \{T > 0 : \forall t \in (0, T),\, y'(t) > 0\}.$$

This means that $y'(t) > 0$ for all $t \in (0, T_{\max})$.

5. Let $0 < t_0 < t < T_{\max}$. Prove that

$$t - t_0 = \int_{y(t_0)}^{y(t)} \frac{\mathrm{d}s}{\sqrt{2s(1-s)(1+s)}} \leq \int_0^1 \frac{\mathrm{d}s}{\sqrt{2s(1-s)(1+s)}} < +\infty.$$

Deduce that $T_{\max} < +\infty$, $y'(T_{max}) = 0$ and $y(T_{\max}) = 1$.

6. For $t \in \mathbb{R}$, define $w(t) = y(t + 2\,T_{\max})$. Show that $w(-T_{\max}) = y(-T_{\max})$ and that $w'(-T_{\max}) = y'(-T_{\max})$. Deduce that $y(t) = y(t + 2T_{\max})$ for all $t \in \mathbb{R}$.

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Sheet 3 on ODEs - Equilibria

Let $A \in M_2(\mathbb{R})$. Consider the following differential equation on $\mathbb{R}^2$:

$$(\star) \qquad X'(t) = AX(t), \quad t \in \mathbb{R}.$$

The set $\{X(t) : t \in \mathbb{R}\}$ is called the *trajectory* associated to a solution X. It is simply the image of the parametrized curve $t \in \mathbb{R} \mapsto X(t) = (x(t), y(t)) \in \mathbb{R}^2$. The aim of this first series of exercises is to represent on the plane some remarkable solutions of system $(\star)$.

Exercice 1. *Trajectories*

1. State the Cauchy-Lipschitz theorem for linear ODEs.

2. Show that a solution is either reduced to a singleton $\{X_0\}$, or is the image of a *regular* parametrized curve $t \mapsto X(t)$ (*i.e.* $X'(t) \neq 0$ for all $t \in \mathbb{R}$).

3. Show that if Γ_1 and Γ_2 are two distinct trajectories, then their intersection $\Gamma_1 \cap \Gamma_2$ is empty.

4. Show that if $\Gamma \subset \mathbb{R}^2$ is a trajectory, then for all $\lambda \in \mathbb{R}$, the set $\lambda\Gamma = \{\lambda(x, y) : (x, y) \in \Gamma\}$ is a trajectory as well.

5. Let $P \in \mathrm{GL}_2(\mathbb{R})$ and $B = PAP^{-1}$. Consider the equation

$$(\star\star) \qquad Y'(t) = BY(t), \quad t \in \mathbb{R}.$$

Show that Y is a solution to $(\star\star)$ if and only if $P^{-1}Y$ is a solution of $(\star)$.

Exercice 2. *Explicit resolutions*

For each of the following matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

1. Compute explicitly the space of solutions of the equation $(\star)$.

2. Draw some remarkable trajectories in the plane.

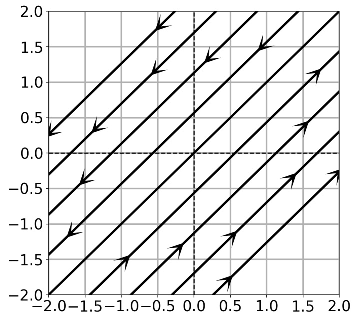
Exercice 3. *Identify phase portraits*

To each of the following matrices

$$M_1 = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} -0.5 & 2 \\ 1 & -0.5 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0.4 & 1 \\ -1 & 0 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 0.2 & -1 \\ 1 & 0.8 \end{pmatrix}, \quad M_5 = \begin{pmatrix} -0.5 & -0.5 \\ 1 & 0.2 \end{pmatrix}, \quad M_6 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix},$$

associate the corresponding phase portrait among the ones given below. All answers must be concisely justified. It is advisable to avoid long and tedious calculations.



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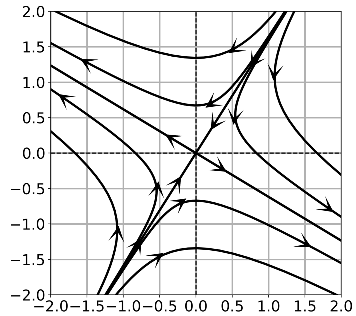

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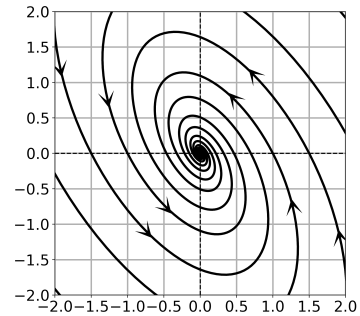

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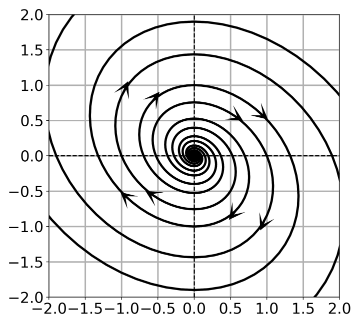

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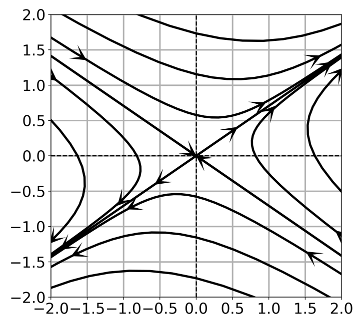

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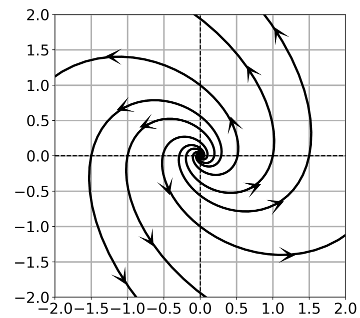

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Exercice 4. *Draw phase portraits*
Sketch the phase portrait of ($\star$) when the matrix A is given by:

$$\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ -6 & 4 \end{pmatrix}, \quad \begin{pmatrix} -1 & -5 \\ 1 & 3 \end{pmatrix}.$$

The explicit expression of the solutions is not required.

From now on, we will study the equilibria of non-linear equations.

Exercice 5. *Potential flow and minimizers*
Let f be a twice continuously differentiable function on $\mathbb{R}^n$ and let $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$. Consider the Cauchy problem

$$\begin{cases} x'(t) = -\nabla f(x(t)), \\ x(0) = x_0. \end{cases}$$

Additionally, assume that $f(x) \to +\infty$ as $\|x\| \to +\infty$.

