TD 1: Topology issues in product spaces and Banach spaces

Exercise 1 (General topology).

1. Let $f : E \to F$ be a continuous map between topological spaces. Show that f is sequently continuous. Namely, show that if the sequence $(x_n)_n$ converges to x in E then the sequence $(f(x_n))_n$ converges to $f(x)$ in F. Can we claim that if f is sequently continuous then f is continuous ?
2. Let $f : E \to F$ be a map between topological spaces. The function f is said to be continuous at $x \in E$ if for all open set $\mathcal{V}$ containing $f(x)$, there exists an open set $\mathcal{U}$ containing x and such that $f(\mathcal{U}) \subset \mathcal{V}$. Check that, in this definition, "open set" can be replaced by "neighbourhood".
3. Let X be a set, $(F_i)_{i\in I}$ be a family of topological spaces and $f_i : X \to F_i$ be some functions.
   (a) Prove that the "coarsest topology that makes the functions f_i continuous" exists.
   (b) Let $g : E \to X$ be a function defined on a topological space E. Check that g is continuous if and only if for all $i \in I$, $f_i \circ g$ is continuous.
   (c) Let $(x_n)_n$ be a sequence in X. Prove that $(x_n)_n$ converges to x if and only if for all $i \in I$, $(f_i(x_n))_n$ converges to $f_i(x)$.
4. Let $(F_i)_{i\in I}$ be a family of topological spaces. We define the product topology on $\prod_{i\in I} F_i$ as the "coarsest topology" making the projections continuous. Show that this topology is generated by the cylinder sets, *i.e.* the sets of the form $C_J = \prod_{i\in I} U_i$, where each U_i is open in F_i and $U_i = F_i$, except for a finite number of indexes $i \in J$.

Exercise 2 (A theorem of Hörmander). Let $1 \leq p, q < \infty$ and

$$T : (L^p(\mathbb{R}^n), \|\cdot\|_p) \to (L^q(\mathbb{R}^n), \|\cdot\|_q),$$

be a continuous linear operator which commutes with the translations, that is, which satisfies $\tau_h T = T\tau_h$ for all $h \in \mathbb{R}^n$, where $\tau_h f = f(\cdot - h)$. The purpose of this exercice is to prove the following property: if $q < p < \infty$, then the operator T is trivial.

1. Let u be a function in $L^p(\mathbb{R}^n)$. Prove that $\|u + \tau_h u\|_p \to 2^{1/p}\|u\|_p$ as $\|h\| \to \infty$.
   Hint: you may decompose u as the sum of a compactly supported function and of a function with arbitrarily small L^p norm.
2. Check that if C stands for the norm of operator T, then we have that for all $u \in L^p(\mathbb{R}^n)$,
   $$\|Tu\|_q \leq 2^{1/p-1/q} C\|u\|_p,$$
   and conclude.
3. Can you give the example of a non-trivial such operator T when $p \leq q$?

Exercise 3 (Fourier coefficients of L^1 functions). For any function f in $L^1(\mathbb{T})$, we define the function $\hat{f} : \mathbb{Z} \to \mathbb{C}$ by

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,\mathrm{d}t, \quad n \in \mathbb{Z}.$$

We denote by c_0 the space of complex valued functions on $\mathbb{Z}$ tending to 0 at $\pm\infty$.

1. Check that $(c_0, \|\cdot\|_\infty)$ is a Banach space.
2. Prove that, for all $f \in L^1(\mathbb{T})$, $\hat{f} \in c_0$.
   Hint: Recall that the trigonometric polynomials $\sum_{k=-n}^{n} a_k e^{ikt}$ are dense in $L^1(\mathbb{T})$.

Now we study the converse question: is every element of c_0 the sequence of Fourier coefficients of a function in $L^1(\mathbb{T})$?

3. Prove that $\Lambda : f \to \hat{f}$ defines a bounded linear map from $L^1(\mathbb{T})$ to c_0.
4. Prove that the function Λ is injective.
5. Show that the function Λ is not onto.
   Hint: You may use the Dirichlet kernel $D_n(t) = \sum_{k=-n}^{n} e^{ikt}$, whose $L^1(\mathbb{T})$ norm goes to $+\infty$ as $n \to +\infty$.

Exercise 4 (Equivalence of norms).

1. Let E be a vector space endowed with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ such that both $(E, \|\cdot\|_1)$ and $(E, \|\cdot\|_2)$ are Banach spaces. Assume the existence of a finite constant $C > 0$ such that
$$\forall x \in E, \quad \|x\|_1 \leqslant C\|x\|_2.$$
   Prove that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.
2. Let K be a compact subset of $\mathbb{R}^n$. We consider a norm N on the space $\mathcal{C}^0(K, \mathbb{R})$ such that $(\mathcal{C}^0(K, \mathbb{R}), N)$ is a Banach space, and satisfying that any sequence of functions $(f_n)_n$ in $\mathcal{C}^0(K, \mathbb{R})$ that converges for the norm N also converges pointwise to the same limit. Prove that the norm N is then equivalent to the norm $\|\cdot\|_\infty$.

Exercise 5 (A Rellich-like theorem). Let us consider E the following subspace of $L^2(\mathbb{R})$
$$E = \big\{u \in \mathcal{C}^1(\mathbb{R}) : \|u\|_E < +\infty\big\}, \quad \text{where} \quad \|u\|_E = \|(\sqrt{1+x^2})u\|_{L^2(\mathbb{R})} + \|u'\|_{L^2(\mathbb{R})}.$$
The aim of this exercice is to prove that the unit ball B_E of E is relatively compact in $L^2(\mathbb{R})$, with
$$B_E = \big\{u \in \mathcal{C}^1(\mathbb{R}) : \|u\|_E \leq 1\big\}.$$
In the following, we denote by ϕ a non-negative $\mathcal{C}^\infty$ function such that $\phi^{-1}(\{0\}) = \mathbb{R} \setminus [-2, 2]$ and $\phi^{-1}(\{1\}) = [-1, 1]$.

1. Considering the cut-off $\phi_R(x) = \phi(x/R)$, show that $\sup_{u \in B_E} \|(1 - \phi_R)u\|_{L^2(\mathbb{R})}$ converges to 0 as $R \to +\infty$.
2. We define $\psi_\varepsilon(x) = \frac{1}{\varepsilon}\phi(\frac{x}{\varepsilon})$ and τ_h the translation operator (see Exercice 2). Show that for all $R \geq 1$ and $\varepsilon > 0$, there exists $C_{\varepsilon,R} > 0$ such that for all $h \in \mathbb{R}$ and $u \in E$,
$$\|\tau_h((\phi_R u) * \psi_\varepsilon) - (\phi_R u) * \psi_\varepsilon\|_{L^\infty(\mathbb{R})} \leq C_{\varepsilon,R}|h|\|u\|_E \quad \text{and} \quad \|(\phi_R u) * \psi_\varepsilon\|_{L^\infty(\mathbb{R})} \leq C_{\varepsilon,R}\|u\|_E.$$
3. Show that for any sequence $(u_n)_n$ in B_E, there exists a subsequence $(u_{n'})_{n'}$ such that for any $R, \varepsilon^{-1} \in \mathbb{N}^*$, the sequence $((\phi_R u_{n'}) * \psi_\varepsilon)_{n'}$ converges in $L^2(\mathbb{R})$ as $n' \to \infty$.
   Hint: Use Cantor's diagonal argument.
4. Conclude.
5. Let us now consider the set $B_{H^1} \subset L^2(\mathbb{R})$ defined by
$$B_{H^1} = \big\{u \in \mathcal{C}^1(\mathbb{R}) : \|u\|_{L^2(\mathbb{R})} + \|u'\|_{L^2(\mathbb{R})} \leq 1\big\}.$$
   Is B_{H^1} relatively compact in $L^2(\mathbb{R})$?

TD 2: Locally convex topological vector spaces and Fréchet spaces

In the following, "locally convex topological vector space" will be abbreviated as l.c.t.v.s.

