
TD 1: Topology issues in product spaces and Banach spaces

Exercise 1 (General topology).

1. Let $f : E \to F$ be an application between topological spaces. The function f is said to be continuous at $x \in E$ if for all open set $\mathcal{V}$ containing $f(x)$, there exists an open set $\mathcal{U}$ containing x and such that $f(\mathcal{U}) \subset \mathcal{V}$. Check that, in this definition, "open set" can be replaced by "neighbourhood".
2. Let X be a set, $(F_i)_{i\in I}$ be a family of topological spaces and $f_i : X \to F_i$ be some functions.
   (a) Prove that the "coarsest topology that makes the functions f_i continuous" exists.
   (b) Let $g : E \to X$ be a function defined on a topological space E. Check that g is continuous if and only if for all $i \in I$, $f_i \circ g$ is continuous.
   (c) Let $(x_n)_n$ be a sequence in X. Prove that $(x_n)_n$ converges to x if and only if for all $i \in I$, $(f_i(x_n))_n$ converges to $f_i(x)$.
3. Let $(F_i)_{i\in I}$ be a family of topological spaces. We define the product topology on $\prod_{i\in I} F_i$ as the "coarsest topology" making the projections continuous. Show that this topology is generated by the cylinder sets, *i.e.* the sets of the form

$$C_J = \prod_{i\in I} U_i,$$

   where each U_i is open in F_i and $U_i = F_i$, except for a finite number of indexes $i \in J$.

Exercise 2 (A theorem of Hörmander). Let $1 \leq p, q < \infty$ and

$$T : (L^p(\mathbb{R}^n), \|\cdot\|_p) \to (L^q(\mathbb{R}^n), \|\cdot\|_q),$$

be a continuous linear operator which commutes with the translations, that is, which satisfies $\tau_h T = T\tau_h$ for all $h \in \mathbb{R}^n$, where $\tau_h f = f(\cdot - h)$. The purpose of this exercice is to prove the following property: if $q < p < \infty$, then the operator T is trivial.

1. Let u be a function in $L^p(\mathbb{R}^n)$. Prove that $\|u + \tau_h u\|_p \to 2^{1/p}\|u\|_p$ as $\|h\| \to \infty$.
   Hint: you may decompose u as the sum of a compactly supported function and of a function with arbitrarily small L^p norm.
2. Check that if C stands for the norm of operator T, then we have that for all $u \in L^p(\mathbb{R}^n)$,

$$\|Tu\|_q \leq 2^{1/p-1/q} C\|u\|_p,$$

   and conclude.
3. Can you give the example of a non-trivial such operator T when $p \leq q$?

Exercise 3 (Fourier coefficients of L^1 functions). For any function f in $L^1(\mathbb{T})$, we define the function $\hat{f} : \mathbb{Z} \to \mathbb{C}$ by

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,\mathrm{d}t, \quad n \in \mathbb{Z}.$$

We denote by c_0 the space of complex valued functions on $\mathbb{Z}$ tending to 0 at $\pm\infty$.

1. Check that $(c_0, \|\cdot\|_\infty)$ is a Banach space.
2. Prove that, for all $f \in L^1(\mathbb{T})$, $\hat{f} \in c_0$.
   Hint: Recall that the trigonometric polynomials $\sum_{k=-n}^{n} a_k e^{ikt}$ are dense in $L^1(\mathbb{T})$.

Now we study the converse question: is every element of c_0 the sequence of Fourier coefficients of a function in $L^1(\mathbb{T})$?

2. Prove that $\Lambda : f \to \hat{f}$ defines a bounded linear map from $L^1(\mathbb{T})$ to c_0.
3. Prove that the function Λ is injective.
4. Show that the function Λ is not onto.
   Hint: You may use the Dirichlet kernel $D_n(t) = \sum_{k=-n}^{n} e^{ikt}$, whose $L^1(\mathbb{T})$ norm goes to $+\infty$ as $n \to +\infty$.

Exercise 4 (Equivalence of norms).

1. Let E be a vector space endowed with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ such that both $(E, \|\cdot\|_1)$ and $(E, \|\cdot\|_2)$ are Banach spaces. Assume the existence of a finite constant $C > 0$ such that
$$\forall x \in E, \quad \|x\|_1 \leqslant C\|x\|_2.$$
   Prove that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.
2. Let K be a compact subset of $\mathbb{R}^n$. We consider a norm N on the space $\mathcal{C}^0(K, \mathbb{R})$ such that $(\mathcal{C}^0(K, \mathbb{R}), N)$ is a Banach space, and satisfying that any sequence of functions $(f_n)_n$ in $\mathcal{C}^0(K, \mathbb{R})$ that converges for the norm N also converges pointwise to the same limit. Prove that the norm N is then equivalent to the norm $\|\cdot\|_\infty$.

Exercise 5 (A Rellich-like theorem). Let us consider E the following subspace of $L^2(\mathbb{R})$
$$E = \big\{u \in \mathcal{C}^1(\mathbb{R}) : \|u\|_E < +\infty\big\}, \quad \text{where} \quad \|u\|_E = \|(\sqrt{1+x^2})u\|_{L^2(\mathbb{R})} + \|u'\|_{L^2(\mathbb{R})}.$$
The aim of this exercice is to prove that the unit ball B_E of E is relatively compact in $L^2(\mathbb{R})$, with
$$B_E = \big\{u \in \mathcal{C}^1(\mathbb{R}) : \|u\|_E \le 1\big\}.$$
In the following, we denote by ϕ a non-negative $\mathcal{C}^\infty$ function such that $\phi^{-1}(\{0\}) = \mathbb{R} \setminus [-2, 2]$ and $\phi^{-1}(\{1\}) = [-1, 1]$.

1. Considering the cut-off $\phi_R(x) = \phi(x/R)$, show that $\sup_{u \in B_E} \|(1 - \phi_R)u\|_{L^2(\mathbb{R})}$ converges to 0 as $R \to +\infty$.
2. We define $\psi_\varepsilon(x) = \frac{1}{\varepsilon}\phi(\frac{x}{\varepsilon})$ and τ_h the translation operator (see Exercice 2). Show that for all $R \ge 1$ and $\varepsilon > 0$, there exists $C_{\varepsilon,R} > 0$ such that for all $h \in \mathbb{R}$ and $u \in E$,
$$\|\tau_h((\phi_R u) * \psi_\varepsilon) - (\phi_R u) * \psi_\varepsilon\|_{L^\infty(\mathbb{R})} \le C_{\varepsilon,R}|h|\|u\|_E \quad \text{and} \quad \|(\phi_R u) * \psi_\varepsilon\|_{L^\infty(\mathbb{R})} \le C_{\varepsilon,R}\|u\|_E.$$
3. Show that for any sequence $(u_n)_n$ in B_E, there exists a subsequence $(u_{n'})_{n'}$ such that for any $R, \varepsilon^{-1} \in \mathbb{N}^*$, the sequence $((\phi_R u_{n'}) * \psi_\varepsilon)_{n'}$ converges in $L^2(\mathbb{R})$ as $n' \to \infty$.
   Hint: Use Cantor's diagonal argument.
4. Conclude.
5. Let us now consider the set $B_{H^1} \subset L^2(\mathbb{R})$ defined by
$$B_{H^1} = \big\{u \in \mathcal{C}^1(\mathbb{R}) : \|u\|_{L^2(\mathbb{R})} + \|u'\|_{L^2(\mathbb{R})} \le 1\big\}.$$
   Is B_{H^1} relatively compact in $L^2(\mathbb{R})$?

TD 2: L^p compactness and Banach spaces

Exercise 1 (F. Riesz's theorem). Let E be a normed vector space.

