École Normale Supérieure de Lyon **Année 2023 - 2024**
Département de Mathématiques **M1 - EDP**

Homework problem: Observability of the Schrödinger equation

The following problem is devoted to the study of the observability properties of the Schrödinger equation on the one-dimensional torus $\mathbb{T} \simeq \mathbb{R}/(2\pi\mathbb{Z})$ (whose size plays no role here). Precisely, the main theorem we aim at proving is the following:

Theorem 1. *Let $T > 0$ be a positive time and $\omega \subset \mathbb{T}$ be an open set. Then, there exists a positive constant $C_{\omega,T} > 0$ such that for every $u_0 \in L^2(\mathbb{T}^1)$,*

$$\|u_0\|^2_{L^2(\mathbb{T})} \le C_{\omega,T} \int_0^T \left\|e^{it\partial_x^2} u_0\right\|^2_{L^2(\omega)} \,\mathrm{d}t.$$

The original proof of this kind of result by S. Jaffard [2] (dealing in fact with the two-dimensional torus $\mathbb{T}^2$) was based on results for lacunary Fourier series by J.-P. Kahane. Here, we present a more modern approach in a microlocal spirit, based on the survey [3] by C. Laurent, which is more robust, and allows to consider the Schrödinger equation on general Riemannian compact manifold without boundary. Let us mention that Theorem 1 is also valid for any mesurable subset $\omega \subset \mathbb{T}$ with positive measure as proven recently by N. Burq and M. Zworski [1].

In the above statement, $(e^{it\partial_x^2})_{t\in\mathbb{R}}$ denotes the Schrödinger group on $L^2(\mathbb{T})$. For all purposes, let us recall that for any function $u \in L^2(\mathbb{T})$ and for non-negative real number $s \ge 0$, the H^{-s}-norm of the function u is defined by

$$\|u\|^2_{H^{-s}(\mathbb{T})} = \sum_{k\in\mathbb{Z}} \langle k\rangle^{-2s} |\widehat{u}(k)|^2,$$

where $\widehat{u}$ denotes the Fourier transform of the function u. Moreover, given a positive time $T > 0$, the space $L^2([0,T], H^{-s}(\mathbb{T}))$ is endowed with the following norm

$$\|u\|^2_{L^2([0,T],H^{-s}(\mathbb{T}))} = \int_0^T \|u(t,\cdot)\|^2_{H^{-s}(\mathbb{T})} \,\mathrm{d}t.$$

1 Propagation of compactness

Given a positive time $T > 0$ and a open subset $\omega \subset \mathbb{T}$, the aim of this first section is to prove that there exists a positive constant $C_{\omega,T} > 0$ such that for all $u_0 \in L^2(\mathbb{T})$,

$$(1) \qquad \|u_0\|^2_{L^2(\mathbb{T})} \le C_{\omega,T} \int_0^T \left\|e^{it\partial_x^2} u_0\right\|^2_{L^2(\omega)} \,\mathrm{d}t + C_{\omega,T}\|u_0\|^2_{H^{-2}(\mathbb{T})}.$$

Notice that this is a weaker observability estimate than the one stated in Theorem 1, due to the presence of the H^{-2}-norm. Proceeding by contradiction, we consider a sequence $(u_{n,0})_n$ in $L^2(\mathbb{T})$ such that $\|u_{n,0}\|_{L^2(\mathbb{T})} = 1$ and

$$(2) \qquad \int_0^T \left\|e^{it\partial_x^2} u_{n,0}\right\|^2_{L^2(\omega)} \,\mathrm{d}t + \|u_{n,0}\|^2_{H^{-2}(\mathbb{T})} \le \frac{1}{n}.$$

In the following, we set $u_n(t) = e^{it\partial_x^2} u_{n,0}$ for any $t \in [0,T]$. Notice from the estimate (2) that the sequence $(u_n)_n$ converges strongly to 0 in the space $L^2([0,T], L^2(\omega))$. The proof consists essentially in proving that the sequence $(u_n)_n$ in fact converges strongly in $L^2_{loc}([0,T], L^2(\mathbb{T}))$, that is, satisfies that for all $\psi \in C_0^\infty(0,T)$,

$$\langle \psi(t) u_n, u_n \rangle_{L^2([0,T],L^2(\mathbb{T}))} \underset{n\to+\infty}{\to} 0. \tag{3}$$

This is a propagation of compactness result.