Exercise 1. Let E be a locally convex topological vector space whose topology is induced by a (separating) countable family of semi-norms $(p_n)_{n\in\mathbb{N}}$. We define

$$d(x,y) = \sum_{n=0}^{+\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1+p_n(x-y)}, \quad x,y \in E.$$

Let us prove that the topology induced by d and the topology induced by the family of seminorms $(p_n)_{n\in\mathbb{N}}$ coincide.

1. Show that $g : [0,\infty) \to \mathbb{R}$ defined by $g(t) = \frac{t}{1+t}$ is an increasing sub-additive function and give its image. Deduce that d is a translation invariant distance on E.
2. Give a basis of neighbourhoods of 0_E for the topology induced by the family of semi-norms, and show that every neighbourhood of 0_E contains an open ball for the distance d.
3. Show that every open ball for the distance d centered on 0_E contains a neighbourhood of 0_E for the topology induced by the family of semi-norms.
4. Conclude.

More generally, let us consider a continuous bounded function $g : [0,+\infty) \to \mathbb{R}_+$ and

$$d_g(x,y) = \sum_{n=0}^{+\infty} \frac{1}{2^n} g(p_n(x-y)), \quad x,y \in E$$

5. Under what condition on g does d_g defines a distance on E whose topology coincide with the one induced by the family of seminorms $(p_n)_{n\in\mathbb{N}}$?

Exercise 2. Let X and Y be l.c.t.v.s. We consider $(p_\alpha)_{\alpha\in A}$ (resp. $(q_\beta)_{\beta\in B}$) a countable family of continuous semi-norms which is separating and generates the topology of X (resp. of Y). Let $T : X \to Y$ be a linear map. Prove that T is continuous if and only if for all $\beta \in B$, there exists a finite set $I \subset A$ and a positive constant $c > 0$ such that for all $u \in X$,

$$q_\beta(Tu) \leq c \sum_{\alpha\in I} p_\alpha(u).$$

Exercise 3 (Space of continuous functions). Let U be an open subset of $\mathbb{R}^d$ and $(K_n)_n$ be an exhaustive sequence of compacts of U.

1. Prove that $C^0(U)$ is a Fréchet space for the distance

$$d(f,g) = \sum_{n=0}^{+\infty} \frac{1}{2^n} \min(1, p_n(f-g)),$$

defined by the semi norms $p_n(f) = \sup_{x\in K_n} |f(x)|$.

2. Recall that a subset $B \subset C^0(U)$ is said to be bounded if for any neighborhood V of 0, there exists $\lambda > 0$ such that $\lambda B \subset V$. Prove that if B is a subset of equibounded functions of $C^0(U)$, that is $\sup_{f \in B} \|f\|_\infty < \infty$, then B is bounded.
3. Let us consider $(f_n)_n$ a sequence of continuous function on U such that $f_n : U \to [0, n]$ with $f_n = 0$ on K_n and $f_n = n$ on $U \setminus K_{n+1}$. Show that $\cup_n \{f_n\}$ is a bounded subset of $C^0(U)$.
4. Prove that the space $C^0(\mathbb{R})$ is not locally bounded, that is, the origin does not have a bounded neighborhood.

Exercise 4 (Space of C^∞ functions). We consider the vector space $E = C^\infty([0,1], \mathbb{R})$ equipped with the following metric

$$d(f, g) = \sum_{k \geq 0} \frac{1}{2^k} \min\big(1, \|f^{(k)} - g^{(k)}\|_\infty\big).$$

1. Check that E is a Fréchet space.
2. Prove that any closed and bounded (*cf* the previous exercise) subset of E is compact.
3. Can the topology of E be defined by a norm ?

Exercise 5 (L^p spaces with $0 < p < 1$). Let $p \in (0, 1)$ and L^p be the set of real-valued measurable functions u defined over $[0, 1]$, modulo almost everywhere vanishing functions, for which the following quantity is finite:

$$\|u\|_p = \left(\int_0^1 |u(x)|^p \,\mathrm{d}x \right)^{\frac{1}{p}}.$$

1. Show that L^p is a vector space and that $d(u, v) = \|u - v\|_p^p$ is a distance. Prove that (L^p, d) is complete.
2. Let $f \in L^p$ and $n \geq 1$ be a positive integer. Prove that there exist some points $0 = x_0 < x_1 < \ldots < x_n = 1$ such that for all $i = 0, \ldots, n-1$,

$$\int_{x_i}^{x_{i+1}} |f|^p \,\mathrm{d}x = \frac{1}{n} \int_0^1 |f|^p \,\mathrm{d}x.$$

3. Prove that the only convex open domain in L^p containing $u \equiv 0$ is L^p itself. Deduce that the space L^p is not locally convex.
Hint: Introduce the functions $g_i^n = n f \mathbb{1}_{[x_i, x_{i+1}]}$.
4. *Bonus*: Show that the (topological) dual space of L^p reduces to $\{0\}$.

TD 3: Hahn-Banach theorems

Exercise 1 (Hahn-Banach Theorem without the axiom of choice).

1. Let (E, d_E) and (F, d_F) be metric spaces, (F, d_F) being complete, $D \subset E$ be a dense subset and $f : (D, d_E) \to (F, d_F)$ be a uniformly continuous function. Then, there exists a unique continuous function $F : (E, d_E) \to (F, d_F)$ such that $F_{|D} = f$. Moreover, prove that the function F is uniformly continuous.
2. Let E be a real separable Banach space and p be a continuous seminorm on E. Let M be a linear subspace of E and $\varphi : M \to \mathbb{R}$ be a linear functional which is dominated by p. Without using the axiom of choice, prove that φ can be extended to a linear functional $E \to \mathbb{R}$ which remains dominated by p.

Exercise 2 (Separation in Hilbert spaces without the Hahn-Banach theorem). In this exercise, the use of the axiom of choice is prohibited. Let H be an Hilbert space.

1. Let $C \subset H$ be a convex, closed and non-empty set. Prove that any $v \notin C$ can be strictly separated by C by a closed hyperplane, *i.e.* there exists $v_0 \in H$ such that
$$\forall u \in C, \quad \langle v_0, u \rangle < \langle v_0, v \rangle.$$
2. Let $C_1, C_2 \subset H$ be convex, closed and non-empty disjoint sets, C_1 being moreover compact. Prove that C_1 and C_2 can be strictly separated by a closed hyperplane, *i.e.* there exists $u_0 \in H$ such that
$$\sup_{u \in C_1} \langle u_0, u \rangle < \inf_{u \in C_2} \langle u_0, u \rangle.$$

Exercise 3 (First uses of the Hahn-Banach theorem). Let E be a normed vector space.

1. Let G be a vector subspace of E and $g : G \to \mathbb{R}$ be a continuous linear form. Recall why there exists a continuous linear form f over E that extends g, and such that
$$\|f\|_{E^*} = \|g\|_{G^*}.$$
When E is an Hilbert space, prove that this extension is unique.
2. Assume that $E = \ell^1(\mathbb{N})$. Give the example of a continuous linear form of norm 1, defined on a strict vector subspace of E, which admits an infinite number of linear continuous extensions of norm 1 over E.
3. Assume that E is a Banach space.
   (a) Prove that for all $x \in E$,
   $$\|x\| = \max_{f \in E^* : \|f\|_{E^*} \leq 1} |f(x)|.$$
   (b) Let B be a subset of E such that
   $$\forall f \in E^*, \quad \sup_{x \in B} f(x) < +\infty.$$
   Prove that B is bounded.

Exercise 4 (Convex sets that cannot be separated). Let H be the Hilbert space $L^2([-1,1])$. For every $\alpha \in \mathbb{R}$, let $C_\alpha \subset H$ be the subset of continuous functions $u : [-1,1] \to \mathbb{R}$ such that $u(0) = \alpha$. Prove that C_α is a convex dense subset of H. Deduce that, if $\alpha \neq \beta$, then C_α and C_β are convex disjoint subsets that cannot be separated by a continuous linear form.

Exercise 5 (Banach limit).