1. Prove that if M is a closed subspace of E, with $M \neq E$, then for all $\varepsilon > 0$, there exists $u \in E$ of norm $\|u\| = 1$ such that $d(u, M) \geq 1 - \varepsilon$.
2. Deduce that if E is infinite-dimensional, then its unit ball $\mathcal{B}$ is not compact, with

$$\mathcal{B} = \big\{x \in E : \|x\| \leq 1\big\}.$$

Exercise 2 (Norm on the quotient space). Let E be a Banach space and M be a closed vector subspace of E. Let us consider $N : E/M \to \mathbb{R}$ defined by

$$N(\xi) = \inf_{\xi = \overline{x}} \|x\|.$$

Prove that N defines a norm on E/M, and that E/M is a Banach space.
Hint: Prove that if $(u_n)_n$ is a Cauchy sequence, then one can extract a subsequence $(n_k)_k$ such that

$$\forall k \geq 0, \quad \|u_{n_{k+1}} - u_{n_k}\| \leq \frac{1}{2^k}.$$

Exercise 3 (Characterization of equi-integrability). Let (X, μ) be a measured space and $\mathcal{F} \subset L^1(X, \mu)$ being bounded. Prove that the following assertions are equivalent:

1. $\mathcal{F}$ is equi-integrable,
2. For all $\varepsilon > 0$, there exists some $\eta > 0$ such that for any measurable set A,

$$\mu(A) < \eta \Rightarrow \sup_{u \in \mathcal{F}} \int_A |u| \, \mathrm{d}\mu < \varepsilon.$$

3. There exists an increasing function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{x\to\infty} \Phi(x)/x = \infty$ and

$$\sup_{u \in \mathcal{F}} \int_X \Phi(|u|) \, \mathrm{d}\mu < \infty.$$

Hint: to show 2. $\Rightarrow$ 3., consider the sequence $(M_n)_n$ such that

$$\sup_{u \in \mathcal{F}} \int_X |u| \mathbb{1}_{|u| > M_n} \, \mathrm{d}\mu < 2^{-n}.$$

Exercise 4 (Vitali's convergence theorem). We consider $(X, \mathcal{A}, \mu)$ a σ-finite measured space. Let $1 \leq p < +\infty$ and $(u_n)_n$ be a sequence in $L^p(X)$. Assume that

1. $(u_n)_n$ is a Cauchy sequence in measure, meaning that for all $\varepsilon > 0$, there exists $n_0 \geq 0$ such that

$$\forall m, n \geq n_0, \quad \mu\big(|u_n - u_m| \geq \varepsilon\big) < \varepsilon.$$

2. $(u_n)_n$ is equi-integrable in $L^p(X)$,
3. for all $\varepsilon > 0$, there exists a measurable set Γ of finite measure such that

$$\forall n \geq 0, \quad \|u_n \mathbb{1}_{X\setminus\Gamma}\|_{L^p(X)} \leq \varepsilon.$$

Prove that $(u_n)_n$ is a Cauchy sequence in $L^p(X)$ (and therefore converges in this space).

Exercise 5 (Obstructions to strong convergence). The purpose of this exercise is to present three obstructions to strong convergence in $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{T}^d)$. In the following, $\varphi \in C_c^\infty(\mathbb{R}^d)$ denotes a compactly supported smooth function being not identically equal to zero.

1. (Loss of mass) Let ν be a vector of norm 1. Prove that the sequence $(\varphi(\cdot - n\nu))_n$ does not converge in $L^2(\mathbb{R}^d)$.
2. (Concentration) Prove that the sequence $(n^{d/2}\varphi(n\,\cdot))_n$ does not converge in $L^2(\mathbb{R}^d)$.
3. (Oscillations) We now consider $w \in L^2(\mathbb{T}^d)$ a non-constant function. Prove that the sequence $(w(n\,\cdot))_n$ does not converge in $L^2(\mathbb{T}^d)$.

Exercise 6 (Averaging lemma). Let $u \in \mathscr{S}(\mathbb{R}^d_x \times \mathbb{R}^d_v)$ be a Schwartz function. For any function $\phi \in C_c^\infty(\mathbb{R}^d)$, we consider the moment

$$\rho_\phi(x) := \int_{\mathbb{R}^d} \phi(v) u(x,v)\,\mathrm{d}v.$$

1. Let us define $\hat{u}(\xi, v)$ as the Fourier transform of the function u with respect to the space variable $x \in \mathbb{R}^d$. Considering the function $w := (1 + v \cdot \nabla_x)u$, show that for all $\xi \in \mathbb{R}^d$,

$$|\hat{\rho}_\phi(\xi)|^2 \leq \bigg(\int_{\mathbb{R}^d} |\hat{w}|^2(\xi, v)\,\mathrm{d}v\bigg)\bigg(\int_{\mathbb{R}^d} \frac{\phi^2(v)\,\mathrm{d}v}{1 + |v\cdot\xi|^2}\bigg).$$

2. Deduce that

$$\|\rho_\phi\|^2_{H^{1/2}(\mathbb{R}^d)} := \int_{\mathbb{R}^d} (1 + |\xi|^2)^{1/2} |\hat{\rho}_\phi|^2(\xi)\,\mathrm{d}\xi \leq C_\phi\big(\|u\|^2_{L^2(\mathbb{R}^{2d})} + \|v\cdot\nabla_x u\|^2_{L^2(\mathbb{R}^{2d})}\big),$$

where the constant $C_\phi > 0$ only depends on the function ϕ.

TD 3: Hahn-Banach theorem and locally convex topological vector spaces

Exercise 1 (Towards duality). Let E be a normed vector space.

1. Let G be a vector subspace of E and $g : G \to \mathbb{R}$ be a continuous linear form. Show that there exists a continuous linear form f over E that extends g, and such that
$$\|f\|_{E^*} = \|g\|_{G^*}.$$
When E is an Hilbert space, prove that this extension is unique.
2. Assume that $E = \ell^1(\mathbb{N})$. Give the example of a continuous linear form of norm 1, defined on a strict vector subspace of E, which admits an infinite number of linear continuous extensions of norm 1 over E.
3. Assume that E is a Banach space. Let B be a subset of E such that
$$\forall f \in E^*, \quad \sup_{x \in B} f(x) < +\infty.$$
Prove that B is bounded.

Exercise 2 (Hahn-Banach theorems for complex spaces). Let E be a vector space over $\mathbb{C}$. Let M be a vector subspace of E and let $f : M \to \mathbb{C}$ be a $\mathbb{C}$-linear form. Suppose that there is a semi-norm $p : E \to [0, \infty)$ such that
$$\forall x \in M, \quad |f(x)| \leq p(x).$$
Prove that there there exists a linear form $F : E \to \mathbb{C}$ extending f, and such that $|F| \leq p$.

Exercise 3 (Hahn-Banach Theorem without the axiom of choice.). Let E be a real separable Banach space and p be a norm on E. Let M be a linear subspace of E and $\varphi : M \to \mathbb{R}$ be a linear functional which is dominated by p. Prove that φ can be extended to a linear functional $E \to \mathbb{R}$ which remains dominated by p.

Exercise 4 (Separation of convex sets in Hilbert spaces). Let H be an Hilbert space.

1. Let $C \subset H$ be a convex, closed and non-empty set. Prove that any $v \notin C$ can be strictly separated by C by a closed hyperplane, *i.e.* there exists $u_0 \in H$ such that
$$\forall u \in C, \quad \langle u_0, u \rangle < \langle u_0, v \rangle.$$
2. Let $C_1, C_2 \subset H$ be convex, closed and non-empty disjoint sets, C_1 being moreover compact. Prove that C_1 and C_2 can be strictly separated by a closed hyperplane, *i.e.* there exists $u_0 \in H$ such that
$$\sup_{u \in C_1} \langle u_0, u \rangle < \inf_{u \in C_2} \langle u_0, u \rangle.$$

Exercise 5 (Convex sets that cannot be separated). Let H be the Hilbert space $L^2([-1,1])$. For every $\alpha \in \mathbb{R}$, let $C_\alpha \subset H$ be the subset of continuous functions $u : [-1,1] \to \mathbb{R}$ such that $u(0) = \alpha$. Prove that C_α is a convex dense subset of H. Deduce that, if $\alpha \neq \beta$, then C_α and C_β are convex disjoint subsets that cannot be separated by a continuous linear form.

Exercise 6 (Banach limit).

1. Let $s : \ell^\infty(\mathbb{N}) \to \ell^\infty(\mathbb{N})$ be the shift operator, defined by $s(x)_i = x_{i+1}$ for all $i \in \mathbb{N}$ and $x \in \ell^\infty(\mathbb{N})$. Prove the existence of a continuous linear function $\Lambda \in (\ell^\infty(\mathbb{N}))'$ satisfying $\Lambda \circ s = \Lambda$ and
$$\forall u \in \ell^\infty(\mathbb{N}), \quad \liminf_{n\to+\infty} u_n \leq \Lambda(u) \leq \limsup_{n\to+\infty} u_n.$$
Such a linear form Λ is called Banach limit.
Hint: Consider the vector space of bounded sequences that converge in the sense of Cesàro.
2. Deduce that there exists a function $\mu : \mathcal{P}(\mathbb{N}) \to \mathbb{R}_+$ which satisfies
   - (i) $\mu(\mathbb{N}) = 1$,
   - (ii) μ is finitely additive: $\forall A, B \subset \mathbb{N}$ with $A \cap B = \emptyset$, $\mu(A \cup B) = \mu(A) + \mu(B)$,
   - (iii) μ is left-invariant: $\forall k \in \mathbb{N}$ and $A \subset \mathbb{N}$, $\mu(k + A) = \mu(A)$.