1. Show that this problem has a unique maximal solution x on an interval (T_*, T^*).

2. Show that $t \mapsto f(x(t))$ is a non-increasing function. Deduce that $T^* = +\infty$ and that x is bounded on $[0, +\infty)$.

3. Do we necessarily have $T_* = -\infty$? (one can consider the example where $n = 1$ and $f(x) = x^4$.)

4. From now on, assume f to be strictly convex. Show that f admits a unique global minimum m reached at some point x^m, *i.e.* $f(x^m) = m$.

5. Show that x^m is the only point where $\|\nabla f\|$ vanishes. One can use the following inequality:

$$\forall y, z \in \mathbb{R}^n, \quad f(y) \geq f(z) + \nabla f(z) \cdot (y - z).$$

6. We now want to show that $x(t) \to x^m$ as $t \to +\infty$, for every initial condition x_0.

- **a.** Prove that the function $r : t \mapsto f(x(t))$ converges to a finite limit ℓ at $+\infty$.
- **b.** Show that for all $\varepsilon > 0$, there exists $\alpha > 0$ such that we have $\|\nabla f(a)\| \geq \alpha$ for every point $a \in \mathbb{R}^n$ satisfying

$$\|a\| \leq \|x\|_{L^\infty(]0,+\infty[)} \quad \text{and} \quad f(a) \geq m + \varepsilon.$$

- **c.** Show that for all $\varepsilon > 0$, there exists $\eta > 0$ such that $r'(t) \leq -\eta$ for all $t \in [0, +\infty)$ that satisfies $m + \varepsilon \leq f(x(t))$.
- **d.** Deduce that $\ell = m$.
- **e.** Conclude.

Exercice 6. *Species in competition*

Consider, for some $\alpha, \beta > 0$, the following system :

$$\begin{cases} x'(t) = x(1 - x - \beta y), \\ y'(t) = y(1 - y - \alpha x), \\ x(0) = x_0, \\ y(0) = y_0, \end{cases}$$

This system models the evolution of two populations that compete against each other (their survival depends on a shared common resource, therefore they must compete for it).

1. Show that this system satisfies the assumptions of the local Cauchy-Lipschitz theorem.

2. Given x_0, y_0 some non-negative initial conditions, let us denote by $(J, (x, y))$ the corresponding maximal solution. Show that the functions x and y are non-negative on J.

3. Also for non-negative initial conditions x_0 and y_0, show that for every $t \geq 0$,

$$x(t) \leq x_0 e^t \quad \text{and} \quad y(t) \leq y_0 e^t.$$

Deduce that $[0, +\infty) \subset J$.

4. Discuss, depending on the parameters α and β, the stability of the following equilibrium points:

$$P_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Can we interpret these results from the perspective of the model ?

5. Show that if $\alpha > 1$ and $\beta > 1$, or if $\alpha < 1$ and $\beta < 1$, then there exists another equilibrium point P_4 in the first quadrant $(0, +\infty) \times (0, +\infty)$. Determine this equilibrium point and study its stability in both cases.

Exercice 7. *Stability by linearization?*

Consider the following equations

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 - y_1\,(y_1^2 + y_2^2) \\ -y_1 - y_2\,(y_1^2 + y_2^2) \end{pmatrix}, \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 + y_1\,(y_1^2 + y_2^2) \\ -y_1 + y_2\,(y_1^2 + y_2^2) \end{pmatrix}.$$

1. Show that, for any initial condition, both of these equations have a unique maximal solution.
2. Show that both of these equations have the same unique equilibrium point.
3. Do we have a conclusive answer on the stability of this equilibrium if we linearize the system ?
4. Study the stability of this equilibrium by using the function $r : t \mapsto y_1(t)^2 + y_2(t)^2$.

Exercice 8. *Pendulum*
Let us study the following equation

$$\theta'' + \mu\theta' + \sin\theta = 0,$$

where $\mu > 0$ is a given constant.

1. By rewriting this equation in the form of a first order system of equations

$$Y' = f(Y) \quad \text{where} \quad Y = \begin{pmatrix} \theta \\ \theta' \end{pmatrix},$$

show that for every initial condition $\theta(0) = \theta_0$ and $\theta'(0) = \theta_1$ there exists a unique global solution.

2. Find the equilibrium points of the system.

3. We want to show that the equilibrium $(0, 0)$ is asymptotically stable. Let $\theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\theta_1 \in \mathbb{R}$ such that $\frac{1}{2}\theta_1^2 - \cos\theta_0 < 0$. For a solution θ, we define the corresponding energy

$$E(t) = \frac{1}{2}\theta'(t)^2 - \cos(\theta(t)).$$

  a. Show that E is non-increasing and that it admits a finite limit at $+\infty$.
  b. Show that $\theta(t) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for all $t \in \mathbb{R}_+$.
  c. Show that θ' and θ'' are bounded on $\mathbb{R}_+$.
  d. Deduce that $\lim_{t\to+\infty} \theta'(t) = 0$.
  e. Deduce that θ converges to 0 at $+\infty$.

4. Show that we can find the same result as the previous question by studying the linearized system at $(0, 0)$.

5. What can we say about the equilibrium point $(\pi, 0)$?

Exercice 9. *RLC Circuit*
The analysis of RLC circuits leads us to the study of ODEs of the form

$$\begin{cases} L\dfrac{\mathrm{d}i}{\mathrm{d}t} = v - h(i), \\ C\dfrac{\mathrm{d}v}{\mathrm{d}t} = -i, \end{cases}$$

where $h : \mathbb{R} \mapsto \mathbb{R}$ is continuously differentiable. The variables v and i correspond to voltage and intensity respectively, and the positive constants L and C to resistance and capacity.

1. Find the unique equilibrium of the ODE and discuss its stability depending on the function h.

2. Suppose that h satisfies $xh(x) > 0$ for all $x \neq 0$. The goal of this question is to show that the equilibrium is asymptotically stable. To that end, we introduce the energy

$$E(i, v) = \frac{1}{2}(Li^2 + Cv^2).$$

Denote by φ the corresponding flow of the system. Let us fix some $(i_0, v_0) \in \mathbb{R}^2$.

a. By studying the variations of $\theta : t \mapsto E(\varphi(t,(i_0,v_0)))$, show that for all $(i_0,v_0) \in \mathbb{R}^2$, the interval on which $\varphi(\cdot,(i_0,v_0))$ is defined contains $\mathbb{R}_+$.

b. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence that goes to $+\infty$ and such that the sequence $(\varphi(t_n,(i_0,v_0)))_{n\in\mathbb{N}}$ converges to a point (i_∞,v_∞). Prove that for all $s \geq 0$,

$$\lim_{n\to+\infty} \theta(s+t_n) = E((i_\infty,v_\infty)).$$

Deduce that for all $s \geq 0$,

$$(\star) \qquad E(\varphi(s,(i_\infty,v_\infty))) = E((i_\infty,v_\infty)).$$

c. Show that $(i_\infty,v_\infty) = (0,0)$ (one can differentiate $(\star)$ with respect to s).

d. Conclude.