1. Let $s : \ell^\infty(\mathbb{N}) \to \ell^\infty(\mathbb{N})$ be the shift operator, defined by $s(x)_i = x_{i+1}$ for all $i \in \mathbb{N}$ and $x \in \ell^\infty(\mathbb{N})$. Prove the existence of a continuous linear function $\Lambda \in (\ell^\infty(\mathbb{N}))'$ satisfying $\Lambda \circ s = \Lambda$ and
$$\forall u \in \ell^\infty(\mathbb{N}), \quad \liminf_{n\to+\infty} u_n \leq \Lambda(u) \leq \limsup_{n\to+\infty} u_n.$$
Such a linear form Λ is called Banach limit.
Hint: Consider the vector space of bounded sequences that converge in the sense of Cesàro.
2. Deduce that there exists a function $\mu : \mathcal{P}(\mathbb{N}) \to \mathbb{R}_+$ which satisfies
   - (i) $\mu(\mathbb{N}) = 1$,
   - (ii) μ is finitely additive: $\forall A, B \subset \mathbb{N}$ with $A \cap B = \emptyset$, $\mu(A \cup B) = \mu(A) + \mu(B)$,
   - (iii) μ is left-invariant: $\forall k \in \mathbb{N}$ and $A \subset \mathbb{N}$, $\mu(k + A) = \mu(A)$.

Exercise 6 (Finite-dimensional case).

1. Let $C \subset \mathbb{R}^d$ be a convex set such that $C \neq \mathbb{R}^d$, and $x_0 \notin C$. Prove that there exists an affine hyperplane that separates C and $\{x_0\}$.
2. Does this result hold in an infinite dimensional space ?

Exercise 7 (Convex hull). Let E be a locally convex topological vector space (abbreviated l.c.t.v.s. in the following). One says that H is a closed half-space if there exists a $\varphi \in E^*$ and $a \in \mathbb{R}$ such that $H = \{u \in E \mid \varphi(u) \leq a\}$.

1. If C is a convex subset of E, show that its closure $\overline{C}$ is also convex.
2. Let A be a closed convex subset of E. Show that A is the intersection of the closed half-spaces containing A.
3. Deduce that $\overline{co(A)}$ is the intersection of the closed half-spaces containing A for any subset A of E, where $co(A)$ denotes the convex hull of the set A, that is, the smallest convex set that contains A.

Exercise 8 (Density criterion).

1. Let E be a real normed vector space and $F \subset E$ be a vector subspace such that $\overline{F} \neq E$. Prove that there exists $\varphi \in E' \setminus \{0\}$ such that $\varphi(u) = 0$ for all $u \in F$.
2. *Application:* Let $(a_n)_n$ be a sequence in $]1, +\infty[$ that diverges to $+\infty$. Prove that the set
$$W = \operatorname{span}\left\{x \in [0,1] \mapsto \frac{1}{x - a_n} : n \geq 0\right\},$$
is dense in the space $\mathcal{C}^0([0,1])$ equipped with the norm $\|\cdot\|_\infty$.
Hint: While considering a continuous linear form that vanishes on W, introduce a generating function.

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TD 4: Weak topologies

Exercise 1.

1. Let E be a l.c.t.v.s whose topology is generated by a separating family of seminorms $(p_\alpha)_{\alpha \in I}$. Prove that a sequence $(x_n)_n$ of elements in E converges to some $x \in E$ if and only if for all $\alpha \in I$, the sequence $(p_\alpha(x - x_n))_n$ converges to 0.
2. Let E be a Banach space. By using the previous question, give a characterization of weakly converging sequences in terms of continuous linear forms.

Exercise 2. Let X be a normed vector space.

1. Let $(u_n)_n$ be a weakly convergent sequence in X. Justify that (u_n) is bounded and that the weak limit u of $(u_n)_n$ satisfies $\|u\| \leq \liminf_{n \to +\infty} \|u_n\|$.
2. Suppose that the sequence $(\varphi_n)_n$ in X^* is converging strongly to some $\varphi \in X^*$. Show that for any sequence $(u_n)_n$ in X that converges weakly to $u \in X$, then the sequence $(\varphi_n(u_n))_n$ converges to $\varphi(u)$.
3. Assume that X is a Hilbert space. Let $(u_n)_n$ be a sequence in X that converges weakly to $u \in X$ and such that $(\|u_n\|)_n$ converges to $\|u\|$. Prove that $(u_n)_n$ converges strongly to u.

Exercise 3. The purpose of this exercise is to present three obstructions to strong convergence in $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{T}^d)$. In the following, $\varphi \in C_c^\infty(\mathbb{R}^d)$ denotes a compactly supported smooth function being not identically equal to zero.

1. (Loss of mass) Let ν be a vector of norm 1. Prove that the sequence $(\varphi(\cdot - n\nu))_n$ converges weakly to zero in $L^2(\mathbb{R}^d)$, but not strongly.
2. (Concentration) Prove that the sequence $(n^{d/2}\varphi(n\,\cdot))_n$ converges weakly to zero in $L^2(\mathbb{R}^d)$, but not strongly.
3. (Oscillations) We now consider $w \in L^2(\mathbb{T}^d)$ a non-constant function. Prove that the sequence $(w(n\,\cdot))_n$ converges weakly but not strongly to $\frac{1}{2\pi}\int_0^{2\pi} w$ in $L^2(\mathbb{T}^d)$.

Exercise 4. Let E be a Banach space.

1. Show that if E is finite-dimensional, then the weak topology $\sigma(E, E^*)$ and the strong topology coincide.
2. We assume that E is infinite-dimensional.
   (a) Show that every weak open subset of E contains a straight line.
   (b) Deduce that $B = \{x \in E : \|x\| < 1\}$ has an empty interieur for the weak topology.
   (c) Let $S = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. What is the weak closure of S ?

Exercise 5. Let E be an infinite-dimensional Banach space. Prove that the weak topology on E is not metrizable.
Hint: Recall that any open weak set contains a line.

Exercise 6.

1. (Mazur's lemma) Let E be a Banach space and $(u_n)_n$ be a sequence in E weakly converging to $u_\infty \in E$. Show that u_∞ is a strong limit of finite convex combinations of the u_n.
2. (Banach-Sacks' property) Show that if E is in addition a Hilbert space, we can extract a subsequence converging to u_∞ strongly in the sens of Cesàro.

Exercise 7 (Schur's property for $\ell^1(\mathbb{N})$).

1. Recall why weak and strong topologies always differ in an infinite dimensional norm vector space.

The aim is to prove that a sequence of $\ell^1(\mathbb{N})$ converges weakly if and only if it converges strongly. Take $(u^n)_n$ a sequence in $\ell^1(\mathbb{N})$ weakly converging to 0.

2. Show that for all k, $\lim_{n\to\infty} u_k^n \to 0$.
3. Show that if $u_n \nrightarrow 0$ in $\ell^1(\mathbb{N})$, one can additionally assume that $\|u^n\|_{\ell^1} = 1$.
4. Define via a recursive argument two increasing sequences of $\mathbb{N}$, $(a_k)_k$ and $(n_k)_k$, such that

$$\forall k \geq 0, \qquad \sum_{j=a_k}^{a_{k+1}-1} |u_j^{n_k}| \geq \frac{3}{4}.$$

5. Show that there exists $v \in \ell^\infty(\mathbb{N})$ such that $(v, u^{n_k})_{\ell^2} \geq \frac{1}{2}$ for all k. Conclude.

TD 5: Weak topologies (II)

Exercise 1. Let E and F be two Banach spaces, and $T : E \to F$ be a linear map. Show that T is strongly continuous (*i.e.* continuous from $(E, \|\cdot\|_E)$ to $(F, \|\cdot\|_F)$) if and only if T is weakly continuous (*i.e.* continuous from $(E, \sigma(E, E^*))$ to $(F, \sigma(F, F^*))$).

Exercise 2. Let E be a separable real normed vector space. Let $(u_n)_n$ be a dense sequence in $B_E(0,1)$. By considering the following metric d on the unit ball of E^*,

$$d(f,g) = \sum_{n=0}^{+\infty} \frac{1}{2^n} |(f-g)(u_n)|, \quad f,g \in B_{E^*}(0,1),$$

prove that the weak-$*$ topology on $B_{E^*}(0,1)$ is metrizable.