Exercise 7 (L^p spaces with $0 < p < 1$). Let $p \in (0,1)$ and L^p be the set of real-valued measurable functions u defined over $[0,1]$, modulo almost everywhere vanishing functions, for which the following quantity is finite:
$$\|u\|_p = \left(\int_0^1 |u|^p \, dx \right)^{\frac{1}{p}}.$$

1. Show that L^p is a vector space and that $d(u,v) = \|u - v\|_p^p$ is a distance. Prove that (L^p, d) is complete.
2. Let $f \in L^p$ and $n \geq 1$ be a positive integer. Prove that there exist some points $0 = x_0 < x_1 < \ldots < x_n = 1$ such that for all $i = 0, \ldots, n-1$,
$$\int_{x_i}^{x_{i+1}} |f|^p \, dx = \frac{1}{n} \int_0^1 |f|^p \, dx.$$
3. Prove that the only convex open domain in L^p containing $u \equiv 0$ is L^p itself. Deduce that the space L^p is not locally convex.
Hint: Introduce the functions $g_i^n = n f \mathbb{1}_{[x_i, x_{i+1}]}$.
4. Show that the (topological) dual space of L^p reduces to $\{0\}$.

TD 4: Geometric Hahn-Banach theorem and Fréchet spaces

Exercise 1 (Finite-dimensional case). Let $C \subset \mathbb{R}^d$ be a convex set such that $C \neq \mathbb{R}^d$, and $x_0 \notin C$. Prove that there exists an affine hyperplane that separates C and $\{x_0\}$.

Exercise 2 (Convex hull). Let E be a locally convex topological vector space (abbreviated l.c.t.v.s. in the following). One says that H is a closed half-space if there exists a $\varphi \in E^*$ and $a \in \mathbb{R}$ such that $H = \{u \in E \mid \varphi(u) \leq a\}$.

1. If C is a convex subset of E, show that its closure $\overline{C}$ is also convex.
2. Let A be a closed convex subset of E. Show that A is the intersection of the closed half-spaces containing A.
3. Deduce that $\overline{co(A)}$ is the intersection of the closed half-spaces containing A for any subset A of E, where $co(A)$ denotes the convex hull of the set A, that is, the smallest convex set that contains A.

Exercise 3 (Density criterion).

1. Let E be a real normed vector space and $F \subset E$ be a vector subset such that $\overline{F} \neq E$. Prove that there exists $\varphi \in E' \setminus \{0\}$ such that $\varphi(u) = 0$ for all $u \in F$.
2. *Application:* Let $(a_n)_n$ be a sequence in $]1, +\infty[$ that diverges to $+\infty$. Prove that the set

$$W = \text{vect}\left\{x \in [0,1] \mapsto \frac{1}{x - a_n} : n \geq 0\right\},$$

   is dense in the space $\mathcal{C}^0([0,1])$ equipped with the norm $\|\cdot\|_\infty$.
   Hint: While considering a continuous linear form that vanishes on W, introduce a generating function.

Exercise 4 (Extreme points). Let K be a subset of a vector space E. A point $a \in K$ is called an *extremal point* of K if, whenever $a = \theta b + (1-\theta)c$ with $\theta \in (0,1)$ and $b, c \in K$, one has $b = c$. A subset[1] S of K is called an *extremal subset* of K if, for all a in S such that $a = \theta b + (1-\theta)c$ with $\theta \in (0,1)$ and $b, c \in K$, one has $b \in S$ and $c \in S$.

1. In a Hilbert space, what are the extremal points of the unit closed ball ? What about the open ball ?
2. Let c_0 denote the space of real sequences $(a_n)_{n\in\mathbb{N}}$ converging to zero. We endow c_0 with the norm $\|\cdot\|_\infty$. Show that the closed unit ball of c_0 does not admit extremal points.
3. Let $I \subset \mathbb{R}$ be an interval. Show that the closed unit ball of $L^1(I)$ does not admit extremal points.

Exercise 5 (Krein-Milman theorem). The aim of this exercise is to prove the following statement.

Theorem 1 (Krein-Milman)**.** *Let E be a l.c.t.v.s. and K be a non-empty convex compact subset of E. Then K coincides with the closed convex envelop of its extremal points.*

[1] This notion is only used in Exercice 5

1. The first step is to show the existence of an extremal point in K. Let $\mathcal{P}$ be the set of non-empty closed extremal subsets of K, endowed with the order "$A \prec B$ if and only if $B \subset A$". Show that $\mathcal{P}$ admits a maximal element which is reduced to a point.
   Hint: If a maximal element S is composed of more than one point, choose a continuous linear form separating points of S and consider the set of points reaching the maximum of this form on S.
2. Define $\tilde{K} = \overline{co}(ext(K))$ the closed convex hull of the extremal points of K, and show that $\tilde{K}$ and K coincide.
3. *Application:* An $n \times n$ matrix with real entries is bi-stochastic if its entries are non-negative, and the sum of the entries of either rows or columns equals 1. One denotes $SM_n(\mathbb{R})$ the set of bistochastic matrices. Show that every matrix in $SM_n(\mathbb{R})$ is actually a convex combination of permutation matrices.

Exercise 6. Let X and Y be l.c.t.v.s. We consider $(p_\alpha)_{\alpha \in A}$ (resp. $(q_\beta)_{\beta \in B}$) a countable family of continuous semi-norms which is separating and generates the topology of X (resp. of Y). Let $T : X \to Y$ be a linear map. Prove that T is continuous if and only if for all $\beta \in B$, there exists a finite set $I \subset A$ and a positive constant $c > 0$ such that for all $u \in X$,

$$q_\beta(Tu) \leq c \sum_{\alpha \in I} p_\alpha(u).$$

Exercise 7 (Space of continuous functions). Let U be an open subset of $\mathbb{R}^d$ and $(K_n)_n$ be an exhaustive sequence of compacts of U.

1. Prove that $C^0(U)$ is a Fréchet space for the distance

$$d(f,g) = \sum_{n=0}^{+\infty} \frac{1}{2^n} \min(1, p_n(f-g)),$$

   defined by the semi norms $p_n(f) = \sup_{x \in K_n} |f(x)|$.
2. A subset $B \subset C^0(U)$ is said to be bounded if for any neighborhood V of 0, there exists $\lambda > 0$ such that $\lambda B \subset V$. Prove that if B is a subset of equibounded functions of $C^0(U)$, that is $\sup_{f \in B} \|f\|_\infty < \infty$, then B is bounded.
3. Let us consider $(f_n)_n$ a sequence of continuous function on U such that $f_n : U \to [0,n]$ with $f_n = 0$ on K_n and $f_n = n$ on $U \setminus K_{n+1}$. Show that $\cup_n \{f_n\}$ is a bounded subset of $C^0(U)$.
4. Prove that the space $C^0(\mathbb{R})$ is not locally bounded, that is, the origin does not have a bounded neighborhood.

Exercise 8 (Space of C^∞ functions). We consider the $E = C^\infty([0,1],\mathbb{R})$ equipped with the following metric

$$d(f,g) = \sum_{k \geq 0} \frac{1}{2^k} \min\left(1, \|f^{(k)} - g^{(k)}\|_\infty\right).$$

1. Check that E is a Fréchet space.
2. Prove that any closed and bounded (*cf* the previous exercise) subset of E is compact.
3. Can the topology of E be defined by a norm ?

TD 5: Weak topology

Exercise 1 (Properties of weakly convergent sequences). Let X be a normed vector space.

1. Let $(u_n)_n$ be a weakly convergent sequence in X. Justify that (u_n) is bounded and that the weak limit u of $(u_n)_n$ satisfies $\|u\| \leq \liminf_{n\to+\infty} \|u_n\|$.
2. Suppose that the sequence $(\varphi_n)_n$ in X^* is converging strongly to some $\varphi \in X^*$. Show that for any sequence $(u_n)_n$ in X that converges weakly to $u \in X$, then the sequence $(\varphi_n(u_n))_n$ converges strongly to $\varphi(u)$.
3. Assume that X is a Hilbert space. Let $(u_n)_n$ be a sequence in X that converges weakly to $u \in X$ and such that $(\|u_n\|)_n$ converges to $\|u\|$. Prove that $(u_n)_n$ converges strongly to u.