Exercice 10. *Spirals*

Consider the vector field defined by $V(0,0) = (0,0)$ and for all $(x,y) \neq (0,0)$,

$$V(x,y) = \begin{pmatrix} -y + x\cos(x^2+y^2)\exp\left(-\frac{1}{x^2+y^2}\right) \\ x + y\cos(x^2+y^2)\exp\left(-\frac{1}{x^2+y^2}\right) \end{pmatrix}.$$

1. Show that $V \in C^\infty(\mathbb{R}^2)$ and deduce that the following Cauchy problem

$$\begin{cases} Y' = V(Y), \\ Y(0) = Y_0, \end{cases}$$

is locally well-posed.

2. Prove the existence of an infinite number of circles $(C_k)_{k\geq 0}$, with radii $R_k \nearrow +\infty$, corresponding to trajectories of solutions of the system. Deduce that all the solutions are global (one can try to decompose the vector field into its radial and tangential part).

3. Find the equilibrium points of the system and study their linear stability. What can be deduced from this ?

4. Compute the scalar product $V(x,y)\cdot(x,y)$ for all $(x,y) \in \mathbb{R}^2$ and draw the vector field V. If $Y(0)$ is strictly between two circles C_k and C_{k+1}, how does the corresponding trajectory behave when $t \to \pm\infty$? (one can try using polar coordinates) What if $Y(0) \neq 0$ is in the interior of the circle C_1? What information can be drawn from this in relation to question 3 ?

Sheet 1 on transport equations - Models with constant velocities

Exercice 1. *Homogeneous equation*
Let $c \in \mathbb{R}$ and $u_0 \in C^1(\mathbb{R})$. Consider the equation

$$(1) \qquad \begin{cases} \partial_t u(t,x) + c\partial_x u(t,x) = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,\cdot) = u_0. \end{cases}$$

1. Suppose that $u \in C^1(\mathbb{R} \times \mathbb{R})$ is a solution of (1). For $(t,y) \in \mathbb{R} \times \mathbb{R}$ we define

$$\tilde{u}(t,y) = u(t, y+ct).$$

In other words, we are simply making a change of variables $x = y + ct$.

- **a.** Show that $\tilde{u}$ is continuously differentiable on $\mathbb{R} \times \mathbb{R}$.
- **b.** Show that
$$\forall (t,y) \in \mathbb{R} \times \mathbb{R}, \quad \partial_t \tilde{u}(t,y) = 0.$$
- **c.** Express $\tilde{u}$ in terms of u_0.
- **d.** Deduce an expression of u in terms of u_0.
- **e.** Show that the function u obtained in the previous question is indeed a solution of (1).
- **f.** Conclude on the existence and uniqueness of continuously differentiable solutions for this problem.
- **g.** Draw the lines on the plane (x,t) along which the solution u is constant for any initial condition u_0.

2. Denote u the solution of the problem (1).

- **a.** Suppose that u_0 is C^k on $\mathbb{R}$ for some $k \geq 1$. Show that u is also C^k but on $\mathbb{R} \times \mathbb{R}$.
- **b.** Suppose that $\operatorname{supp} u_0 \subset [a,b]$ for some $a < b$. Show that for all $t \in \mathbb{R}$ the support of the function $x \mapsto u(t,x)$ is contained in $[a+ct, b+ct]$.
- **c.** Show that for all $p \in [1,+\infty]$ and $t \in \mathbb{R}$ we have $\|u(t,\cdot)\|_{L^p} = \|u_0\|_{L^p}$.

Exercice 2. *Inhomogeneous equation*
Let $c \in \mathbb{R}$, $u_0 \in C^1(\mathbb{R})$ and $f \in C^1(\mathbb{R} \times \mathbb{R})$. Consider the equation

$$(2) \qquad \begin{cases} \partial_t u + c\partial_x u = f(t,x), & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,\cdot) = u_0. \end{cases}$$

By using the same strategy as in the previous exercise, show that problem (2) admits a unique C^1 solution. Give the explicit expression of this solution in terms of the given data.

Exercice 3. *Additional zero-order term*

Let $c \in \mathbb{R}$, $u_0 \in C^1(\mathbb{R})$, $f \in C^1(\mathbb{R} \times \mathbb{R})$ and $a \in C^1(\mathbb{R} \times \mathbb{R})$. Consider the equation

$$\begin{cases} \partial_t u + c\partial_x u + a(t,x)u = f(t,x), & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,\cdot) = u_0. \end{cases} \tag{3}$$

By proceeding as in previous exercises, show that problem (3) admits a unique solution. Give the explicit expression of this solution in terms of the given data.

Exercice 4. *Domain with boundary*

Let $c \in \mathbb{R}$, $u_0 \in C^1(\mathbb{R})$. Consider the equation

$$\begin{cases} \partial_t u + c\partial_x u = 0, & (t,x) \in [0,+\infty[\times[0,+\infty[, \\ u(0,\cdot) = u_0. \end{cases} \tag{4}$$

1. Set $D_c = \{(t,x) \in \mathbb{R} \times \mathbb{R} : t \geq 0, x \geq \min(ct, 0)\}$. Show that there exists a unique $u \in C^1(D_c)$ solution of (4). Gives an explicit expression of u.

2. Deduce that equation (4) admits a unique solution $u \in C^1([0,+\infty[\times[0,+\infty[)$ if and only if $c \leq 0$.

We now focus on the case $c > 0$. Let $g \in C^1(\mathbb{R})$.

3. Find a necessary and sufficient condition for the existence of $u \in C^1([0,+\infty[\times[0,+\infty[)$ satisfying (4) and the additional condition $u(t,0) = g(t)$, $t \geq 0$.

Exercice 5. *Higher dimension*

Let $n \in \mathbb{N}^*$, $v \in \mathbb{R}^n$, $u_0 \in C^1(\mathbb{R}^n)$, $f \in C^1(\mathbb{R} \times \mathbb{R}^n)$ and $a \in C^1(\mathbb{R} \times \mathbb{R}^n)$. Consider the equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u + a(t,x)u = f(t,x), & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0,\cdot) = u_0. \end{cases} \tag{5}$$

By using the same strategy as in the previous exercises, show that the problem (5) admits a unique C^1 solution. Give the explicit expression of this solution in terms of the given data.