Exercise 3 (Goldstine lemma). Let X be a Banach space. For any $x \in X$, let us define the evaluation $\mathrm{ev}_x : \varphi \in X^* \mapsto \varphi(x) \in \mathbb{R}$. We can therefore consider the following application

$$J : \begin{cases} X & \to & X^{**} \\ x & \mapsto & \mathrm{ev}_x \end{cases}$$

For any normed vector space E, we denote by B_E its closed unit ball.

1. Check that J is an isometry and that $J(X)$ is strongly closed in X^{**}.
2. Let E be a normed vector space. Determine all the linear forms on E^* which are continuous for the weak-$*$ topology $\sigma(E^*, E)$.
3. By using the Hahn-Banach theorem, prove that $J(B_X)$ is dense in $B_{X^{**}}$ for the weak-$*$ topology $\sigma(X^{**}, X^*)$.

Exercise 4.

1. In $\ell^\infty(\mathbb{N})$ we consider
$$C = \left\{ x \in \ell^\infty(\mathbb{N}) : \liminf_n x_n \geq 0 \right\}.$$
Show that C is strongly closed but not weakly-$*$ closed.
2. Let E be a normed vector space. Show that an hyperplane $H \subset E^*$ which is closed for the weak-$*$ topology $\sigma(E^*, E)$ is the kernel of $\mathrm{ev}_x : \varphi \mapsto \varphi(x)$ for some $x \in E$.

Exercise 5. Let $(E, \|\cdot\|)$ be a reflexive space and B_E be its unit ball. Show that for all $\varphi \in E^*$, there exists $x_\varphi \in B_E$, such that $\|\varphi\|_{E^*} = |\varphi(x_\varphi)|$, i.e. the supremum in the definition of the norm operator is in fact a maximum.

Exercise 6. The aim of this exercise is to prove by two different methods that the space $(C^0([0,1]), \|\cdot\|_\infty)$ of continuous real-valued functions on $[0,1]$ is not reflexive.

1. Method by compactness.
   (a) Define $\varphi \in C^0([0,1])^*$ by
   $$\varphi(f) = \int_0^{1/2} f(t)\,\mathrm{d}t - \int_{1/2}^1 f(t)\,\mathrm{d}t, \quad f \in C^0([0,1]),$$
   and show that $\|\varphi\| = 1$.
   (b) Prove that $|\varphi(f)| < 1$ for all $f \in C^0([0,1])$ such that $\|f\|_\infty \leq 1$.
   (c) Conclude that the space $C^0([0,1])$ is not reflexive.
2. Method by separability.
   (a) Prove that if E is a Banach space and its dual E^* is separable, then E is separable.
   (b) Show that $C([0,1])$ is separable.
   (c) Prove that $C([0,1])^*$ is not separable.
   Hint: Consider the functions $\delta_t : C^0([0,1]) \to \mathbb{R}$ *defined by* $\delta_t(f) = f(t)$ *for any* $t \in [0,1]$.
   (d) Conclude that $C^0([0,1])$ is not isomorphic to $C^0([0,1])^{**}$ as Banach spaces.
   Remark: This is stronger than not being reflexive.

Exercise 7.

1. Let E be a reflexive, separable Banach space. Let $(u_n)_n$ be a bounded sequence in E. Show that one can extract a subsequence $(u_{n'})_{n'}$ which converges weakly in E.
2. Does this result hold when E is not reflexive ?

Exercise 8. Let E be a reflexive Banach space and $I : E \to \mathbb{R}$ be a continuous, convex and coercive functional, in the sense that there exist $\alpha > 0$ and $M \geq 0$ such that for all $x \in E$,
$$I(x) \geq \alpha \|x\|_E - M.$$
We also consider $A \subset E$ a non-empty, closed and convex set. Prove that the functional I admits a minimum on A.

Exercise 9. Let B denote the closed unit ball of $L^1([0,1])$. Recall that a function $f \in B$ is called an extreme point if, whenever $f = \theta f_1 + (1-\theta) f_2$ with $\theta \in (0,1)$ and $f_1, f_2 \in B$, one has $f_1 = f_2$. Prove that B does not admit extremal points. Deduce that there is no isometry between $L^1([0,1])$ and the topological dual of a normed vector space.
Hint: We admit Krein-Milman's theorem, stating that any non-empty convex compact subset of any l.c.t.v.s coincides with the closed convex envelop of its extremal points.

École Normale Supérieure de Lyon **Année 2022 - 2023**

Département de Mathématiques **M1 - Analyse avancée**

TD 6: Compactness in L^p spaces

Exercise 1 (Equi-integrability). Let $(X, \mathcal{A}, \mu)$ be a measured space and $\mathcal{F} \subset L^1(X)$ being bounded. Prove that the following assertions are equivalent:

1. For all $\varepsilon > 0$, there exists some $M > 0$ such that
$$\sup_{f \in \mathcal{F}} \int_{\{|f|>M\}} |f| \, \mathrm{d}\mu < \varepsilon.$$
2. For all $\varepsilon > 0$, there exists some $\eta > 0$ such that for any measurable set A,
$$\mu(A) < \eta \Rightarrow \sup_{f \in \mathcal{F}} \int_A |f| \, \mathrm{d}\mu < \varepsilon.$$
3. There exists an increasing function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{x \to \infty} \Phi(x)/x = \infty$ and
$$\sup_{f \in \mathcal{F}} \int_X \Phi(|f|) \, \mathrm{d}\mu < \infty.$$

When one of the above conditions is satisfied, the set $\mathcal{F}$ is said to be *equi-integrable*.
Hint: to show 2. ⇒ 3., consider the sequence $(M_n)_n$ *such that*
$$\sup_{f \in \mathcal{F}} \int_X |f| \mathbb{1}_{|u|>M_n} \, \mathrm{d}\mu < 2^{-n}.$$

In the following two exercices, the notion of equi-integrability introduced in the previous exercice will be considered. When $p \in [1, +\infty)$, a set $\mathcal{F} \subset L^p(X)$ will be said to be equi-integrable when the set $\{|f|^p : f \in \mathcal{F}\}$ is equi-integrable in $L^1(X)$.

Exercise 2 (Vitali's convergence theorem). We consider $(X, \mathcal{A}, \mu)$ a σ-finite measure space. Let $p \in [1, +\infty)$ and $(f_n)_n$ be a sequence in $L^p(X)$. Assume that

1. The sequence $(f_n)_n$ is a Cauchy sequence in measure, meaning that for all $\varepsilon > 0$, there exists $n_0 \geq 0$ such that
$$\forall m, n \geq n_0, \quad \mu\big(|f_n - f_m| \geq \varepsilon\big) < \varepsilon.$$
2. The sequence $(f_n)_n$ is equi-integrable in $L^p(X)$,
3. For all $\varepsilon > 0$, there exists a measurable set $\Gamma \subset X$ of finite measure such that
$$\forall n \geq 0, \quad \|f_n \mathbb{1}_{X \setminus \Gamma}\|_{L^p(X)} \leq \varepsilon.$$

Prove that $(u_n)_n$ is a Cauchy sequence in $L^p(X)$ (and therefore converges in this space).

Exercise 3 (Dunford-Pettis' Theorem). The objective of the exercise is to prove Dunford-Pettis' theorem:

> Let $\Omega \subset \mathbb{R}^d$ be a bounded set and $(f_n)_n$ be a bounded sequence in $L^1(\Omega)$. Then, the set $\{f_n\}$ is sequentially compact for the weak topology $\sigma(L^1, L^\infty)$ if and only if the sequence $(f_n)_n$ is equi-integrable.

First we prove the reciprocal: let $(f_n)_n$ be a bounded and equi-integrable sequence in $L^1(\Omega)$.

1. Show that we can reduce to the case where the f_n are non-negative.
2. Let $f_n^k = \mathbb{1}_{f_n \le k} f_n$. Show that $\sup_n \|f_n - f_n^k\|_{L^1} \to 0$.
3. Show that there exists an extraction (n') such that for all $k \in \mathbb{N}$, $f_{n'}^k \rightharpoonup f^k$ in $L^1(\Omega)$.
4. Prove that $(f^k)_k$ is an increasing sequence and deduce that there exists some $f \in L^1(\Omega)$ such that $f^k \to f$ in $L^1(\Omega)$.
5. Conclude that $f_{n'} \rightharpoonup f$ in $L^1(\Omega)$.