1. Justify that it suffices to establish the convergence (3) in order to end the proof of the weak observability estimate (1).

Admit for a moment that we have for every $\varphi \in C^\infty(\mathbb{T}^1)$

$$\langle \psi(t) \varphi'(x) u_n, u_n \rangle_{L^2([0,T],L^2(\mathbb{T}))} \underset{n\to+\infty}{\to} 0. \tag{4}$$

2. Give a characterization of the smooth functions $f \in C^\infty(\mathbb{T})$ that can be written $f = \varphi'$ with $\varphi \in C^\infty(\mathbb{T})$.
3. Deduce from (4) that for every $f \in C_0^\infty(\omega)$ and $x_0 \in \mathbb{T}$, we have

$$\langle \psi(t) f(\cdot - x_0) u_n, u_n \rangle_{L^2([0,T],L^2(\mathbb{T}))} \underset{n\to+\infty}{\to} 0,$$

and conclude to the estimate (3).

The rest of this section is devoted to the proof of the convergence (4). In order to alleviate the writing, we will simply denote by $\langle \cdot, \cdot \rangle$ the scalar product in $L^2([0,T], L^2(\mathbb{T}))$.

4. Prove that the sequence $(u_{n,0})_n$ converges strongly to 0 in $H^{-1}(\mathbb{T})$. Deduce that the sequence $(u_n)_n$ converges strongly to 0 in the space $L^2([0,T], H^{-1}(\mathbb{T}))$.
5. Let us consider the operator $\mathcal{D}^{-1}$ defined on $L^2(\mathbb{T})$ by

$$\widehat{\mathcal{D}^{-1}u}(k) = \begin{cases} k^{-1}\widehat{u}(k) & \text{if } k \in \mathbb{Z}^*, \\ \widehat{u}(0) & \text{if } k = 0. \end{cases}$$

Check that the operator $-i\partial_x \mathcal{D}^{-1}$ is the orthogonal projection onto the subspace of functions $u \in L^2(\mathbb{T})$ with $\widehat{u}(0) = 0$. Deduce that

$$\langle \psi(t)\varphi'(x) u_n, u_n \rangle = \langle \psi(t)\varphi'(x)\widehat{u_n}(t,0), u_n \rangle - i\langle \psi(t)\varphi'(x)\partial_x \mathcal{D}^{-1} u_n, u_n \rangle. \tag{5}$$

6. Check that the first scalar product in the right-hand side of the estimate (5) converges to 0.
7. We introduce the operator $A = \psi(t)\varphi(x)\mathcal{D}^{-1}$ on $L^2(\mathbb{T})$. By using the fact that the functions u_n are solutions to the Schrödinger equation $i\partial_t u_n + \partial_x^2 u_n = 0$, prove that

$$\langle [A, \partial_x^2] u_n, u_n \rangle = i\langle \psi'(t)\varphi(x)\mathcal{D}^{-1} u_n, u_n \rangle \underset{n\to+\infty}{\to} 0,$$

where $[A, \partial_x^2] = A \circ \partial_x^2 - \partial_x^2 \circ A$ denotes the commutator between the operators A and ∂_x^2.
8. Check that

$$[A, \partial_x^2] = -2\psi(t)\varphi'(x)\partial_x \mathcal{D}^{-1} - \psi(t)\varphi''(x)\mathcal{D}^{-1}.$$

9. Conclude to the convergence (4).

2 A unique continuation result

Given a positive time $T > 0$ and an open subset $\omega \subset \mathbb{T}$, we consider the following vector space

$$\mathcal{N}_{\omega,T} = \big\{u_0 \in L^2(\mathbb{T}) : e^{it\partial_x^2} u_0 = 0 \text{ on } (0,T) \times \omega\big\}.$$

The aim of the following questions is to prove that $\mathcal{N}_{\omega,T} = \{0\}$. Reasoning by contradiction, we assume that this is not the case.