Exercice 6. *Homogeneous system*

In this exercise, let us consider the resolution of a system of mono-dimensional transport equations with constant coefficients of the form

$$\begin{cases} \partial_t U + A\partial_x U = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ U(0,\cdot) = U_0. \end{cases} \tag{6}$$

where U is the unknown function taking values in $\mathbb{R}^2$ and A is a 2×2 matrix with constant coefficients. Assume that the initial condition U_0 is continuous and bounded on $\mathbb{R}$.

1. Further assume that A is diagonalizable with real eigenvalues, and write $A = P^{-1}DP$ with $D = \mathrm{diag}(\lambda_1, \lambda_2)$ and P an invertible matrix. Show that problem (6) admits a unique solution U. Give the explicit expression of this solution in terms of U_0, P and the λ_i's.

2. Show that in the previous case, there exists a constant $C > 0$, depending on A, such that

$$\|U\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})} \leq C\|U_0\|_{L^\infty(\mathbb{R})}.$$

3. Assume now that A is real symmetric and that U_0 has compact support. Prove that for every $t \geq 0$, the solution $U(t, \cdot)$ of the problem (6) also has compact support $K(t)$ and satisfies

$$\|U(t,\cdot)\|_{L^2(\mathbb{R})} = \|U_0\|_{L^2(\mathbb{R})}.$$

4. We would like to understand what happens in the case of a non-diagonalizable matrix. From now on, the function U_0 is assumed to be bounded and C^2 with bounded derivative.

a. In the case where

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

give the explicit expression of the solution of problem (6) in terms of U_0. Is the solution U bounded on $\mathbb{R}_+ \times \mathbb{R}$? Prove the following local-in-time estimate

$$\forall T > 0,\ \exists C_T > 0, \quad \|U\|_{L^\infty([0,T]\times\mathbb{R})} \leq C_T(\|U_0\|_{L^\infty(\mathbb{R})} + \|U_0'\|_{L^\infty(\mathbb{R})}). \tag{7}$$

b. Consider the case where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Show that, for all $n \geq 1$, the function U_n defined by

$$U_n(t,x) = e^{tn} \begin{pmatrix} \sin(nx) \\ -\cos(nx) \end{pmatrix},$$

is solution to the system (6) for a well chosen initial condition U_0. Deduce that an inequality of the same type as (7) cannot be true in this case.

Exercice 7. *Square non-linearity*

Let $c \in \mathbb{R}$ and $u_0 \in C^1(\mathbb{R}, \mathbb{R})$ be a bounded function. Let us study the following non-linear equation

$$\begin{cases} \partial_t u(t,x) + c\partial_x u(t,x) - u(t,x)^2 = 0, & (t,x) \in \mathbb{R}_+^* \times \mathbb{R}, \\ u(0,\cdot) = u_0. \end{cases} \tag{8}$$

1. For this question, suppose that u_0 is non-positive. Show that problem (8) admits a unique C^1 solution defined on $\mathbb{R}_+ \times \mathbb{R}$. Give the explicit expression of this solution using the given data.

2. For this question, suppose that u_0 is non-negative (and non identically zero). This time, show that the problem (8) admits a unique C^1 solution defined on a domain containing $[0, T^*) \times \mathbb{R}$, for some value of T^*. Give the maximal value of T^* based on the given data.

3. Redo the two previous questions in the case where the velocity field c is no longer constant but a bounded and C^1 function $(t,x) \mapsto c(t,x)$. What can we say on the existence of T^*?

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Master 1 - EDO et EDP **Semestre de printemps**

Sheet 2 on transport equations - Models with velocities depending on time and space

Exercice 1. *Homogeneous equation*
Let $c = c(t, x)$ be a continuously differentiable function on $\mathbb{R} \times \mathbb{R}$. Consider the equation

$$\partial_t u(t,x) + c(t,x)\partial_x u(t,x) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{1}$$

Assume that $u \in C^1(\mathbb{R} \times \mathbb{R})$ is a solution of (1). Show that if the function $X \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is solution of a certain ODE (with respect to t), then independently of the initial condition, we have

$$\forall (t,y) \in \mathbb{R} \times \mathbb{R}, \quad \frac{\mathrm{d}}{\mathrm{d}t} u\big(t, X(t,y)\big) = 0.$$

Compare this to the change of variables that has been made in the case where c was constant and solve explicitly the equation (1) with initial condition $u(0,\cdot) = u_0 \in C^1(\mathbb{R})$ when $c(t,x) = t - x$.

Exercice 2. *Higher-dimensional equation*
Let $n \in \mathbb{N}^*$, $v \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, $u_0 \in C^1(\mathbb{R}^n)$, $f \in C^1(\mathbb{R} \times \mathbb{R}^n)$ and $a \in C^1(\mathbb{R} \times \mathbb{R}^n)$. In addition, assume that v is bounded. Consider the following equation

$$\begin{cases} \partial_t u + v(t,x) \cdot \nabla_x u + a(t,x) u = f(t,x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0,\cdot) = u_0. \end{cases} \tag{2}$$

1. Let $(t_0, y_0) \in \mathbb{R} \times \mathbb{R}^n$. Consider the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t, t_0, y_0) = v\big(t, X(t, t_0, y_0)\big), \quad X(t_0, t_0, y_0) = y_0. \tag{3}$$

a. Show that (3) admits a unique solution X which is well-defined on $\mathbb{R}$.
b. Show that X is a continuously differentiable function on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$.
c. Show that for every $y_0 \in \mathbb{R}^n$ and $t_1, t_2, t_3 \in \mathbb{R}$, we have

$$X\big(t_1, t_2, X(t_2, t_3, y_0)\big) = X(t_1, t_3, y_0).$$

d. By differentiating with respect to t the equality

$$X\big(s, t, X(t, s, x)\big) = x,$$

show that for all $(t, t_0, y_0) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$, we have

$$\partial_{t_0} X(t, t_0, y_0) + \mathrm{d}X(t, t_0, \cdot)(y_0) \cdot v(t_0, y_0) = 0.$$

