Homework problem: Duality of L^p spaces

The following problem is devoted to the study of the duality of the Lebesgue spaces. Precisely, the main theorem we aim at proving is the following:

Theorem 1. *Let $(X, \mathcal{F}, \mu)$ be a measure space and $p, q \in (1, +\infty)$ satisfying $1/p + 1/q = 1$. Then, for all $L \in L^p(X)'$, there exists a unique $f \in L^q(X)$ such that*

$$\forall g \in L^p(X), \quad L(g) = \int_X fg \, \mathrm{d}\mu.$$

We will check moreover that this result also holds in the case $p = 1$ when the measure μ is σ-finite, but fails when $p = +\infty$ even under this assumption.

In the above statement, $L^p(X)$ denotes the standard Lebesgue space associated with $(X, \mathcal{F}, \mu)$ (the dependence with respect to the σ-algebra $\mathcal{F}$ and the measure μ will always be omitted) and $L^p(X)'$ stands for the associated topological dual, $L^p(X)$ being endowed with the usual norm $\|\cdot\|_{L^p}$.

1 Duality of the spaces $l^p(\mathbb{N})$

First of all, we study the duality of the $l^p(\mathbb{N})$ spaces, which corresponds to the case where $X = \mathbb{N}$, $\mathcal{F} = \mathcal{P}(\mathbb{N})$ and μ is the counting measure (which is σ-finite). Let $p \in [1, +\infty]$. For all $y \in l^q(\mathbb{N})$, we set

$$F_y(x) = \sum_{n=0}^{+\infty} x_n y_n, \quad x \in l^p(\mathbb{N}).$$

1. Check that the linear map $F : y \in l^q(\mathbb{N}) \mapsto F_y \in (l^p(\mathbb{N}))'$ is well-defined and is an isometry.
2. In the case where $p \in [1, +\infty)$, prove that F is onto.
3. By considering the space of converging sequences and the Hahn-Banach theorem, prove that F is not onto when $p = +\infty$.
   Remark: This proves that Theorem 1 fails in the case $p = +\infty$ even under the assumption that the measure μ is σ-finite.

Let us consider $c_0(\mathbb{N})$ the subspace of $l^\infty(\mathbb{N})$ composed of sequences which converge to 0, endowed with the norm $\|\cdot\|_\infty$.

4. Give an onto isometry between $l^1(\mathbb{N})$ and $c_0(\mathbb{N})'$.

2 Uniformly convex spaces

This section is devoted to study the notion of uniform convexity, defined as follows

Definition 2. A normed vector space (E, N) is said to be *uniformly convex* if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in E$,

$$\big(N(x) \leq 1,\ N(y) \leq 1,\ N(x-y) > \varepsilon\big) \Rightarrow N\left(\frac{x+y}{2}\right) < 1 - \delta.$$

5. Check that a Hilbert space is uniformly convex.
6. Prove that $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$ are not uniformly convex.

2.1 Projection theorem

As for the Hilbert spaces, there is a projection theorem for the uniformly convex Banach spaces:

Theorem 3. *Let (E, N) be a uniformly convex Banach space and $C \subset E$ be a non-empty closed convex set. For all $x \in E$, there exists a unique $p_C(x) \in C$ such that*

$$N(x - p_C(x)) = d(x, C) := \inf_{y \in C} N(x - y). \tag{1}$$

The map $x \in E \mapsto p_C(x) \in C$ defined this way is continuous.

7. Check that for all $x \in E$, $d(x, C) = 0$ if and only if $x \in C$.
8. What is $p_C(x)$ when $x \in C$?
9. *Uniqueness.* Prove that the infimum in (1) can not be reached at two different points.
10. *Existence.* Let $x \in E \setminus C$. Prove that the infimum in (1) is reached at some point $p_C(x) \in C$.

2.2 James' theorem

The topological dual of the uniformly convex Banach spaces can be nicely described, under a "reasonable" assumption of differentiability for the associated norm. This is James' theorem which we prove in this subsection.

Definition 4. Let E, F be normed vector spaces and let $x \in E$. A function $f : E \to F$ is said to be Gâteaux-differentiable at x when there exists a continuous linear operator $A_x \in \mathcal{L}_c(E, F)$ such that for all $y \in E$

$$f(x + ty) = f(x) + tA_x(y) + o(t) \quad \text{as } t \to 0.$$

We denote $A_x = Df(x)$.