Exercise 2 (Examples of weakly convergent sequences).

1. Let H be a separable Hilbert space and $(e_n)_n$ be a Hilbert basis of H. Prove that $(e_n)_n$ converges weakly to 0 but not strongly.
2. Let $K \subset \mathbb{R}^d$ be a compact set. Show that weak convergence in $C(K)$ is equivalent to bounded pointwise convergence.
3. Let $\Omega \subset \mathbb{R}^d$ and $(u_n)_n$, $(v_n)_n$ be two sequences in $L^2(\Omega)$ such that $(u_n)_n$ converges weakly and $(v_n)_n$ strongly. Show that the sequence $(u_n v_n)_n$ converges weakly in $L^1(\Omega)$. What happens if the two sequences converge weakly ?

Exercise 3 (Weak topology). Let X be a topological vector space. Show that X, endowed with the weak topology, is a locally convex topological vector space.

Exercise 4. Let E be a Banach space.

1. Show that if E is finite-dimensional, then the weak topology $\sigma(E, E^*)$ and the strong topology coincide.
2. We assume that E is infinite-dimensional.
   (a) Show that every weak open subset of E contains a straight line.
   (b) Deduce that $B = \{x \in E : \|x\| < 1\}$ is not open for the weak topology.
   (c) Let $S = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. What is the weak closure of S ?

Exercise 5. Let $p, q \in [1, +\infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. We introduce the canonical family of sequences e^k in $\ell^p(\mathbb{N})$, for which every term is zero, except the k^{th} which is 1. We also consider the map

$$\begin{array}{rccc} J_p : & \ell^q(\mathbb{N}) & \to & (\ell^p(\mathbb{N}))^* \\ & (a_n)_n & \mapsto & \left((x_n)_n \mapsto \displaystyle\sum_{n=0}^{+\infty} a_n x_n \right) \end{array}$$

1. When $p \in [1, \infty)$, show that J_p is a surjective isometry.
2. Show that J_∞ is a non-surjective isometry.

3. When $p \in (1,\infty)$, prove that the sequence $(e^k)_k$ converges weakly but not strongly in $\ell^p(\mathbb{N})$ towards the null sequence.
4. Still assuming that $p \in (1,\infty)$, we consider the following subset of $\ell^p(\mathbb{N})$:

$$E = \{e^n + ne^m : n, m \in \mathbb{N},\ m > n\}.$$

   (a) Show that E is closed for the strong topology in $\ell^p(\mathbb{N})$.
   (b) Show that 0 is in the weak closure of E.
   (c) Show that a sequence of E cannot converge weakly towards 0.
   (d) Deduce that the weak topology on ℓ^p is not metrizable.

Exercise 6.

1. (Mazur's lemma) Let E be a Banach space and $(u_n)_n$ be a sequence in E weakly converging to $u_\infty \in E$. Show that u_∞ is a strong limit of finite convex combinations of the u_n.
2. (Banach-Sacks' property) Show that if E is in addition a Hilbert space, we can extract a subsequence converging to u_∞ strongly in the sens of Cesàro.

Exercise 7 (Schur's property for $\ell^1(\mathbb{N})$).

1. Recall why weak and strong topologies always differ in an infinite dimensional norm vector space.

The aim is to prove that a sequence of $\ell^1(\mathbb{N})$ converges weakly if and only if it converges strongly. Take $(u^n)_n$ a sequence in $\ell^1(\mathbb{N})$ weakly converging to 0.

2. Show that for all k, $\lim_{n\to\infty} u^n_k \to 0$.
3. Show that if $u_n \nrightarrow 0$ in $\ell^1(\mathbb{N})$, one can additionally assume that $\|u^n\|_{\ell^1} = 1$.
4. Define via a recursive argument two increasing sequences of $\mathbb{N}$, $(a_k)_k$ and $(n_k)_k$, such that

$$\forall k \geq 0, \qquad \sum_{j=a_k}^{a_{k+1}-1} |u^{n_k}_j| \geq \frac{3}{4}.$$

5. Show that there exists $v \in \ell^\infty(\mathbb{N})$ such that $(v, u^{n_k})_{\ell^2} \geq \frac{1}{2}$ for all k. Conclude.

TD 6: Weak-$*$ topology

Exercise 1. (Warm-up exercise) Let E and F be two Banach spaces, and $T : E \to F$ be a linear map. Show that T is strongly continuous (*i.e.* continuous from $(E, \|\cdot\|_E)$ to $(F, \|\cdot\|_F)$) if and only if T is weakly continuous (*i.e.* continuous from $(E, \sigma(E, E^*))$ to $(F, \sigma(F, F^*))$).

Exercise 2 (Weak-$*$ topology and metrics). Let E be a separable real normed vector space. Let $(u_n)_n$ be a dense sequence in $B_E(0, 1)$. By considering the following metric d on the unit ball of E^*,

$$d(f, g) = \sum_{n=0}^{+\infty} \frac{1}{2^n} |(f - g)(u_n)|, \quad f, g \in B_{E^*}(0, 1),$$

prove that the weak-$*$ topology on $B_{E^*}(0, 1)$ is metrizable.

Exercise 3 (Weak-$*$ closed hyperplanes).

1. In $\ell^\infty(\mathbb{N})$ we consider
$$C = \left\{u \in \ell^\infty(\mathbb{N}) : \liminf_n u_n \geq 0\right\}.$$
Show that C is strongly closed but not weakly-$*$ closed.

Let us now consider E a normed vector space.

2. Let $\varphi : E^* \to \mathbb{R}$ a linear form continuous for the $\sigma(E^*, E)$ topology. Show that:
$$\exists u \in E, \forall \ell \in E^*, \quad \varphi(\ell) = \ell(u).$$
3. Show that an hyperplane $H \subset E^*$ which is closed for the weak-$*$ topology is the kernel of $\mathrm{ev}_u : \varphi \mapsto \varphi(u)$ for some $u \in E$.

Exercise 4 (Eberlein-Šmulian's theorem). The aim of the exercice is to prove the following result:

> Let A a subset of a Banach space E. If A is relatively compact for the weak topology, then A is sequentially relatively compact (still for the weak topology of E).

1. Recall why the result is direct if E^* is separable.
2. Let $(a_n)_n$ be a sequence in A. We denote $F := \overline{\mathrm{vect}\{a_n : n \in \mathbb{N}\}}$. Show that there exists a sequence of linear continuous form $(\phi_n)_n$ such that for any $u \in F$,
$$\|u\| = \sup_n |\phi_n(u)|.$$
Show that $(F, \sigma(F, F^*))$ is metrisable on any weak compact of F.
3. Conclude.
4. Show that the result is wrong for the weak-$*$ topology.
Hint: Work in the space $\ell^\infty(\mathbb{N})^*$.

Remark: the converse implication is also true.

Exercise 5 (Dunford-Pettis' Theorem). The objective of the exercise is to prove Dunford-Pettis' theorem:

> Let $\Omega \subset \mathbb{R}^d$ be a bounded set and $(f_n)_n$ be a bounded sequence in $L^1(\Omega)$. Then, the set $\{f_n\}$ is sequentially compact for the weak topology $\sigma(L^1, L^\infty)$ if and only if the sequence $(f_n)_n$ is equi-integrable.

1. Recall the definition of equi-integrability.

First we prove the reciprocal: let $(f_n)_n$ be a bounded and equi-integrable sequence in L^1.

2. Show that we can reduce to the case where the f_n are non-negative.
3. Let $f_n^k = \mathbf{1}_{f_n \le k} f_n$. Show that $\sup_n \|f_n - f_n^k\|_{L^1} \to 0$.
4. Show that there exists an extraction (n') such that for all $k \in \mathbb{N}$, $f_{n'}^k \rightharpoonup f^k$ in L^1.
5. Prove that $(f^k)_k$ is an increasing sequence and deduce that there exists some $f \in L^1$ such that $f^k \to f$ in L^1.
6. Conclude that $f_{n'} \rightharpoonup f$ in L^1.