2. Assume that u is a solution of (2). For $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, compute

$$\frac{\mathrm{d}}{\mathrm{d}t} u\big(t, X(t, t_0, x_0)\big).$$

3. Show that the problem (2) admits a unique solution $u \in C^1(\mathbb{R} \times \mathbb{R}^n)$ given by

$$u(t,x) = \exp\Big(- \int_0^t a\big(s, X(s,t,x)\big)\, \mathrm{d}s \Big) u_0\big(X(0,t,x)\big) \\ + \int_0^t f\big(s, X(s,t,x)\big) \exp\Big(- \int_s^t a\big(r, X(r,t,x)\big)\, \mathrm{d}r \Big)\, \mathrm{d}s.$$

Exercice 3. *Domain with boundary*
Let $v \in C^1(\mathbb{R} \times [0,1])$ such that for $t \in \mathbb{R}$, we have

$$v(t,0) = v(t,1) = 0.$$

Let u_0 be a continuously differentiable function on $[0,1]$. Consider the problem

$$\begin{cases} \partial_t u(t,x) + v(t,x)\partial_x u(t,x) = 0, & (t,x) \in \mathbb{R} \times [0,1], \\ u(0,\cdot) = u_0. \end{cases} \tag{4}$$

1. Let $(t_0, x_0) \in \mathbb{R} \times [0,1]$. Consider the maximal solution $X(\cdot, t_0, x_0)$ of the ODE (3), just like in the previous exercise. Show that X is well-defined on $\mathbb{R}$ and takes its values in $[0,1]$.

2. Show that the problem (4) admits a unique solution $u \in C^1(\mathbb{R} \times [0,1])$. Give the explicit expression of the solution in terms of X and u_0.

Exercice 4. *Conservation law*
Consider the following non-linear equation

$$\begin{cases} \partial_t u(t,x) + \partial_x (F(u(t,x))) = 0, & (t,x) \in \mathbb{R}_+^* \times \mathbb{R}, \\ u(0,\cdot) = u_0. \end{cases} \tag{5}$$

where $u_0 : \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ are two given functions. Suppose that F is twice continuously differentiable on $\mathbb{R}$.

1. Suppose u is a C^1 solution. Define the characteristic curves $X(\cdot, x)$ for all $x \in \mathbb{R}$ as

$$\begin{cases} \partial_t X(t,x) = F'(u(t, X(t,x))), & t > 0, \\ X(0,x) = x. \end{cases}$$

Give an expression of $u(t, X(t,x))$ and $X(t,x)$ in terms of u_0.

2. In the following, assume that u_0 is C^1, bounded and with bounded derivative on $\mathbb{R}$. Define T^* as

$$T^* = \begin{cases} +\infty & \text{if } F'(u_0) \text{ is non-decreasing,} \\ -\dfrac{1}{\inf_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}x}(F'(u_0))} & \text{otherwise.} \end{cases}$$

Draw the characteristic curves on the plan (x,t) in both cases, and give a geometric interpretation of T^*.

3. Show that there exists $Y \in C^1([0, T^*[\times \mathbb{R})$ such that for every $t \in [0, T^*[$, $x \in \mathbb{R}$ and $y \in \mathbb{R}$,

$$y = X(t,x) \quad \Longleftrightarrow \quad x = Y(t,y).$$

4. Show that there exists a unique solution $u \in C^1([0, T^*[\times \mathbb{R})$ to the problem (5).

Sheet 3 on transport equations - Weak solutions

Preamble In the following, c stands for a continuously differentiable and bounded function on $\mathbb{R} \times \mathbb{R}$. Recall that, in this case, we can define for all $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}$ the function $t \mapsto X(t, t_0, x_0)$ as *the global* solution of the ODE

$$\partial_t X(t, t_0, x) = c(t, X(t, t_0, x)), \quad X(t_0, t_0, x) = x.$$

Additionally, the function $(t, t_0, x) \mapsto X(t, t_0, x)$ is continuously differentiable on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Finally, we have that the function $t \mapsto \partial_x X(t, t_0, x)$ is also continuously differentiable and satisfies

$$\partial_t(\partial_x X)(t, t_0, x) = \partial_x c(t, X(t, t_0, x))\partial_x X(t, t_0, x).$$

Let $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R})$. Recall that $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R})$ is a weak solution of the problem

$$\begin{cases} \partial_t u(t, x) + c(t, x)\partial_x u(t, x) = 0, & (t, x) \in \mathbb{R}_+^* \times \mathbb{R}, \\ u(0, \cdot) = u_0, \end{cases} \tag{1}$$

if, for every $\varphi \in C^1_c(\mathbb{R} \times \mathbb{R})$, we have

$$\int_{\mathbb{R}_+ \times \mathbb{R}} u(t, x)(\partial_t \varphi(t, x) + \partial_x(c\varphi)(t, x))\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} \varphi(0, x)u_0(x)\,\mathrm{d}x = 0.$$

Exercice 1. *Constant velocity*
Let us consider $c \in \mathbb{R}$.

1. Let f be a measurable and non-negative function defined on $\mathbb{R}$. Show that for all $A > 0$, we have

$$\int_{[0,A] \times \mathbb{R}} f(x - ct)\,\mathrm{d}x\,\mathrm{d}t = A \int_{\mathbb{R}} f(y)\,\mathrm{d}y.$$

2. Let $u_0, \tilde{u}_0 : \mathbb{R} \to \mathbb{R}$ be two measurable functions. Define the functions $u, \tilde{u} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ by $u(t, x) = u_0(x - ct)$, $\tilde{u}(t, x) = \tilde{u}_0(x - ct)$, for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$. Suppose that u_0 and $\tilde{u}_0$ are equal almost everywhere on $\mathbb{R}$. Show that u and $\tilde{u}$ are also equal almost everywhere on $\mathbb{R}_+ \times \mathbb{R}$.

3. Suppose that u_0 is locally integrable on $\mathbb{R}$. Show that u is also locally integrable on $\mathbb{R}_+ \times \mathbb{R}$.

4. Deduce that the map that sends $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R})$ to the function $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R})$, where $u : (t, x) \in \mathbb{R}_+ \times \mathbb{R} \mapsto u_0(x - ct)$, is well defined.

5. Show that a function u as such is a weak solution of $\partial_t u + c\partial_x u = 0$.

Exercice 2. *Composition by the flow*
We keep the notations of the preamble.

1. Show that for all $t \in \mathbb{R}$, $y \mapsto X(t, 0, y)$ is an increasing function on $\mathbb{R}$. Give the range of this function.