11. Explain the difference between the notions of Gâteaux differentiability and Fréchet differentiability (studied last year).

Let us now consider (E, N) a uniformly convex Banach space.

12. Explain why N is never Gâteaux-differentiable at 0 and prove that when N is Gâteaux-differentiable at $x \in E \setminus \{0\}$, then $\|DN(x)\| = 1$.
13. Assume that N is Gâteaux-differentiable on $E \setminus \{0\}$. Consider C a non-empty closed convex in E and p_C the associated projection. Prove that for all $x \in E \setminus C$, the point $p_C(x) \in C$ is characterized by
$$\forall z \in C, \quad DN(x - p_C(x)) \cdot (z - p_C(x)) \leq 0.$$

We can now prove the following result

Theorem 5 (James)**.** *Let (E, N) be a uniformly convex Banach space whose norm N is Gâteaux-differentiable on $E \setminus \{0\}$. Then, for all continuous linear form $L \in E'$, there exists $x \in E$ and $\lambda \in \mathbb{R}$ such that $L = \lambda DN(x)$.*

14. An auxilliary result: Let E be a vector space and $L, L_1, \ldots, L_p$ be linear forms on E such that $\bigcap_{1 \leq j \leq p} \ker L_j \subset \ker L$. Prove that $L \in \operatorname{span}(L_1, \ldots, L_p)$.
15. By considering the projection onto $\ker L$, with $L \in E'$, conclude to Theorem 5.

3 Duality of L^p spaces

In all this section, we consider $(X, \mathcal{F}, \mu)$ a measure space.

3.1 Proof of Theorem 1

Considering $p \in (1, +\infty)$, our aim is first to derive Theorem 1 from Theorem 5. On the one hand, we need to prove that the space $L^p(X)$ is uniformly convex. We will only focus on the case $p \geq 2$ in the following two questions (the case $1 < p < 2$ is treated with other arguments).

16. Prove that for all real numbers $a, b \in \mathbb{R}$,

$$\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \leq \frac{|a|^p}{2} + \frac{|b|^p}{2}.$$

17. Conclude to the uniform convexity of the space $L^p(X)$.

It now only remains to differentiate the norm $\|\cdot\|_{L^p}$.

18. Check that for all $f \in L^p(X) \setminus \{0\}$ the norm $\|\cdot\|_{L^p}$ is Gâteaux differentiable on f, with

$$D\|\cdot\|_{L^p}(f) \cdot g = \frac{1}{\|f\|_{L^p}^{p-1}} \int_X \operatorname{sgn}(f)|f|^{p-1} g \, \mathrm{d}\mu, \quad g \in L^p(X).$$

19. Conclude to Theorem 1.

3.2 Case $p = 1$

In this subsection, we check that Theorem 1 does not hold in the case where $p = 1$ when the measure μ is not σ-finite, and holds under this extra assumption.

20. Assume that $X = \{a, b\}$, with $a \neq b \in \mathbb{R}$, $\mathcal{F} = \mathcal{P}(X)$ and that the measure μ is given by

$$\mu(\{a\}) = +\infty \quad \text{and} \quad \mu(\{b\}) = 0.$$

Notice that the measure space $(X, \mathcal{F}, \mu)$ is not σ-finite. Check that there is no onto isometry between $L^1(X)'$ and $L^\infty(X)$.

21. Assume that the measure μ is finite. By using Theorem 1 for $p = 2$, give an onto isometry between $L^1(X)'$ and $L^\infty(X)$.
22. *Bonus:* Same question when the measure μ is only assumed to be σ-finite.

3.3 Duality in norm

23. Prove that for all $p \in [1, +\infty)$ and $g \in L^p(X)$,

$$\|g\|_{L^p} = \sup_{\|f\|_{L^q}=1} \left| \int_X fg \, \mathrm{d}\mu \right|, \tag{2}$$

and that the above supremum is reached when $p \in [1, +\infty)$. Prove that (2) also holds when $p = +\infty$, provided that the mesure μ is σ-finite. Discuss this result in view of Theorem 1.