Now we want to prove the direct implication. Let $(f_n)_n$ be a bounded sequence in $L^1(\Omega)$ satisfying $f_n \rightharpoonup f \in L^1(\Omega)$. We consider $\mathcal{X}$ the set of indicator functions and, for a fixed $\varepsilon > 0$, we also consider the sets X_n defined for all $n \ge 0$ by:

$$X_n := \left\{ \mathbf{1}_A \in \mathcal{X} : \forall k \ge n, \ \left| \int_A (f_k - f) \, \mathrm{d}x \right| \le \varepsilon \right\}.$$

7. Show that $\mathcal{X}$ and X_n are closed in $L^1(\Omega)$.
8. Using a Baire's argument, show that $(f_n)_n$ is equi-integrable.
9. Conclude.

Exercise 6 (Egorov's theorem).

1. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and $(g_n)_n$ be a sequence of measurable functions such that $(g_n)_n$ converge a.e. to some measurable function g. Show that for all $\varepsilon > 0$, there exists a measurable set $E_\varepsilon \subset \Omega$ such that $\mu(E_\varepsilon^c) < \varepsilon$ and $(g_n)_n$ converges uniformly in E_ε.
2. Let $(f_n)_n$ be a sequence in $L^1(\Omega)$ with $f_n \rightharpoonup f \in L^1(\Omega)$, and $(g_n)_n$ be a bounded sequence in $L^\infty(\Omega)$ satisfying $g_n \to g$ a.e. Show that $f_n g_n \rightharpoonup fg$ in $L^1(\Omega)$.
   Hint: Use Dunford-Pettis' theorem.

Exercise 7 (L^1 is not a dual space). Show that the closed unit ball of $L^1([0,1])$ does not admit extremal points. Deduce that $L^1([0,1])$ is not the dual space of a normed vector space.
Hint: Use Krein-Milman's theorem.

TD 7: Reflexivity

Exercise 1. Let $(E, \|\cdot\|)$ be a reflexive space and B_E be its unit ball. Show that for all $f \in E^*$, there exists $x_f \in B_E$, such that $\|f\|_{E^*} = |f(x_f)|$, i.e. the supremum in the definition of the norm operator is in fact a maximum.

Exercise 2. The aim of this exercise is to prove by two different methods that the space $(C^0([0,1]), \|\cdot\|_\infty)$ of continuous real-valued functions on $[0,1]$ is not reflexive.

1. Method by compactness.
   (a) Define $\varphi \in C([0,1])^*$ by
   $$\varphi(f) = \int_0^{\frac{1}{2}} f(t)\,\mathrm{d}t - \int_{\frac{1}{2}}^1 f(t)\,\mathrm{d}t, \quad f \in C^0([0,1]),$$
   and show that $\|\varphi\| = 1$.
   (b) Prove that $|\varphi(f)| < 1$ for all $f \in C^0([0,1])$ such that $\|f\|_\infty \leq 1$.
   (c) Conclude that the space $C^0([0,1])$ is not reflexive.
2. Method by separability.
   (a) Prove that if E is a Banach space and its dual E^* is separable, then E is separable.
   (b) Show that $C([0,1])$ is separable.
   (c) Prove that $C([0,1])^*$ is not separable.
   Hint: Consider the functions $\delta_t : C([0,1]) \to \mathbb{R}$ defined by $\delta_t(f) = f(t)$ for any $t \in [0,1]$.
   (d) Conclude that $C([0,1])$ is not isomorphic to $C([0,1])^{**}$ as Banach spaces.
   Remark: This is stronger than not being reflexive.

Exercise 3.

1. Let E be a reflexive, separable Banach space. Let $(u_n)_n$ be a bounded sequence in E. Show that one can extract a subsequence $(u_{n'})_{n'}$ which converges weakly in E.
   Remark: the condition "separable" is not necessary thanks to exercise 5.
2. Does this result hold when E is not reflexive ?

Exercise 4. Let E be a normed vector space. Show that any weakly compact set of E is bounded for the norm.

Exercise 5 (Eberlein-Šmulian's theorem). The aim of the exercise is to prove the following result:

> Let A be a subset of a normed vector space E. If A is weakly compact, then A is weakly sequentially compact.

1. Assume that E^* is separable. Recall the key argument that gives the result.

Let $(a_n)_n$ be a sequence in A. We set $F := \overline{\mathrm{vect}\{a_n : n \in \mathbb{N}\}}$ and set $\tilde{A} := A \cap F$.

2. Show that $\tilde{A}$ is weakly compact in F.
3. Show that the unit ball of F^* admits a countable subset $\{\phi_k : k \in \mathbb{N}\}$ such that

$$\forall x \in F, \quad \|x\| = \sup_k |\phi_k(x)|.$$

In the following, we denote by σ the weak topology on $\tilde{A}$ and by τ the topology generated by the semi-norms $|\phi_k|$, $k \in \mathbb{N}$.

4. Show that that $(\tilde{A}, \tau)$ is Hausdorff and that the identity map $\mathrm{Id}_{\sigma,\tau} : (\tilde{A}, \sigma) \to (\tilde{A}, \tau)$ is continuous.
5. Deduce that $(\tilde{A}, \tau)$ is compact and that $\mathrm{Id}_{\sigma,\tau}$ is an homeomorphism.
   Hint: show that the image of a closed set by $Id_{\sigma,\tau}$ is closed.
6. Show that $(\tilde{A}, \sigma(F, F^*))$ is metrizable.
7. Show that one can extract a subsequence $(a_{n_k})_k$ converging weakly in F (to some limit a), and that $(a_{n_k})_k$ converges also weakly to a in E.
8. Show that the result is wrong for the weak-$*$ topology.
   Hint: consider the dual of $\ell^\infty(\mathbb{N})$.

TD 8: Distributions

Exercise 1 (Warming).

1. Let H be the Heaviside function. Show that $H' = \delta_0$ in $\mathcal{D}'(\mathbb{R})$.
2. Give an example of distribution of order n for all $n \in \mathbb{N}$.
3. Let $U \subset \mathbb{R}^d$ be an open set and $T \in \mathcal{D}'(U)$. We consider $f \in C^\infty(U)$ which vanishes on the support of T. Do we have $fT = 0$ in $\mathcal{D}'(U)$?

Exercise 2. Let $U \subset \mathbb{R}^d$ be an open set. Prove that we have an injection of $L^1_{loc}(U)$ in $\mathcal{D}'(U)$.

Exercise 3 (An example of distribution). Show that the formula

$$\langle \alpha, u \rangle = \sum_{n \geq 0} u^{(n)}(n), \quad u \in \mathcal{D}(\mathbb{R}),$$

defines a distribution $\alpha \in \mathcal{D}'(\mathbb{R})$. What about its order ?

Exercise 4 (Convergence of distributions). Do the following series

$$\sum_{n \geq 0} \delta_n^{(n)} \quad \text{and} \quad \sum_{n \geq 0} \delta_0^{(n)},$$

converge in $\mathcal{D}'(\mathbb{R})$?

Exercise 5 (Non-negative distributions).

1. Check that distributions of order 0 are locally signed measures.
2. Let $U \subset \mathbb{R}^d$ be an open set and $\alpha \in \mathcal{D}'(U)$. We say that α is non-negative if and only if for all non-negative test function $u \in \mathcal{D}(U)$, we have $\langle \alpha, u \rangle \geq 0$. Deduce from the previous question that any non-negative distribution is a locally signed measure.

Exercise 6 (Principal value of $1/x$). We define $\mathrm{p.v.}(1/x)$ as follows

$$\forall u \in \mathcal{D}(\mathbb{R}), \quad \langle \mathrm{p.v.}(1/x), u \rangle = \lim_{\varepsilon \to 0} \left(\int_{|x| > \varepsilon} \frac{u(x)}{x} \, \mathrm{d}x \right).$$

1. Show that the above limit exists and defines a distribution. Compute its order.
2. Show that $\mathrm{p.v.}(1/x)$ is the derivative of $\log|x|$ in the sense of distributions.
3. Compute $x \, \mathrm{p.v.}(1/x)$.
4. Let $\alpha \in \mathcal{D}'(\mathbb{R})$ which satisfies $x\alpha = 1$. Show that there exists a constant $c \in \mathbb{R}$ such that $\alpha = \mathrm{p.v.}(1/x) + c\,\delta_0$.
5. Show that $|x|^{\alpha - 2} x \to \mathrm{p.v.}(1/x)$ in $\mathcal{D}'(\mathbb{R})$ as $\alpha \to 0^+$.

Exercise 7. Solve the equation $\alpha' = 0$ in $\mathcal{D}'(\mathbb{R})$.