10. First of all, we will prove that the set $\mathcal{N}_{\omega,T}$ is invariant under the action of ∂_x^2, and that there exists a positive constant $c > 0$ such that

$$\forall u_0 \in \mathcal{N}_{\omega,T}, \quad \|\partial_x^2 u_0\|_{L^2(\mathbb{T}^1)} \leq c\|u_0\|_{L^2(\mathbb{T}^1)}. \tag{6}$$

   (a) Check that for all $u_0 \in \mathcal{N}_{\omega,T}$ and for all $\varepsilon > 0$ small enough, we have $u_\varepsilon := \varepsilon^{-1}(e^{i\varepsilon\partial_x^2} u_0 - u_0) \in \mathcal{N}_{\omega,T/2}$ and

$$\|u_\varepsilon\|^2_{L^2(\mathbb{T}^1)} \leq C\|u_\varepsilon\|^2_{H^{-2}(\mathbb{T}^1)},$$

   for some positive constant $C > 0$ not depending on $\varepsilon > 0$ nor on $u_0 \in \mathcal{N}_{\omega,T}$.

   (b) By passing properly to the limit as $\varepsilon \to 0^+$, conclude to the estimate (6).

   (c) Prove that $\mathcal{N}_{\omega,T}$ is invariant under the action of the operator ∂_x^2.

11. By using a compactness argument, prove that $\mathcal{N}_{\omega,T}$ is finite-dimensional and then deduce that there exists $\lambda \in \mathbb{R}$ and $u_\lambda \in \mathcal{N}_{\omega,T} \setminus \{0\}$ such that $-\partial_x^2 u_\lambda = \lambda u_\lambda$ in $L^2(\mathbb{T})$.
12. Justify that the function u_λ cannot vanish on an open subset of $\mathbb{T}$.
13. Deduce that $\mathcal{N}_{\omega,T} = \{0\}$.

3 Removing the H^{-2}-norm

Arguing by contradiction, we can now conclude to Theorem 1. Let $(u_{n,0})_n$ be a sequence in $L^2(\mathbb{T})$ satisfying $\|u_{n,0}\|_{L^2(\mathbb{T})} = 1$ and

$$\int_0^T \big\|e^{it\partial_x^2} u_{n,0}\big\|^2_{L^2(\omega)} \,\mathrm{d}t \leq \frac{1}{n}.$$

14. Justify that there exists a function $u_0 \in L^2(\mathbb{T})$ such that, up to a subsequence, the sequence $(u_{n,0})_n$ converges weakly to u_0 in $L^2(\mathbb{T})$ and converges strongly to u_0 in $H^{-2}(\mathbb{T})$.
15. Check that for every $t \in [0,T]$, the sequence $(e^{it\partial_x^2} u_{n,0})_n$ converges weakly to $e^{it\partial_x^2} u_0$ in $L^2(\mathbb{T})$ and deduce that $u_0 = 0$.
16. Conclude.

References

[1] N. Burq & M. Zworski, *Rough controls for Schrödinger operators on 2-tori*, Ann. H. Lebesgue **2** (2019), 331–347.

[2] S. Jaffard, *Contrôle interne exact des vibrations d'une plaque rectangulaire*, Port. Math. **47** (1990), 423–429.

[3] C. Laurent, *Internal control of the Schrödinger equation*, Math. Control Relat. Fields **4** (2014), 161–186.