2. Let f be a positive and measurable function on $\mathbb{R}$. Show that for all $t \in \mathbb{R}$,

$$\int_{\mathbb{R}} f(X(0,t,x))\,\mathrm{d}x = \int_{\mathbb{R}} f(y)\partial_x X(t,0,y)\,\mathrm{d}y.$$

Deduce that for all $A > 0$, we have

$$\int_{[0,A]\times\mathbb{R}} f(X(0,t,x))\,\mathrm{d}t\,\mathrm{d}x = \int_{[0,A]\times\mathbb{R}} f(y)\partial_x X(t,0,y)\,\mathrm{d}t\,\mathrm{d}y.$$

3. Prove that the map that sends u_0 to $u : (t,x) \mapsto u_0(X(0,t,x))$ is well defined as a map from $L^1_{loc}(\mathbb{R})$ to $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R})$.

Exercice 3. *Non-constant velocity*

With the same notations as in the preamble, let $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R})$ and $\varphi \in C^1_c(\mathbb{R}\times\mathbb{R})$.

1. Show that for all compact set $K \subset \mathbb{R}_+ \times \mathbb{R}$, the set

$$\big\{(t,y) \in \mathbb{R}_+ \times \mathbb{R} : (t, X(t,0,y)) \in K\big\}$$

is a compact set in $\mathbb{R}_+ \times \mathbb{R}$.

2. Show the equality (start by proving that each side is well-defined):

$$\int_{\mathbb{R}_+\times\mathbb{R}} u_0(X(0,t,x))\partial_t\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x = \int_{\mathbb{R}_+\times\mathbb{R}} u_0(y)\partial_t\varphi(t,X(t,0,y))\partial_x X(t,0,y)\,\mathrm{d}t\,\mathrm{d}y.$$

3. Show the equality (start again by proving that each side is well-defined):

$$\int_{\mathbb{R}_+\times\mathbb{R}} u_0(X(0,t,x))\partial_x(c\varphi)(t,x)\,\mathrm{d}t\,\mathrm{d}x = \int_{\mathbb{R}_+\times\mathbb{R}} u_0(y)\partial_x(c\varphi)(t,X(t,0,y))\partial_x X(t,0,y)\,\mathrm{d}t\,\mathrm{d}y.$$

4. Deduce that

$$\begin{aligned}\int_{\mathbb{R}_+\times\mathbb{R}} u_0(X(0,t,x))(\partial_t\varphi+\partial_x(c\varphi))(t,x)\,\mathrm{d}t\,\mathrm{d}x &= \int_{\mathbb{R}_+\times\mathbb{R}} u_0(y)\partial_t(\varphi(t,X(t,0,y)))\partial_x X(t,0,y)\,\mathrm{d}t\,\mathrm{d}y \\ &+ \int_{\mathbb{R}_+\times\mathbb{R}} u_0(y)\varphi(t,X(t,0,y))\partial_x c(t,X(t,0,y))\partial_x X(t,0,y)\,\mathrm{d}t\,\mathrm{d}y.\end{aligned}$$

5. Conclude that $(t,x) \mapsto u_0(X(0,t,x))$ is a weak solution to the problem (1).

Exercice 4. *Inhomogeneous equation*

Let $c > 0$ and $\psi \in C^1_c(]0,+\infty[\times\mathbb{R})$. Define the function

$$\varphi(t,x) = -\int_t^{+\infty} \psi(s, x + c(s-t))\,\mathrm{d}s, \quad (t,x) \in \mathbb{R}\times\mathbb{R}.$$

1. Show that $\varphi \in C^1(\mathbb{R}\times\mathbb{R})$.

2. Show that on $\mathbb{R}_+ \times \mathbb{R}$, we have

$$\partial_t\varphi + c\partial_x\varphi = \psi.$$

3. Prove the existence of a function $\theta \in C^1_c(\mathbb{R})$ such that the function $\widetilde{\varphi} : (t,x) \mapsto \theta(t)\varphi(t,x)$ belongs to the space $C^1_c(\mathbb{R}\times\mathbb{R})$ and satisfies on $\mathbb{R}_+\times\mathbb{R}$ that

$$\partial_t\widetilde{\varphi} + c\partial_x\widetilde{\varphi} = \psi.$$

4. Show that the problem (1) has at most one weak solution when $c(t,x) = c$.

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Master 1 - EDO et EDP **Semestre de printemps**

Sheet 1 on elliptic equations - Sobolev spaces in dimension 1

Exercice 1. *Lack of coercivity in C^1*

Let

$$X = \{u \in C^1([-1,1]) : u(-1) = u(1) = 0\},$$

endowed with its usual norm. Define on X the following functional

$$E(u) = \frac{1}{2} \int_{-1}^{1} |u'|^2 \, \mathrm{d}x, \quad u \in X.$$

Show that there exists a sequence $(u_n)_n$ in X of the form

$$u_n(x) = \frac{1}{\sqrt{n}} \varphi(nx),$$

with a well-chosen φ, such that the sequence $(E(u_n))_n$ is bounded but $\|u_n\|_X \to +\infty$ as $n \to +\infty$.

Exercice 2. *Lack of completeness*

1. Find a Cauchy sequence in $(C^0([-1,1]), \|\cdot\|_{L^2})$ that does not converge to a continuous function.

2. Show that the sequence $(u_n)_{n\geq 1}$ defined by

$$u_n(x) = \sqrt{x^2 + \frac{1}{n}},$$

is a Cauchy sequence that does not converge in $(C^1([-1,1]), \|\cdot\|_{H^1})$.

Exercice 3. *Examples of functions in H^1*

1. Let $f \in L^2(0,1)$. Define the function $F : [0,1] \to \mathbb{R}$ by

$$F(x) = \int_0^x f(t) \, \mathrm{d}t, \quad x \in [0,1].$$

Show that F is in $H^1(0,1)$. What is the weak derivative of F ?

2. Conversely, if F is a function in $H^1(0,1)$, show that there exists $f \in L^2(0,1)$ and $c \in \mathbb{R}$ such that for almost all $x \in [0,1]$,

$$F(x) = c + \int_0^x f(t) \, \mathrm{d}t.$$

Hint: Recall that if a function $h \in H^1(0,1)$ has a weak derivative that is zero everywhere, then there exists $c \in \mathbb{R}$ such that $h = c$ almost everywhere.

3. Let $g : [0,1] \to \mathbb{R}$ be a continuous function and piecewise continuously differentiable. Show that g belongs to $H^1(0,1)$.

Exercice 4. *C^0 is dense in H^1*

1. Let $(f_n)_n$ be a sequence of functions in $H^1(0,1)$. Suppose that $(f_n)_n$ converges in $L^2(0,1)$ to a function $f \in L^2(0,1)$ while $(f_n')_n$ converges in $L^2(0,1)$ to a function $g \in L^2(0,1)$. Show that $f \in H^1(0,1)$ and that $f' = g$.