Exercise 8 (Jump formula). Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class C^1 on $\mathbb{R}^*$. We say that f has a jump at 0 if the limits $f(0^\pm) = \lim_{x\to 0^\pm} f(x)$ exist, and we denote by $[[f(0)]] = f(0^+) - f(0^-)$ the height of the jump. We denote by $\{f'\}$ the derivative of the regular part of f, *i.e.*

$$\{f'\}(x) = \begin{cases} f'(x) & \text{if } f \text{ is differentiable at } x \\ 0 & \text{otherwise} \end{cases}$$

1. Show that in the sense of distributions:

$$f' = \{f'\} + [[f(0)]]\delta_0.$$

2. Let $(x_n)_{n\in\mathbb{Z}}$ be an increasing sequence such that $\lim_{n\to-\infty} x_n = -\infty$ and $\lim_{n\to+\infty} x_n = +\infty$. Let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise C^1 function presenting jumps at every x_n. Show that in the sense of distributions,

$$f' = \{f'\} + \sum_{n\in\mathbb{Z}} [[f(x_n)]]\delta_{x_n}.$$

Exercise 9 (Punctual support). Let $\alpha \in \mathcal{D}'(\mathbb{R}^d)$ such that $\operatorname{supp}\alpha = \{0\}$. We consider $\psi \in \mathcal{D}(\mathbb{R}^d)$ such that $\psi = 1$ in a neighborhood of $\overline{B(0,1)}$ and $\operatorname{supp}\psi \subset B(0,2)$. We set $\psi_r(x) = \psi(x/r)$ for all $r > 0$ and $x \in \mathbb{R}^n$.

1. Recall why α has a finite order, which will be denoted $m \geq 0$ in the following.
2. Show that for all $r > 0$, $\psi_r\alpha = \alpha$.
3. Let $u \in \mathcal{D}(\mathbb{R}^d)$ satisfying that for all $p \in \mathbb{N}^n$ with $|p| \leq m$, $\partial^p u(0) = 0$. Check that $\langle \alpha, u\rangle = 0$.
4. Prove that there exist some real numbers $a_p \in \mathbb{R}$ such that $\alpha = \sum_{|p|\leq m} a_p \delta_0^{(p)}$.

Exercise 10 (Support and order). Let α be the linear map defined for all $u \in \mathcal{D}(\mathbb{R})$ by

$$\langle \alpha, u\rangle = \lim_{n\to+\infty} \left(\sum_{j=1}^{n} u\left(\frac{1}{j}\right) - nu(0) - (\log n)u'(0) \right).$$

1. Check that $\langle \alpha, u\rangle$ is well defined for all $u \in \mathcal{D}(\mathbb{R})$, and that α is a distribution of order less than or equal to 2.
2. What is the support S of α ?
3. What is the order of α ?
Hint: Use test functions of the form

$$u_k(x) = \psi(x) \int_0^x \int_0^y \varphi(kt)\, \mathrm{d}t\mathrm{d}y,$$

where $\varphi \in \mathcal{D}(0,1)$ *has integral* 1 *and* $\psi \in \mathcal{D}(-1,2)$ *satisfies* $0 \leq \psi \leq 1$ *and* $\psi = 1$ *on* $[0,1]$.

TD 9: CONVOLUTION OF DISTRIBUTIONS

EXERCISE 1 (Examples of convolutions). Compute the following convolutions:

1. $\delta_a * \delta_b$ in $\mathbb{R}^d$,
2. $T * \delta_a$, with $T \in \mathcal{D}'(\mathbb{R}^d)$,
3. $(x^p \delta_0^{(q)}) * (x^m \delta_0^{(n)})$,
4. $\delta_0^{(k)} * (x^m H)$,
5. $\mathbb{1}_{[a,b]} * \mathbb{1}_{[c,d]}$,
6. $\mathbb{1}_{[0,1]} * (xH)$.

EXERCISE 2 (Associativity and convolution). Show that the convolution product is not associative without assumptions on the supports by considering the distributions 1, δ_0' and H in $\mathcal{D}'(\mathbb{R})$, where H is the Heaviside function.

EXERCISE 3. We will study the behavior of the convolution of distributions with respect to the convolution product.

1. Let $T \in \mathcal{D}'(\mathbb{R}^d)$ be compactly supported, $V \in \mathcal{D}'(\mathbb{R}^d)$ and $(V_n)_n$ be a sequence of distributions in $\mathcal{D}'(\mathbb{R}^d)$. Prove that if $V_n \to V$ in $\mathcal{D}'(\mathbb{R}^d)$, then $V_n * T \to V * T$ in $\mathcal{D}'(\mathbb{R}^d)$.
2. Show that there exist two sequences of distributions T_n and V_n tending to 0 in $\mathcal{D}'(\mathbb{R})$ and such that $T_n * V_n \to \delta_0$.

EXERCISE 4 (Regularization by polynomials). For $n \in \mathbb{N}^*$, we define the polynomial P_n on $\mathbb{R}^d$ by

$$P_n(x) = \frac{n^d}{\pi^{d/2}} \left(1 - \frac{|x|^2}{n}\right)^{n^3}.$$

1. What is the limit in $\mathcal{D}'(\mathbb{R}^d)$ of the sequence $(P_n)_n$?
2. Deduce that any compactly supported distribution is the limit in $\mathcal{D}'(\mathbb{R}^d)$ of a sequence of polynomials.

EXERCISE 5 (Convolution and translations). Let $F : \mathcal{D}(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ be a continuous linear map. We say that F commutes with translations when $\tau_x \circ F = F \circ \tau_x$ for all $x \in \mathbb{R}^d$.

1. Check that if there exists $T \in \mathcal{D}'(\mathbb{R}^d)$ such that, for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $F(\varphi) = T * \varphi$, then F commutes with translations.
2. Show that for all $T \in \mathcal{D}'(\mathbb{R}^d)$, and all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we have $\langle T, \varphi \rangle = T * \check{\varphi}(0)$, where $\check{\varphi}(x) = \varphi(-x)$.
3. Prove that if F commutes with translations, then there exists $T \in \mathcal{D}'(\mathbb{R}^d)$ such that, for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $F(\varphi) = T * \varphi$.

EXERCISE 6 (The extension of the convolution).

1. Let $\varphi \in C^\infty(\mathbb{R}^d)$ and $T \in \mathcal{D}'(\mathbb{R}^d)$ such that $\operatorname{supp}(T) \cap \operatorname{supp}(\varphi)$ is compact. Show that $\langle T, \varphi \rangle$ can be defined in a meaningful way.

2. Let $T, S \in \mathcal{D}'(\mathbb{R}^d)$ satisfying the following property: for every compact K in $\mathbb{R}^d$,

$$D_K = \left\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : x \in \operatorname{supp} T,\ y \in \operatorname{supp} S,\ x + y \in K\right\}$$

is compact. Show that in this case, $T * S$ and $S * T$ are well-defined and are equal.
3. Compute the distribution $(x^p H) * (x^q H)$ for all $p, q \in \mathbb{N}$, where H is the Heaviside function.

Exercise 7 (Linear differential equations). Define $\mathcal{D}'_+(\mathbb{R}) = \{T \in \mathcal{D}'(\mathbb{R}) : \operatorname{supp} T \subset \mathbb{R}_+\}$.

1. By using Exercice 6, show that the convolution of two elements of $\mathcal{D}'_+(\mathbb{R})$ is well-defined and gives an element of $\mathcal{D}'_+(\mathbb{R})$. *In the following, we admit that $\mathcal{D}'(\mathbb{R}_+)$ is a commutative algebra for the convolution.* What is the identity element for the convolution in $\mathcal{D}'_+(\mathbb{R})$?
2. Show that for all $a \in \mathbb{R}$ and $T, S \in \mathcal{D}'_+(\mathbb{R})$, we have $(e^{ax}T) * (e^{ax}S) = e^{ax}(T * S)$.
3. For any $T \in \mathcal{D}'_+(\mathbb{R})$, let T^{-1} denote the inverse of T in $\mathcal{D}'_+(\mathbb{R})$ for the convolution whenever it exists. Check that T^{-1} is unique when it exists.
4. Compute H^{-1} and $(\delta_0' - \lambda\delta_0)^{-1}$ for all $\lambda \in \mathbb{R}$ whenever they exist.
5. Let P be a polynomial that splits in $\mathbb{R}$, compute $[P(D)\delta_0]^{-1}$.
6. Solve the following system in $\mathcal{D}'_+(\mathbb{R}) \times \mathcal{D}'_+(\mathbb{R})$

$$\begin{cases} \delta_0'' * X + \delta_0' * Y = \delta_0, \\ \delta_0' * X + \delta_0'' * Y = 0. \end{cases}$$

TD 10: Tempered distribution

Exercise 1.