Now we want to prove the direct implication. Let $(f_n)_n$ be a bounded sequence in $L^1(\Omega)$ satisfying $f_n \rightharpoonup f \in L^1(\Omega)$. We consider $\mathcal{X}$ the set of indicator functions and, for a fixed $\varepsilon > 0$, we also consider the sets X_n defined for all $n \ge 0$ by:

$$X_n := \left\{ \mathbb{1}_A \in \mathcal{X} : \forall k \ge n, \ \left| \int_A (f_k - f) \, \mathrm{d}x \right| \le \varepsilon \right\}.$$

6. Show that $\mathcal{X}$ and X_n are closed in $L^1(\Omega)$.
7. Using a Baire's argument, show that the sequence $(f_n)_n$ is equi-integrable.
8. Conclude.

TD 7: Compact operators

Exercise 1. Let H be a Hilbert space.

1. Prove that $\mathcal{K}(H)$ is closed in $\mathcal{L}(H)$.
2. Let $T \in \mathcal{K}(H)$ and $S \in \mathcal{L}(H)$. Prove that the operators TS and ST are also compact.

Exercise 2. Let H be a Hilbert space and $T : H \to H$ be linear and continuous.

1. Prove that the following assertions are equivalent
   - (i) T is compact.
   - (ii) For any sequence $(x_n)_n$ that weakly converges in H, the sequence $(Tx_n)_n$ strongly converges in H.
   - (iii) T is the limit in $\mathcal{L}(H)$ of finite rank operators.
2. We now assume that H is infinite-dimensional and that T is compact. Is the operator T right or left invertible ?
   Application: Study the compactness of the shift operator $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ defined for all $x = (x_n)_n \in l^2(\mathbb{N})$ by $(Tx)_0 = 0$ and $(Tx)_n = x_{n-1}$ for all $n \geq 1$.
3. When T is compact, prove that T^* is also a compact operator.

Exercise 3. Let $(\lambda_n)_n$ a sequence of complex numbers, and $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the operator defined by

$$T((x_n)_n) = (\lambda_n x_n)_n, \quad (x_n)_n \in l^2(\mathbb{N}).$$

1. Check that the operator T is well-defined and bounded on $l^2(\mathbb{N})$ if and only if the sequence $(\lambda_n)_n$ is bounded.
2. Prove that the operator T is compact if and only if the sequence $(\lambda_n)_n$ converges to 0.

Exercise 4. Let $T : L^2(0,1) \to L^2(0,1)$ be the Volterra operator defined by

$$(Tf)(x) = \int_0^x f(y)\,\mathrm{d}y, \quad f \in L^2(0,1),\ x \in [0,1].$$

1. Check that $Tf \in C^0[0,1]$ for all $f \in L^2(0,1)$.
2. Prove that the operator T is compact.
3. Compute the adjoint of the operator T.
4. Deduce that the operator TT^* is the following
$$(TT^*f)(x) = \int_0^1 \min(x,y) f(y)\,\mathrm{d}y, \quad f \in L^2(0,1),\ x \in [0,1].$$
5. Justify that the operator TT^* is compact, selfadjoint and non-negative.
6. Prove that the set of eigenvalues of the operator T^*T is given by
$$\left\{ \frac{1}{(n\pi + \pi/2)^2} : n \geq 0 \right\}.$$
7. Deduce the value of the norm of the operator T.

TD 8: Distributions

Exercise 1. Let $f : \mathbb{R} \to \mathbb{R}_+$ be a non-negative C^∞ function. Assume that the function f'' is bounded.

1. Prove that the following pointwize estimation holds
$$\forall x \in \mathbb{R}, \quad |f'(x)|^2 \leq 2f(x)\|f''\|_{L^\infty(\mathbb{R})}.$$
2. Can we have an estimate of the following form
$$\forall x \in \mathbb{R}, \quad |f'(x)| \leq c_f f(x),$$
where the positive constant $c_f > 0$ depends on the function f ?

Exercise 2.

1. Let H be the Heaviside function. Show that $H' = \delta_0$ in $\mathcal{D}'(\mathbb{R})$.
2. Give an example of distribution of order n for all $n \in \mathbb{N}$.
3. Let $\Omega \subset \mathbb{R}^d$ be an open set and $T \in \mathcal{D}'(\Omega)$. We consider $f \in C^\infty(\Omega)$ which vanishes on the support of T. Do we have $fT = 0$ in $\mathcal{D}'(\Omega)$?

Exercise 3. Let $\Omega \subset \mathbb{R}^d$ be an open set. Prove that we have an injection of $L^1_{loc}(\Omega)$ in $\mathcal{D}'(\Omega)$.

Exercise 4 (An example of distribution). Show that the formula
$$\langle T, \varphi \rangle = \sum_{n \geq 0} \varphi^{(n)}(n), \quad \varphi \in \mathcal{D}(\mathbb{R}),$$
defines a distribution $T \in \mathcal{D}'(\mathbb{R})$. What about its order ?

Exercise 5 (Convergence of distributions). Do the following series
$$\sum_{n \geq 0} \delta_n^{(n)} \quad \text{and} \quad \sum_{n \geq 0} \delta_0^{(n)},$$
converge in $\mathcal{D}'(\mathbb{R})$?

Exercise 6 (Principal value of $1/x$). We define $\mathrm{p.v.}(1/x)$ as follows
$$\langle \mathrm{p.v.}(1/x), \varphi \rangle = \lim_{\varepsilon \to 0} \left(\int_{|x|>\varepsilon} \frac{\varphi(x)}{x} \,\mathrm{d}x \right), \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

1. Show that the above limit exists and defines a distribution. Compute its order.
2. Show that $\mathrm{p.v.}(1/x)$ is the derivative of $\log|x|$ in the sense of distributions.
3. Compute $x\,\mathrm{p.v.}(1/x)$.

4. Let $T \in \mathcal{D}'(\mathbb{R})$ which satisfies $xT = 1$. Show that there exists a constant $c \in \mathbb{R}$ such that $T = \mathrm{p.v.}(1/x) + c\,\delta_0$.
5. Show that $|x|^{\alpha-2}x \to \mathrm{p.v.}(1/x)$ in $\mathcal{D}'(\mathbb{R})$ as $\alpha \to 0^+$.

Exercise 7. Solve the equation $T' = 0$ in $\mathcal{D}'(\mathbb{R})$.

Exercise 8 (Jump formula). Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class C^1 on $\mathbb{R}^*$. We say that f has a jump at 0 if the limits $f(0^{\pm}) = \lim_{x\to 0^{\pm}} f(x)$ exist, and we denote by $[[f(0)]] = f(0^+) - f(0^-)$ the height of the jump. We denote by $\{f'\}$ the derivative of the regular part of f, *i.e.*

$$\{f'\}(x) = \begin{cases} f'(x) & \text{if } f \text{ is differentiable at } x \\ 0 & \text{otherwise} \end{cases}$$

1. Show that in the sense of distributions:

$$f' = \{f'\} + [[f(0)]]\delta_0.$$

2. Let $(x_n)_{n\in\mathbb{Z}}$ be an increasing sequence such that $\lim_{n\to-\infty} x_n = -\infty$ and $\lim_{n\to+\infty} x_n = +\infty$. Let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise C^1 function presenting jumps at every x_n. Show that in the sense of distributions,

$$f' = \{f'\} + \sum_{n\in\mathbb{Z}} [[f(x_n)]]\delta_{x_n}.$$

Exercise 9 (Punctual support). Let $T \in \mathcal{D}'(\mathbb{R}^d)$ such that $\operatorname{supp} T = \{0\}$. We consider $\psi \in \mathcal{D}(\mathbb{R}^d)$ such that $\psi = 1$ in a neighborhood of $\overline{B(0,1)}$ and $\operatorname{supp}\psi \subset B(0,2)$. We set $\psi_r(x) = \psi(x/r)$ for all $r > 0$ and $x \in \mathbb{R}^n$.