2. Recall that the space $C^0([0,1])$ is dense in $L^2(0,1)$. Prove that $C^1([0,1])$ is dense in $H^1(0,1)$.

Exercice 5. *Differentiation of a product*

Let I be an open and bounded interval of $\mathbb{R}$. Show that the product of two functions $u, v \in H^1(I)$ is an element of $H^1(I)$ and, furthermore, that we have the following identity (in the weak sense)

$$(uv)' = u'v + uv'.$$

In particular, we have an integration by parts formula in the weak sense for every segment $[a,b] \subset \overline{I}$,

$$\int_a^b u'v \, \mathrm{d}x = (uv)(b) - (uv)(a) - \int_a^b uv' \, \mathrm{d}x.$$

Exercice 6. *Differentiation of compositions*

Let I be an open and bounded interval of $\mathbb{R}$. Show that if $F \in C^1(\mathbb{R}, \mathbb{R})$ and $u \in H^1(I)$, then $F \circ u$ is also an element of $H^1(I)$ and that we have almost everywhere

$$(F \circ u)' = (F' \circ u)u'.$$

Exercice 7. *Poincaré inequalities*

In the beginning of this exercise, we will be working on an interval $I = [0, L]$ with $L > 0$.

1. We wish to prove the following Poincaré inequality

$$\forall u \in H_0^1(I), \quad \|u\|_{L^2(I)} \leq \frac{L}{2\pi} \|u'\|_{L^2(I)}. \tag{1}$$

- **a.** Show that if we can prove (1) for functions $u \in C_c^\infty(]0, L[)$, then we get the expected result.
- **b.** Prove inequality (1) for functions in $C_c^\infty(]0, L[)$ by using Fourier series.
- **c.** Show that the constant L/π is optimal, meaning that it is the smallest constant for which inequality (1) is satisfied.

2. Let us introduce the following functional space

$$H_m^1(I) = \left\{ u \in H^1(I) : \int_0^L u(x) \, \mathrm{d}x = 0 \right\}.$$

By proceeding in a similar way as in the previous question, show that the following Poincaré inequality is also satisfied

$$\forall u \in H_m^1(I), \quad \|u\|_{L^2(I)} \leq \frac{L}{2\pi} \|u'\|_{L^2(I)},$$

where the constant $L/2\pi$ is optimal. Deduce that $u \in H_m^1(I) \mapsto \|u'\|_{L^2(I)}$ is a norm on $H_m^1(I)$ that is equivalent to $\| \cdot \|_{H^1(I)}$.

3. Does the following Poincaré inequality hold ?

$$\exists c > 0, \forall u \in H^1(\mathbb{R}), \quad \|u\|_{L^2(\mathbb{R})} \leq c \|u'\|_{L^2(\mathbb{R})}.$$

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Master 1 - EDO et EDP **Semestre de printemps**

Sheet 2 on elliptic equations - One-dimensional problems

Exercice 1. *Revisiting Neumann's problem*
Let $I =]0,1[$ and $f \in L^2(I)$ with zero mean, *i.e.* $\int_0^1 f\,dx = 0$.

1. Let us consider the bilinear form a and the linear form L respectively defined on $H^1(I)$ by

$$a(u,v) = \int_0^1 u'v'\,\mathrm{d}x + \left(\int_0^1 u\,\mathrm{d}x\right)\left(\int_0^1 v\,\mathrm{d}x\right), \quad u,v \in H^1(I), \tag{1}$$

and

$$L(v) = \int_0^1 fv\,\mathrm{d}x, \quad v \in H^1(I).$$

a. Prove that for all $u \in H^1(I)$, we have

$$\int_0^1 \left|u(x) - \int_0^1 u\right|^2 \mathrm{d}x \le \int_0^1 u'(x)^2\,\mathrm{d}x.$$

b. Prove that the Lax-Milgram theorem can be applied to show that the unique solution of the corresponding variational formulation solves the following Neumann problem

$$\begin{cases} -u''(x) = f(x), & x \in I, \\ u'(0) = u'(1) = 0, \\ \int_0^1 u\,\mathrm{d}x = 0, \end{cases}$$

2. Let $\alpha \in \mathbb{R}$ be arbitrary. We still consider the bilinear form a given by (1), but we redefine the linear form L on $H^1(I)$ as follows

$$L(v) = \int_0^1 fv\,\mathrm{d}x + \alpha(v(0) - v(1)), \quad v \in H^1(I).$$

Show that the Lax-Milgram theorem can also be applied in this case and identify the boundary value problem satisfied by the solution u of the corresponding variational formulation.

Exercice 2. *A new boundary condition*
Let $I =]0,1[$, $f \in L^2(I)$ and $\lambda > 0$. Let us consider the bilinear form a and the linear L respectively defined on $H^1(I)$ by

$$a(u,v) = \int_0^1 u'v'\,\mathrm{d}x + \lambda u(0)v(0) + \lambda u(1)v(1), \quad u,v \in H^1(I),$$

and

$$L(v) = \int_0^1 fv\,\mathrm{d}x, \quad v \in H^1(I).$$

Show that the Lax-Milgram theorem can be applied in this context and determine the boundary value problem that is satisfied by the solution u of the corresponding variational formulation.

Exercice 3. *Non-homogeneous Dirichlet conditions*
Let $I =]0,1[$, $f \in C^0([0,1])$ and $k \in C^0([0,1])$ such that $\min_{x\in[0,1]} k(x) > 0$. Let $u_0, u_1 \in \mathbb{R}$. We want to solve the following problem

$$(P_{u_0,u_1}) \qquad \begin{cases} -(ku')' = f, & \text{on } I, \\ u(0) = u_0,\ u(1) = u_1. \end{cases}$$

1. Prove that there exists $r \in C^\infty([0,1])$ such that $r(0) = u_0$ and $r(1) = u_1$.
2. Show that the problem (P_{u_0,u_1}) is equivalent to the following one:

$$(Q_0) \qquad \begin{cases} -(kv')' = f + (kr')', & \text{on } I, \\ v(0) = 0,\ v(1) = 0. \end{cases}$$

3. Prove the existence of a unique solution u in $H^1(I)$ to the variational formulation of (P_{u_0,u_1}).

Exercice 4. *Sturm problem with energy*
Let $I =]0,1[$ and q, f be two continuous functions on $\overline{I}$. Consider the following problem:

$$(2) \qquad \begin{cases} -u'' + q(x)u = f(x), & x \in]0,1[, \\ u(0) = u(1) = 0. \end{cases}$$

1. Show that if $q(x) = -\pi^2$, then the problem is not well-posed in the following sense :
  a. There are some functions f for which (2) has no solution in $C^2([0,1])$,
  b. For $f = 0$, there exist some non-zero solutions u of the problem (2).