1. Let $A \subset \mathbb{R}^d$ be a Borel of finite measure. Show that $\mathcal{F}(\mathbb{1}_A)$ belongs to $L^2(\mathbb{R}^d)$ but not to $L^1(\mathbb{R}^d)$.
2. Does it exist two functions $f, g \in \mathcal{S}(\mathbb{R})$ such that $f * g = 0$? What happens if in addition f and g have compact supports ?

Exercise 2. Prove that the following distributions are tempered and compute their Fourier transform:

1. δ_0 in $\mathbb{R}^d$,
2. $e^{-\frac{|x|^2}{2\sigma}}$ in $\mathbb{R}$ with $\sigma > 0$,
3. 1,
4. H (Heaviside),
5. $\mathrm{p.v.}(1/x)$,
6. $|x|$ in $\mathbb{R}$.

Exercise 3.

1. If $d \geq 3$, show that $u_0(x) = \big(-d(d-2)\mathrm{Vol}(B(0,1))\|x\|^{d-2}\big)^{-1}$ is a fundamental solution for the Laplacian, *i.e.* $\Delta u_0 = \delta_0$ in the sense of distributions.
2. Give a solution of $\Delta u = f$ in the sense of distributions for f in $\mathcal{D}'(\mathbb{R}^d)$ with compact support.
3. What can you say about the regularity of u if f is a function in $\mathcal{S}(\mathbb{R}^d)$?
4. Consider the linear PDE $u - \Delta u = f$ for $f \in \mathcal{S}(\mathbb{R}^d)$. Express a solution in $\mathcal{S}(\mathbb{R}^d)$ in terms of the Bessel kernel $B = \mathcal{F}^{-1}((1+|\xi|^2)^{-1})$.

Exercise 4. Let $k > 0$ and $T \in \mathcal{S}'(\mathbb{R})$ such that $T^{[4]} + kT \in L^2(\mathbb{R})$. Show that for every $j \in \{0, \cdots, 4\}$, $T^{[j]} \in L^2(\mathbb{R})$.

Exercise 5. We investigate the solutions $T \in \mathcal{S}'(\mathbb{R}^4)$ with support in $\mathbb{R}_+ \times \mathbb{R}^3$ of the wave equation

$$\partial_{tt} T - \Delta T = \delta_{(t,x)=(0,0)}, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3.$$

1. Let $\mathcal{F}$ be the partial Fourier transform with respect to x and $\tilde{T} = \mathcal{F}T$. Find an ODE of which $\tilde{T}$ is solution. We denote in the following (E) this equation.
2. Solve this equation with the ansatz

$$\tilde{T}(t,\xi) = H(t)U(t,\xi),$$

where U is solution of the homogenous equation associated with (E).

3. We denote by $\mathrm{d}\sigma_R$ the measure on the sphere of radius R and center 0:

$$\langle \mathrm{d}\sigma_R, \varphi \rangle = \int_{\mathbb{S}(0,R)} \varphi(x)\, \mathrm{d}\sigma_R(x)$$

Show that:

$$\forall \xi \in \mathbb{R}^d, \quad \mathcal{F}\left(\frac{\mathrm{d}\sigma_R}{4\pi R^2}\right)(\xi) = \frac{\sin(R|\xi|)}{R|\xi|}.$$

4. Deduce that for $\varphi \in \mathcal{S}(\mathbb{R}^4)$,

$$\langle T, \varphi \rangle = \int_0^\infty \frac{1}{4\pi t} \int_{\mathbb{S}(0,|t|)} \varphi(t,x)\, \mathrm{d}\sigma_t(x)\, \mathrm{d}t.$$

5. What is the support of T?

Exercise 6. We consider the Schrödinger equation on $\mathbb{R}_t \times \mathbb{R}^d$

$$\begin{cases} i\partial_t u + \Delta u = 0, \\ u_{t=0} = u_0. \end{cases} \tag{1}$$

1. For $u_0 \in \mathcal{S}(\mathbb{R}^d)$, solve the equation (1) in $C^0(\mathbb{R}, \mathcal{S}(\mathbb{R}^d))$.
2. Justify that the Fourier transform of the function $e^{it|\xi|^2}$ is well defined.
3. Show that for $\alpha \in \mathbb{C}$ with positive real part,

$$\mathcal{F}^{-1}(e^{\alpha|\xi|^2}) = \frac{1}{(-4\alpha\pi)^{d/2}} e^{\frac{|x|^2}{4\alpha}}.$$

4. Check that also holds in $\mathcal{S}'(\mathbb{R}^d)$ when $\alpha \in i\mathbb{R}$.
5. Deduce that there exists a constant $C > 0$ such that for all $t > 0$,

$$\|u(t,\cdot)\|_{L^1(\mathbb{R}^d)} \leq \frac{C}{t^{d/2}} \|u_0\|_{L^\infty(\mathbb{R}^d)}.$$

TD 11: Sobolev spaces

Exercise 1 (Warming).

1. Show that $u(x) = |x|$ belongs to $W^{1,2}(-1,1)$ but not to $W^{2,2}(-1,1)$.
2. Check that $v(x) = \frac{\sin(x^2)}{\sqrt{1+x^2}}$ belongs to $L^2(\mathbb{R})$ but not to $W^{1,2}(\mathbb{R})$.
3. Show that $H^1(\mathbb{R}^2)$ is not included in $L^\infty(\mathbb{R}^2)$.
   Hint: Consider a function of the form $x \mapsto \chi(|x|)\,|\log|x||^{1/3}$.

Exercise 2 (Optimality in the Sobolev embeddings). Let $1 \le p < d$ and $\alpha \in [1,\infty]$. By using a homogeneity argument, show that if there exists a continuous injection $W^{1,p}(\mathbb{R}^d) \hookrightarrow L^\alpha(\mathbb{R}^d)$, then necessarily $p \le \alpha \le \frac{dp}{d-p}$.

Exercise 3 (Some properties of $H^s(\mathbb{R}^d)$).

1. Show that $H^{s_1}(\mathbb{R}^d)$ embeds continuously into $H^{s_2}(\mathbb{R}^d)$ for $s_1 \ge s_2$.
2. Check that $\delta_0 \in H^s(\mathbb{R}^d)$ for $s < -d/2$.
3. (a) Prove that if $s > d/2$, the space $H^s(\mathbb{R}^d)$ embeds continuously to $C^0_{\to 0}(\mathbb{R}^d)$, the space of continuous functions u on $\mathbb{R}^d$ satisfying $u(x) \to 0$ as $|x| \to +\infty$.
   (b) State an analogous result in the case where $s > d/2 + k$ for some $k \in \mathbb{N}$. Deduce that $\bigcap_{s\in\mathbb{R}} H^s(\mathbb{R}^d) \subset C^\infty(\mathbb{R}^d)$.
   (c) Let $U \subset \mathbb{R}^d$ be open. Deduce from the above question that $\bigcap_{s\in\mathbb{R}} H^s_{loc}(U) = C^\infty(U)$, where we set
   $$H^s_{loc}(U) = \{u \in L^2(U) : \forall \varphi \in \mathcal{D}(U),\ \varphi u \in H^s(\mathbb{R}^d)\}.$$
4. Let us now consider $s \in (d/2, d/2+1)$.
   (a) Show that for all $\alpha \in [0,1]$ and all $x, y, \xi \in \mathbb{R}^d$:
   $$\left|e^{ix\cdot\xi} - e^{iy\cdot\xi}\right| \le 2^{1-\alpha}|x-y|^\alpha|\xi|^\alpha.$$
   (b) Deduce that for all $\alpha \in (0, s-d/2)$, there exists a constant $C(\alpha) > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$,
   $$\frac{|u(x)-u(y)|}{|x-y|^\alpha} \le C(\alpha)\|u\|_{H^s}.$$
   (c) Conclude that $H^s(\mathbb{R}^d)$ embeds continuously to $C^\alpha(\mathbb{R}^d)$.
5. Assuming that s belongs to $[0, d/2]$, the purpose is now to prove that $H^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$, where $p = 2d/(d-2s)$. To that end, let us recall that for all $u \in L^p(\mathbb{R}^n)$,
   $$\|u\|_{L^p}^p = \int_0^\infty p\lambda^{p-1}\left|\{|u| > \lambda\}\right| \mathrm{d}\lambda.$$
   Considering $u \in \mathcal{S}(\mathbb{R}^d)$ and $A_\lambda > 0$, we set $u_{1,\lambda} = \mathcal{F}^{-1}(\mathbb{1}_{|\xi|<A_\lambda}\hat{u})$ and $u_{2,\lambda} = \mathcal{F}^{-1}(\mathbf{1}_{|\xi|\ge A_\lambda}\hat{u})$.