1. Recall why T has a finite order, which will be denoted $m \geq 0$ in the following.
2. Show that for all $r > 0$, $\psi_r T = T$.
3. Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ satisfying that for all $p \in \mathbb{N}^n$ with $|p| \leq m$, $\partial^p\varphi(0) = 0$. Check that $\langle T, \varphi\rangle = 0$.
4. Prove that there exist some real numbers $a_p \in \mathbb{R}$ such that $T = \sum_{|p|\leq m} a_p \delta_0^{(p)}$.

Exercise 10 (Support and order). Let T be the linear map defined for all $\varphi \in \mathcal{D}(\mathbb{R})$ by

$$\langle T, \varphi\rangle = \lim_{n\to+\infty} \left(\sum_{j=1}^{n} \varphi\left(\frac{1}{j}\right) - n\varphi(0) - (\log n)\varphi'(0) \right).$$

1. Check that $\langle T, \varphi\rangle$ is well defined for all $\varphi \in \mathcal{D}(\mathbb{R})$, and that T is a distribution of order less than or equal to 2.
2. What is the support S of T ?
3. What is the order of T ?
Hint: Use test functions of the form

$$\varphi_k(x) = \psi(x) \int_0^x \int_0^y \varphi(kt)\,\mathrm{d}t\mathrm{d}y,$$

where $\varphi \in \mathcal{D}(0,1)$ *has integral* 1 *and* $\psi \in \mathcal{D}(-1,2)$ *satisfies* $0 \leq \psi \leq 1$ *and* $\psi = 1$ *on* $[0,1]$.

TD 9: Distributions (II)

Exercise 1. Let $\rho \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \leq \rho \leq 1$, $\operatorname{supp}\varphi = \{x \in \mathbb{R}^n : |x| \leq 1\}$ and $\int_{\mathbb{R}^n} \rho = 1$. For all $\varepsilon > 0$, we set $\rho_\varepsilon(x) = \varepsilon^{-n}\rho(x/\varepsilon)$.

1. Prove that for all $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\sup_{x\in\mathbb{R}^n} |(\rho_\varepsilon * \varphi)(x) - \varphi(x)| \underset{\varepsilon\to 0^+}{\to} 0.$$

2. Check that for all $f \in L^p(\mathbb{R}^n)$, $\lim_{\varepsilon\to 0^+} \|\rho_\varepsilon * f - f\|_{L^p(\mathbb{R}^n)} = 0$.

Exercise 2. Let Ω be an open subset of $\mathbb{R}^n$.

1. Let $\varphi \in C^\infty(\Omega \times \mathbb{R}^n)$ and $T \in \mathscr{D}'(\mathbb{R}^n)$. Assume that there exists a compact $K \subset \Omega$ such that

$$\forall y \in \mathbb{R}^n, \quad \operatorname{supp}(\varphi(\cdot, y)) \subset K.$$

Prove then that the function $y \in \mathbb{R}^n \mapsto T(\varphi(\cdot, y))$ is in $C^\infty(\mathbb{R}^n)$, with moreover

$$\forall \alpha \in \mathbb{N}^n, \quad \partial_y^\alpha(T(\varphi(\cdot, y)) = T(\partial_y^\alpha \varphi(\cdot, y)).$$

2. Let $\varphi \in C_0^\infty(\Omega \times \mathbb{R}^n)$ and $T \in \mathscr{D}'(\Omega)$. Prove that

$$\int_{\mathbb{R}^n} T(\varphi(\cdot, y))\,\mathrm{d}y = T\bigg(\int_{\mathbb{R}^n} \varphi(\cdot, y)\,\mathrm{d}y\bigg).$$

Exercise 3.

1. Let $\theta \in C_0^\infty(\mathbb{R})$ such that $\theta(0) = 1$. For all $\varphi \in C_0^\infty(\mathbb{R})$, prove that there exists $\psi \in C_0^\infty(\mathbb{R})$ such that

$$\forall x \in \mathbb{R}, \quad \varphi(x) - \varphi(0)\theta(x) = x\psi(x).$$

2. Solve $xT = 0$ in $\mathscr{D}'(\mathbb{R})$.
3. Solve $xT = 1$ in $\mathscr{D}'(\mathbb{R})$.
4. Solve $(x - 1)T = \delta_0$ and $(x - a)(x - b)T = 1$ with $a \neq b$ in $\mathscr{D}'(\mathbb{R})$.

Exercise 4. For all $x \in \mathbb{R}$ and $\varepsilon > 0$, we set

$$f_\varepsilon(x) = \log(x + i\varepsilon) = \log|x + i\varepsilon| + i\operatorname{Arg}(x + i\varepsilon),$$

the argument being taken in $(-\pi, \pi)$.

1. Prove that as ε goes to zero, the sequence (f_ε) converges in $\mathscr{D}'(\mathbb{R})$ to the locally integrable function $f_0 \in L^1_{loc}(\mathbb{R})$ defined by

$$f_0(x) = \begin{cases} \log(x) & \text{when } x > 0, \\ \log|x| + i\pi & \text{when } x < 0. \end{cases}$$

2. Compute f_0' in $\mathscr{D}'(\mathbb{R})$.
3. Deduce that the following equality holds in $\mathscr{D}'(\mathbb{R})$

$$\frac{1}{x+i0} := \lim_{\varepsilon\to 0^+} \frac{1}{x+i\varepsilon} = -i\pi\delta_0 + \mathrm{p.\,v.}(1/x).$$

4. Show similarly that

$$\frac{1}{x-i0} := \lim_{\varepsilon\to 0^+} \frac{1}{x-i\varepsilon} = i\pi\delta_0 + \mathrm{p.\,v.}(1/x).$$

Exercise 5.

1. What can be said about a distribution $T \in \mathscr{D}'(\mathbb{R})$ which satisfies $T' \in C^0(\mathbb{R})$?
2. Same question with a distribution $T \in \mathscr{D}'(\mathbb{R})$ such that $T^{(n)} = 0$ for some integer $n \in \mathbb{N}$.
3. Let Ω be a measurable subset of $\mathbb{R}^n$, $p \in [1, +\infty)$ and B_p be the unit ball of $L^p(\Omega)$. Prove that if a distribution $T \in \mathscr{D}'(\mathbb{R}^n)$ is bounded on $B_p \cap \mathcal{D}(\Omega)$, then $T \in L^q(\Omega)$, where $q \in (1, +\infty]$ satisfies $1/p + 1/q = 1$.

Exercise 6.

1. Let $T \in \mathscr{D}'(\mathbb{R})$ and $f \in L^1_{loc}(\mathbb{R})$. For all $c \in \mathbb{R}$, we set

$$F_c(x) = c + \int_0^x f(t)\,\mathrm{d}t, \quad x \in \mathbb{R}.$$

   Prove that $T' = f$ if and only if there exists $c \in \mathbb{R}$ such that $T = F_c$.
2. Check that for all $T \in \mathscr{D}'(\mathbb{R})$, the following convergence holds in $\mathscr{D}'(\mathbb{R})$

$$\frac{\tau_{-h}T - T}{h} \underset{h\to 0}{\to} T',$$

   where τ_{-h} denotes the translation operator.
3. Prove that a distribution $T \in \mathscr{D}'(\mathbb{R})$ is a Lipschitz function if and only if $T' \in L^\infty(\mathbb{R})$. *Hint: Use the question 3 of the previous exercice.*

Exercise 7. Let $E_n \in L^1_{loc}(\mathbb{R}^n)$ be the function defined by

$$E_n(x) = \begin{cases} \log(|x|) & \text{when } n = 2, \\ |x|^{2-n} & \text{when } n \geq 3. \end{cases}$$

1. Let $u \in C^2(\mathbb{R}^n \setminus \{0\})$ be a radial function, i.e. $u(x) = U(|x|)$ where $U \in C^2(\mathbb{R}^*)$. Prove that

$$\forall x \in \mathbb{R}^n \setminus \{0\}, \quad (\Delta u)(x) = U''(|x|) + \frac{n-1}{|x|}U'(|x|).$$

2. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$. Justify that

$$(\Delta E_n)(\varphi) = \lim_{\varepsilon\to 0^+} \int_{\Omega_\varepsilon} E_n(x)(\Delta\varphi)(x)\,\mathrm{d}x,$$

   where $\Omega_\varepsilon = \{x \in \mathbb{R}^n : |x| > \varepsilon\}$. By using Green's formula, conclude then that there exists a constant $c_n \in \mathbb{R}$ such that $\Delta E_n = c_n\delta_0$ in $\mathscr{D}'(\mathbb{R}^n)$

TD 10: Fourier transform and tempered distributions

Exercise 1. Let $A \in S_n^{++}(\mathbb{R})$ be a definite positive real matrix. Prove that the function u defined on $\mathbb{R}^n$ by $u(x) = e^{-\langle Ax,x\rangle}$ belongs to the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ and that its Fourier transform is given by

$$\forall \xi \in \mathbb{R}^n, \quad \widehat{u}(\xi) = \sqrt{\frac{\pi^n}{\det A}}\, e^{-\frac{1}{4}\langle A^{-1}\xi,\xi\rangle}.$$

Application: Compute the Fourier transform of the following Gaussian function

$$f_\varepsilon(x) = e^{-\varepsilon|x|^2}, \quad \varepsilon > 0,\, x \in \mathbb{R}^d.$$

Hint: Begin by considering the case $n = 1$, and diagonalize the matrix A to treat the general case.

Exercise 2.

1. Let $A \subset \mathbb{R}^n$ be a measurable subset with finite measure. Prove that $\widehat{\mathbb{1}_A}$ belongs to $L^2(\mathbb{R}^n)$ but not to $L^1(\mathbb{R}^n)$.
2. Are there two functions $f, g \in \mathscr{S}(\mathbb{R}^n)$ not being identically equal to zero and satisfying the relation $f * g = 0$? Same question for some functions f et g with compact supports.
3. Prove that the equation $f * f = f$ has no non trivial solution in $L^1(\mathbb{R}^n)$, but has an infinite number of solutions in $L^2(\mathbb{R}^n)$.

Exercise 3. By computing the Fourier transform of the functions $f = \mathbb{1}_{[-1/2,1/2]}$ and $f * f$, show that

$$\int_{\mathbb{R}} \left(\frac{\sin t}{t}\right)^2 \mathrm{d}t = \pi.$$

Exercise 4. Let $I \subset \mathbb{R}$ be an interval and ρ be a weight function, meaning that ρ is measurable, positive, and satisfies

$$\forall n \in \mathbb{N}, \quad \int_I |x|^n \rho(x)\ \mathrm{d}x < +\infty.$$

Assume that there exists $a > 0$ such that

$$\int_I e^{a|x|}\rho(x)\,\mathrm{d}x < +\infty.$$

Let us denote by $L^2(I,\rho)$ the space of square integrable functions with respect to the measure $\rho\,\mathrm{d}x$.

1. Prove that there exists an orthonormal family of polynomials $(P_n)_{n\geq 0}$ such that $\deg P_n = n$ for all $n \geq 0$.

The aim is now to prove that $(P_n)_{n\geq 0}$ is a Hilbert basis of $L^2(I,\rho)$.

2. Let $f \in L^2(I,\rho)$. Check that the function φ defined by

$$\varphi(x) = \begin{cases} f(x)\rho(x) & \text{if } x \in I, \\ 0 & \text{if } x \notin I, \end{cases}$$

belongs to $L^1(\mathbb{R})$. Prove that its Fourier transform $\widehat{\varphi}$ can be extended to an holomorphic function F on the strip

$$B_a = \big\{z \in \mathbb{C} : |\operatorname{Im} z| < a/2\big\}.$$

3. Assume that the function $f \in L^2(I,\rho)$ is orthogonal to any monomial. By computing the derivatives of the function F at 0, prove that f is identically equal to zero and conclude.

Exercise 5 (Heisenberg's uncertainty principle). Prove that for all $f \in \mathscr{S}(\mathbb{R}^n)$ and $j \in \{1,\dots,n\}$,

$$\inf_{a\in\mathbb{R}} \big\|(x_j - a)f\big\|^2_{L^2(\mathbb{R}^n)} \inf_{b\in\mathbb{R}} \big\|(\xi_j - b)\widehat{f}\big\|^2_{L^2(\mathbb{R}^n)} \geq \frac{(2\pi)^n}{4}\|f\|^2_{L^2(\mathbb{R}^n)},$$

When is this inequality an equality ?

Exercise 6. Let us consider the interval $I = [-1,1]$ and the following subspace of $L^2(I)$

$$\mathrm{BL}^2(I) = \big\{u \in L^2(\mathbb{R}) : \widehat{u} = 0 \text{ almost everywhere on } \mathbb{R}\setminus I\big\}.$$

1. Prove that $\mathrm{BL}^2(I)$ is a Hilbert space.
2. Check that $\mathrm{BL}^2(I) \subset C^0_{\to 0}(\mathbb{R})$ and that the corresponding embedding is continuous.
3. Let us consider the continuous extension of $x \mapsto \sin x/x$, denoted sinc.
   (a) Prove that the family $(\pi^{-1/2}\tau_{2\pi k}\,\mathrm{sinc})_{k\in\mathbb{Z}}$ is a Hilbert basis of $\mathrm{BL}^2(I)$.
   (b) Prove (sampling theorem) that any element $u \in \mathrm{BL}^2(I)$ can be decomposed as follows

$$u(x) = \sum_{k\in\mathbb{Z}} u(2\pi k)\,\mathrm{sinc}(x - 2\pi k),$$

   the convergence being uniform in $\mathbb{R}$, and also holds in $L^2(\mathbb{R})$.

Exercise 7. Prove that the following distributions are tempered and compute their Fourier transform:

1. δ_0
2. 1,
3. H (Heaviside),
4. $\mathrm{p.v.}(1/x)$,
5. $|x|$ in $\mathbb{R}$.

Indication : $\mathrm{p.v.}(1/x)$ *is an odd distribution, so its Fourier transform is also odd.*

Exercise 8. The aim of this exercice is to compute the Fourier transform of the following tempered distribution on $\mathbb{R}^2$

$$\langle T, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \int_{\mathbb{R}} \varphi(x,x)\,\mathrm{d}x, \quad \varphi \in \mathscr{S}(\mathbb{R}^2).$$

1. Let $\psi \in \mathscr{S}(\mathbb{R}^2)$. Prove that

$$\langle \widehat{T}, \psi\rangle_{\mathscr{S}',\mathscr{S}} = \lim_{\varepsilon\to 0^+} I_\varepsilon \quad \text{où} \quad I_\varepsilon = \int_{\mathbb{R}} e^{-\varepsilon x^2}\widehat{\psi}(x,x)\,\mathrm{d}x.$$

2. By using the expression of $\widehat{\psi}(x, x)$, show that

$$I_\varepsilon = 2\sqrt{\pi} \int_{\mathbb{R}^2} e^{-\zeta^2} \psi(\xi, 2\sqrt{\varepsilon}\zeta - \xi) \,\mathrm{d}\xi \,\mathrm{d}\zeta.$$

3. Deduce the expression of $\widehat{T}$.

Exercise 9. Given some real number $s \in \mathbb{R}$, we define the Sobolev space $H^s(\mathbb{R}^d)$ by

$$H^s(\mathbb{R}^d) = \big\{ u \in \mathscr{S}'(\mathbb{R}^d) : \langle \xi \rangle^s \widehat{u} \in L^2(\mathbb{R}^d) \big\},$$

equipped with the following scalar product

$$\langle u, v \rangle_{H^s} = \int_{\mathbb{R}^d} \langle \xi \rangle^{2s} \widehat{u}(\xi) \overline{\widehat{v}(\xi)} \,\mathrm{d}\xi, \quad u, v \in \ H^s(\mathbb{R}^d).$$