  In other words, the problem (2) does not always have solutions, and if they exist, they are not necessarily unique.
2. From now on, we will assume that the function q satisfies

$$\inf_I q > -\pi^2.$$

  Give a variational formulation of the problem (2) on $H_0^1(I)$. Give an expression of an "energy" function $u \mapsto E(u)$ for which a solution of (2), when it exists, would be a minimizer. Show that this functional E is of class C^1 for the topology of $H_0^1(I)$.
3. Show that E is bounded from below on the space $H_0^1(I)$, and then deduce the existence of a minimizing sequence $(u_n)_n$.
4. Prove that this minimizing sequence is Cauchy in $H_0^1(I)$ and deduce the existence of a unique $u \in H_0^1(I)$ such that

$$E(u) = \inf_{v\in H_0^1(I)} E(v) > -\infty.$$

5. Show that the obtained function u is solution to the variational formulation problem, and then prove that it is also a weak solution of (2). Moreover, prove that u is in fact an element of $C^2(\overline{I}) \cap C^1(\overline{I})$ and therefore a solution to problem (2) in the classical sense.
6. Which step(s) of the proof will not work if $q(x) = -\pi^2$?

Exercice 5. *Second order equation*
Let $I =]0,1[$ and $f \in L^2([0,1])$. Solve the following problem in $H_0^1(I)$

$$\begin{cases} -u'' + u' = f & \text{on } I, \\ u(0) = u(1) = 0. \end{cases}$$

Hint: introduce on the space $H_0^1(I)$ the following bilinear form

$$a(u,v) = \int_0^1 u'v' + \int_0^1 u'v, \quad u,v \in H_0^1(I),$$

and prove that $\int_0^1 u'u = 0$.

Exercice 6. *An example of a non-linear boundary value problem*
Define $I =]0,1[$. In this exercise, we will try to solve the following problem

$$\begin{cases} -u'' + \lambda u^3 = f, & \text{on } I, \\ u(0) = u(1) = 0, \end{cases} \tag{3}$$

where $f : I \to \mathbb{R}$ is a continuous function and $\lambda > 0$ a positive real parameter.

1. Uniqueness : Let u_1 and u_2 be two C^2 functions that solve the problem (3). Prove that we have

$$\int_0^1 |u_1' - u_2'|^2 \,\mathrm{d}x + \lambda \int_0^1 (u_1^3 - u_2^3)(u_1 - u_2) \,\mathrm{d}x = 0.$$

Deduce that necessarily, $u_1 = u_2$.

2. Variational formulation and existence of solutions

a. Prove the following inequality

$$\forall a,b \in \mathbb{R}, \quad \left(\frac{a+b}{2}\right)^4 \leq \frac{a^4}{2} + \frac{b^4}{2}.$$

b. Show that every possible solution $u \in C^2([0,1])$ of problem (3) satisfies the following property

$$\forall v \in H_0^1(I), \quad \int_0^1 u'v' \,\mathrm{d}x + \lambda \int_0^1 u^3 v \,\mathrm{d}x = \int_0^1 fv \,\mathrm{d}x. \tag{4}$$

Now, we consider the following problem

$$(\mathcal{P}) \qquad \text{Find } u \in H_0^1(I) \text{ satisfying (4).}$$

c. Why can't we use the Lax-Milgram theorem to solve problem $(\mathcal{P})$?

d. For all $v \in H_0^1(I)$, we introduce the following energy function

$$E(v) = \frac{1}{2}\int_0^1 |v'|^2 \,\mathrm{d}x + \frac{\lambda}{4}\int_0^1 |v|^4 \,\mathrm{d}x - \int_0^1 fv \,\mathrm{d}x, \quad v \in H_0^1(I).$$

Briefly explain why all the terms are well-defined.

e. Prove that E is bounded from below on $H_0^1(I)$.

f. Justify the existence of a minimizing sequence of E in $H_0^1(I)$. Let $(u_n)_n \subset H_0^1(I)$ be such a sequence. By considering $E((u_n + u_{n+p})/2)$, prove the following inequality

$$\forall n, p \geq 0, \quad \frac{1}{8}\|u_n' - u_{n+p}'\|_{L^2(I)}^2 \leq \frac{1}{2}E(u_n) + \frac{1}{2}E(u_{n+p}) - \inf_{v \in H_0^1} E(v).$$

Deduce that $(u_n)_n$ is a Cauchy sequence in $H_0^1(I)$.

g. Conclude on the existence of a function $u \in H_0^1(I)$ such that

$$E(u) = \inf_{v \in H_0^1(I)} E(v).$$

h. Let $v \in H_0^1(I)$ arbitrary. By studying the function $t \in \mathbb{R} \mapsto E(u + tv) \in \mathbb{R}$, prove that the previously obtained function u is solution to problem ($\mathcal{P}$).

i. Prove that $u' \in H^1(I)$ and compute u''. Deduce that u' is continuous and then that u'' is also continuous. Finally, conclude that u is the unique C^2 solution of problem (3).

Exercice 7. *Generalities on the Galerkin methods*

Let $I =]0,1[$ and consider $k \in L^\infty(I)$ and $f \in L^2(I)$. Suppose that there exists $\alpha > 0$ such that $\inf_I k \geq \alpha$. Endow the space $H_0^1(I)$ with the norm $\|u\|_{H_0^1(I)} = \|u'\|_{L^2(I)}$ and consider the problem

$$\begin{cases} -(ku')' = f, & \text{on } I, \\ u(0) = u(1) = 0. \end{cases}$$

1. Write the corresponding variational formulation.

2. Prove the existence of a unique solution $u \in H_0^1(I)$ to the variational formulation.

3. Let $(V_n)_n$ be a sequence of finite dimensional subspaces of $H_0^1(I)$. Prove the existence of a unique solution $u_n \in V_n$ of

$$(FV_n) \qquad \forall v_n \in V_n, \quad \int_I k(x) u_n'(x) v_n'(x)\, \mathrm{d}x = \int_I f(x) v_n(x)\, \mathrm{d}x.$$

4. Prove that the solution u of the variational formulation from question 1 and the solution u_n of (FV_n) satisfy the following estimate

$$d(u, V_n) \leq \|u - u_n\|_{H_0^1} \leq \frac{\|k\|_\infty}{\alpha} d(u, V_n),$$

where $d(u, V_n) = \inf_{v \in V_n} \|u - v\|_{H_0^1}$.