(a) Prove that
$$\forall x \in \mathbb{R}^d, \quad |u_{1,\lambda}(x)| \leq C A_\lambda^{(2d-s)/2} \|u\|_{H^s}.$$
Deduce that there exists some A_λ such that $|\{|u_{1,\lambda}| > \lambda/2\}| = 0$.

(b) Show that for this choice of A_λ,
$$\|u\|_{L^p}^p \leq 4p \int_0^\infty \lambda^{p-3} \|u_{2,\lambda}\|_{L^2}^2 \, \mathrm{d}\lambda.$$

(c) Conclude.

Exercise 4 (Trace on an hyperplane). Let us consider the function
$$\gamma_0 : \varphi(x', x_d) \in C_0^\infty(\mathbb{R}^d) \mapsto \varphi(x', x_d = 0) \in C_0^\infty(\mathbb{R}^{d-1}).$$
Prove that for all $s > 1/2$, the function γ_0 can be uniquely extended as an application mapping $H^s(\mathbb{R}^d)$ to $H^{s-1/2}(\mathbb{R}^{d-1})$.
Hint: For all $\varphi \in C_0^\infty(\mathbb{R}^d)$, begin by computing the Fourier transform of the function $\gamma_0\phi$.

Exercise 5 (An estimate). Let $0 < \alpha < 1$ and $p > 1$ be positive real numbers. Show that there exists a positive constant $C_{\alpha,p} > 0$ such that for all $u \in C_0^\infty(\mathbb{R})$,
$$\left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{|u(x) - u(y)|}{|x-y|^\alpha} \right)^p \frac{\mathrm{d}x\mathrm{d}y}{|x-y|^d} \right)^{1/p} \leq C_{\alpha,p} \|u\|_{L^p(\mathbb{R}^d)}^{1-\alpha} \|\nabla u\|_{L^p(\mathbb{R}^d)}^\alpha.$$
Hint: Consider the two regions $\{|x-y| > R\}$ and $\{|x-y| \leq R\}$, where $R > 0$ is to be chosen.

Exercise 6 (Composition). Let U and U' be two open subset of $\mathbb{R}^d$.

1. Let $H : U' \to U$ be a C^1-diffeomorphism such that the Jacobian $\mathrm{Jac}(H)$ and $\mathrm{Jac}(H^{-1})$ belong to L^∞. Prove that for all $u \in W^{1,p}(\Omega)$, we have $u \circ H \in W^{1,p}(\Omega')$ and that for all $1 \leq i \leq d$,
$$\partial_{y_i}(u \circ H) = \sum_{j=1}^n (\partial_{x_j} u \circ H) \partial_{y_i} H_j.$$
2. Let us now consider a function $G \in C_b^1(\mathbb{R})$ satisfying $G(0) = 0$. Show that for all $u \in W^{1,p}(U)$, we have $G \circ u \in W^{1,p}(U)$ and that for all $1 \leq j \leq n$,
$$\partial_{x_j}(G \circ u) = (G' \circ u) \partial_{x_j} u.$$
3. Do we need to assume that G' is bounded when $d = 1$?

TD 12: Sobolev spaces and PDEs

Exercise 1 (Agmon's and Brezis-Gallouët's type inequalities).

1. Prove that there exists a positive constant $c > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}^3)$,
$$\|u\|_{L^\infty(\mathbb{R}^3)} \leq c\,\|u\|_{H^1(\mathbb{R}^3)}^{1/2}\|u\|_{H^2(\mathbb{R}^3)}^{1/2}.$$
Hint: Setting $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$ *and considering* $R > 0$, *use the following decomposition*
$$\|\widehat{u}\|_{L^1(\mathbb{R}^3)} = \int_{|\xi|\leq R}\langle\xi\rangle|\widehat{u}(\xi)|\frac{\mathrm{d}\xi}{\langle\xi\rangle} + \int_{|\xi|>R}\langle\xi\rangle^2|\widehat{u}(\xi)|\frac{\mathrm{d}\xi}{\langle\xi\rangle^2}.$$
2. Show similarly that there exists a positive constant $c > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}^2)$,
$$\|u\|_{L^\infty(\mathbb{R}^2)} \leq c\Big(1 + \|u\|_{H^1(\mathbb{R}^2)}\sqrt{\log(1+\|u\|_{H^2(\mathbb{R}^2)})}\Big).$$

Exercise 2. Let $U = (0,1)$.

1. Prove that the following continuous embeddings hold
$$W^{1,1}(U) \hookrightarrow C^0(\bar{U}) \quad \text{and} \quad W^{1,p}(U) \hookrightarrow C^{0,1-1/p}(\bar{U}) \quad \text{when } p \in (1,\infty],$$
with the convention $1/\infty = 0$.
2. Prove that for all $1 \leq p < \infty$, the space $W_0^{1,p}(U)$ is given by
$$W_0^{1,p}(U) = \big\{u \in W^{1,p}(U) : u(0) = u(1) = 0\big\}.$$

Exercise 3 (Poincaré's inequality). Let $p \in [1,+\infty)$ and let U be an open subset of $\mathbb{R}^d$.

1. Assume that U is bounded in one direction, meaning that U is contained in the region between two parallel hyperplanes. Prove Poincaré's inequality: there exists $c > 0$ such that for every $f \in W_0^{1,p}(U)$,
$$\|f\|_{L^p(U)} \leq c\|\nabla f\|_{L^p(U)}.$$
As a consequence, $\|\nabla\cdot\|_{L^p(U)}$ defines a norm on $W_0^{1,p}(U)$ which is equivalent to $\|\cdot\|_{W^{1,p}(U)}$.
Hint: Consider first the case $U \subset \mathbb{R}^{d-1}\times[-M,M]$.
2. Assume that U is bounded. Prove Poincaré-Wirtinger's inequality: there exists a constant $c > 0$ such that for any $f \in W^{1,p}(U)$ satisfying $\int_U f = 0$,
$$\|f\|_{L^p(U)} \leq c\|\nabla f\|_{L^p(U)}.$$

Exercise 4 (Duality). Let U be an open subset of $\mathbb{R}^d$ and let $p \in (1, +\infty)$.

1. Prove that for all $F \in W_0^{1,p}(U)'$, there exist $f_0, f_1, \dots, f_d \in L^q(U)$ (with $\frac{1}{p} + \frac{1}{q} = 1$) such that for all $g \in W_0^{1,p}(U)$,
$$\langle F, g \rangle_{W_0^{1,p}(U)', W_0^{1,p}(U)} = \int_U f_0 g \, \mathrm{d}x + \sum_{i=1}^{d} \int_U f_i \partial_i g \, \mathrm{d}x.$$
2. Prove that we also have
$$\|F\|_{W_0^{1,p}(U)'} \leq \left(\sum_{i=0}^{d} \|f_i\|_{L^q(U)}^q \right)^{\frac{1}{q}}.$$
3. Assuming that U is bounded, prove that we may take $f_0 = 0$.

Exercise 5 (A minimization problem). Let $U \subset \mathbb{R}^3$ be open, bounded with smooth boundary. The purpose is to prove that the following elliptic problem has a non-trivial weak solution
$$\begin{cases} -\Delta u = u^3 & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

1. Prove that there exists a solution to the following minimization problem
$$\inf \left\{ \|\nabla v\|_{L^2(U)} : v \in H_0^1(U), \ \|v\|_{L^4(U)} = 1 \right\}. \tag{1}$$
Hint: Since $d = 3$ here, the continuous embedding $H_0^1(U) \hookrightarrow L^q(U)$ holds for all $1 \leq q \leq 6$, and is moreover compact when $1 \leq q < 6$. Moreover, $\|\nabla \cdot\|_{L^2(U)}$ defines a norm on $H_0^1(U)$ which is equivalent to $\|\cdot\|_{W^1(U)}$ as a consequence of Poincaré's inequality, which is proven in Exercise 3.
2. Check that if the function $v \in H_0^1(U)$ solves (1), there exists a positive constant $\lambda > 0$ such that $-\Delta v = \lambda v^3$ in $\mathcal{D}'(U)$.
3. Conclude.