1. Show that $H^{s_1}(\mathbb{R}^d)$ embeds continuously into $H^{s_2}(\mathbb{R}^d)$ for $s_1 \geq s_2$.
2. Check that $\delta_0 \in H^s(\mathbb{R}^d)$ for $s < -d/2$.
3. When $s \in \mathbb{N}^*$ is a nonnegative integer, the Sobolev space is also given by

$$H^s(\mathbb{R}^d) = \big\{ u \in L^2(\mathbb{R}^d) : \forall |\alpha| \leq s,\, \partial^\alpha u \in L^2(\mathbb{R}^d) \big\}.$$

4. Prove that there exists a positive constant $c > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}^3)$,

$$\|u\|_{L^\infty(\mathbb{R}^3)} \leq c \, \|u\|_{H^1(\mathbb{R}^3)}^{1/2} \|u\|_{H^2(\mathbb{R}^3)}^{1/2}.$$

Hint: Considering $R > 0$, use the following decomposition

$$\|\widehat{u}\|_{L^1(\mathbb{R}^3)} = \int_{|\xi| \leq R} \langle \xi \rangle |\widehat{u}(\xi)| \frac{\mathrm{d}\xi}{\langle \xi \rangle} + \int_{|\xi| > R} \langle \xi \rangle^2 |\widehat{u}(\xi)| \frac{\mathrm{d}\xi}{\langle \xi \rangle^2}.$$

5. (a) Prove that if $s > d/2$, the space $H^s(\mathbb{R}^d)$ embeds continuously to $C^0_{\to 0}(\mathbb{R}^d)$, the space of continuous functions u on $\mathbb{R}^d$ satisfying $u(x) \to 0$ as $|x| \to +\infty$.
   (b) State an analogous result in the case where $s > d/2 + k$ for some $k \in \mathbb{N}$. Deduce that $\bigcap_{s \in \mathbb{R}} H^s(\mathbb{R}^d) \subset C^\infty(\mathbb{R}^d)$.

Exercise 10. Let us consider the function

$$\gamma_0 : \varphi(x', x_d) \in C_0^\infty(\mathbb{R}^d) \mapsto \varphi(x', x_d = 0) \in C_0^\infty(\mathbb{R}^{d-1}).$$

Prove that for all $s > 1/2$, the function γ_0 can be uniquely extended as an application mapping $H^s(\mathbb{R}^d)$ to $H^{s-1/2}(\mathbb{R}^{d-1})$.
Hint: For all $\varphi \in \ C_0^\infty(\mathbb{R}^d)$, begin by computing the Fourier transform of the function $\gamma_0 \phi$.

TD 11: Reviews

Exercise 1. We consider the vector space $E = C^\infty([0,1],\mathbb{R})$ equipped with the following metric

$$d(f,g) = \sum_{k\geq 0} \frac{1}{2^k} \min\big(1, \|f^{(k)} - g^{(k)}\|_\infty\big).$$

1. Check that E is a Fréchet space.
2. Prove that any closed and bounded subset of E is compact.
3. Can the topology of E be defined by a norm ?

Exercise 2. For all $n \geq 0$, we set e^n the sequence which every term is zero, except the n^{th} which is 1. Recall that $c_0(\mathbb{N})$ denotes the subspace of $l^\infty(\mathbb{N})$ of sequences that converge to zero. Let

$$S = \left\{\varphi \in c_0(\mathbb{N})^* : \sum_{n=0}^{+\infty} \varphi(e^n) = 0\right\}.$$

1. Justify that S is well-defined and show that S is strongly closed in $c_0(\mathbb{N})^*$.
2. Show that S is weakly closed in $c_0(\mathbb{N})^*$, i.e. closed for the $\sigma(c_0(\mathbb{N})^*, c_0(\mathbb{N})^{**})$-topology.
3. Show that S is not weakly-$*$ closed in $c_0(\mathbb{N})^*$, i.e. not closed for the $\sigma(c_0(\mathbb{N})^*, c_0(\mathbb{N}))$-topology.

Exercise 3 (Banach limit).

1. Let $s : \ell^\infty(\mathbb{N}) \to \ell^\infty(\mathbb{N})$ be the shift operator, defined by $s(x)_i = x_{i+1}$ for all $i \in \mathbb{N}$ and $x \in \ell^\infty(\mathbb{N})$. Prove the existence of a continuous linear function $\Lambda \in (\ell^\infty(\mathbb{N}))'$ satisfying $\Lambda \circ s = \Lambda$ and
$$\forall u \in \ell^\infty(\mathbb{N}), \quad \liminf_{n\to+\infty} u_n \leq \Lambda(u) \leq \limsup_{n\to+\infty} u_n.$$
Such a linear form Λ is called Banach limit.
Hint: Consider the vector space of bounded sequences that converge in the sense of Cesàro.
2. Deduce that there exists a function $\mu : \mathcal{P}(\mathbb{N}) \to \mathbb{R}_+$ which satisfies
   - (i) $\mu(\mathbb{N}) = 1$,
   - (ii) μ is finitely additive: $\forall A, B \subset \mathbb{N}$ with $A \cap B = \emptyset$, $\mu(A\cup B) = \mu(A) + \mu(B)$,
   - (iii) μ is left-invariant: $\forall k \in \mathbb{N}$ and $A \subset \mathbb{N}$, $\mu(k + A) = \mu(A)$.

Exercise 4. Let H be a real Hilbert space and $J : H \to \mathbb{R}$ be a continuous convex functional. We assume that J is coercive, that is, $J(x) \to +\infty$ when $\|x\| \to +\infty$. Prove then that there exists $x_\star$ in H such that $J(x_\star) = \inf_{x\in H} J(x)$.

Exercise 5. Let $T : L^2[0,1] \to L^2[0,1]$ be the operator defined by

$$(Tf)(x) = \int_0^1 e^{-|x-y|} f(y)\,\mathrm{d}y.$$

1. Prove that T is well-defined, selfadjoint, compact and that $\|T\| \leq 1$.
2. Let $g = Tf$, where $f \in C^0[0,1]$. Check that g is in $C^2[0,1]$ and satisfies

$$g'' - g = -2f, \quad g(0) = g'(0), \quad g(1) = -g'(1).$$

3. Reciprocally, let $g \in C^2[0,1]$ satisfying $g(0) = g'(0)$ and $g(1) = g'(1)$. We set $f = (g - g'')/2$. Check that $g = Tf$.
4. Prove that $\operatorname{Im} T$ is dense in $L^2[0,1]$. Is 0 an eigenvalue of T ?
5. Let $f \in C^0[0,1]$ and $g = Tf$. Check that

$$2\langle Tf, f\rangle_{L^2} = |g(0)|^2 + |g(1)|^2 + \int_0^1 |g(x)|^2 \, \mathrm{d}x + \int_0^1 |g'(x)|^2 \, \mathrm{d}x.$$

   Deduce that $2\langle Tf, f\rangle_{L^2} \geq \|Tf\|_{L^2}^2$.
6. Prove that $\sigma(T) \subset [0,1]$.
7. For all $\lambda \in (0,1]$, we set $a_\lambda = \sqrt{(2-\lambda)/\lambda}$. Check that

$$\lambda \in \sigma(T) \cap (0,1] \iff (1 - a_\lambda^2)\sin a_\lambda + 2a_\lambda \cos a_\lambda = 0.$$

8. Deduce that $\sigma(T) = \{0\} \cup \{\lambda_n : n \geq 0\}$, with

$$\frac{2}{1 + (\pi/2 + n\pi)^2} < \lambda_n < \frac{2}{1 + (n\pi)^2}.$$

Exercise 6. Prove that there is no distribution $T \in \mathscr{D}'(\mathbb{R})$ such that

$$T(\varphi) = \int_{\mathbb{R}} \exp\left(\frac{1}{x^2}\right)\varphi(x) \, \mathrm{d}x, \quad \varphi \in C_0^\infty(\mathbb{R} \setminus \{0\}).$$

Hint: Construct a sequence $(\varphi_n)_n$ converging to zero in $C_0^\infty(\mathbb{R})$ such that each φ_n is supported in $\{1/n \leq |x| \leq 2/n\}$ and $(T(\varphi_n))_n$ converges to $+\infty$.