THE VOLUME OF THE NEWTON POLYTOPE OF A DISCRIMINANT

S.Yu. Orevkov

1. Statement of the result. Let $D_n = D_n(x_0, ..., x_n)$ be the *discriminant*, i.e. the polynomial in $x_0, \dots, x_n$, vanishing if and only if the polynomial $\sum_{k=0}^n x_k t^k$ has a multiple root. Example: $D_2(a, b, c) = b^2 - 4ac$.

The *Newton polytope* $\Delta(f)$ of a polynomial $f = \sum a_u x_1^{u_1} \dots x_N^{u_N}$, where $u = (u_1, \dots, u_n)$, is the convex hull in $\mathbf{R}^N$ of the set $\{u \in \mathbf{Z}^N \mid a_u \neq 0\}$. If $V \in \mathbf{R}^N$ is the affine k-plane such that the rank of the lattice $V \cap \mathbf{Z}^N$ equals k, then the k-dimensional volume vol_k on the plane V will be normalized so that the volume of the fundamental parallelepiped of the lattice is equal to one.

Denote $\Delta(D_n)$ by Q_n. Because of the evident homogenity and quasihomogenity of the discriminant, Q_n lies in the $(n-1)$-plane

$$u_0 + \dots + u_n = 2(n-1), \qquad u_1 + 2u_2 + \dots + nu_n = n(n-1). \tag{1}$$

Theorem 1. $\mathrm{vol}_{n-1} Q_n = 2^{n-1} n^{n-2}/n!$.

2. $\mathrm{vol}_{n-2} \Delta(\bar{D}_n) = (n+6)\, 2^{n-3} n^{n-5}/(n-2)!$ *for* $n \geq 3$*, where* $\bar{D}_n(y_0, \dots, y_{n-2})$ *is the discriminant of* $t^n + y_{n-2}t^{n-2} + \dots + y_0$.

Let $A \subset \mathbf{Z}^d$ be an n-point set, P_A its convex hull, $\dim P_A = d$. Following [1], denote by $\mathbf{C}^A$ the space of Laurent polynomials of the form $\sum_{a \in A} x_a t^a$, where $t = (t_1, \dots, t_d)$, $a = (a_1, \dots, a_d)$, $t^a = t_1^{a_1} \dots t_d^{a_d}$, and let us define the discriminant D_A as the polynomial in n variables $(x_a)_{a \in A}$, such that the equation $D_A = 0$ defines a hyperplane in $\mathbf{C}^A$, which is the closure of the set of all polynomials f, for which the hypersurface $\{f = 0\}$ has a singularity in the torus $(\mathbf{C} \setminus 0)^d$. Respectively, the discriminant E_A defines the closure of the set of polynomials which have a degenerate restriction at least to one face of P_A (see details in [1]). Let $N = n - d - 1 = \dim \Delta(D_A) = \dim \Delta(E_A)$.

Theorem 3. $\mathrm{vol}_N \Delta(E_A) > \left(\prod_{k=1}^d (k+1)^{i_k}\right)(N-c)!/N!$*, where* $c = i_0 - d - 1$*, and* i_k *is the number of points in* A*, which are the interior points of* k*-planes of* P_A.

Corollary. *For any* d *there exist* $C_0(d), C_1(d) > 0$*, such that* $\log \mathrm{vol}\, Q_{d,m} \geq C_0(d) + C_1(d)m^d$*, where* $Q_{d,m} = \Delta(D_A)$ *for* $A = \{a \in \mathbf{Z}^d \mid a_i \geq 0, \sum a_i \leq m\}$.

This gives a negative answer to a question of E.I. Shustin about existence of constants $B_0(d)$, $B_1(d)$, such that $\log(N!\, \mathrm{vol}\, Q_{d,m}) \leq B_0(d) + B_1(d)m^d$, where $N = C_{n+d}^d - d - 1 = \dim Q_{d,m}$. An affirmative answer would provide an expected asymptotical upper bound for the number of rigid isotopy types of projective real hypersurfaces of degree m as $m \to \infty$ (of the same order as the lower bound following from the constructions by Viro's method).

Typeset by $\mathcal{AMS}$-TeX

2. Notation. For $k \in \mathbf{Z}$ set $\bar{k} = \{1, \dots, k\}$ ($\bar{0} = \varnothing$). By S_n we denote the symmetric group: $S_n = \{\sigma : \bar{n} \to \bar{n} \,|\, \sigma(\bar{n}) = \bar{n}\}$; by $\pi_n : \mathbf{R}^{n+1} \to \mathbf{R}^{n-1}$ we denote the projection $(u_0, \dots, u_n) \mapsto (u_1, \dots, u_{n-1})$. For a finite set α, we denote its cardinality by $\#\alpha$, and we set $C^k_\alpha = \{\beta \subset \alpha \,|\, \#\beta = k\}$ (then $\#C^k_\alpha = C^k_{\#\alpha}$ is the binomial coeficient). For $\alpha \subset \mathbf{Z}$, let us denote by $\mu_\alpha : \{1, \dots, \#\alpha\} \to \alpha$, the bijection such that $\mu_\alpha(1) < \mu_\alpha(2) < \dots$. The letter m will always denote $n - 1$.

3. Q_n as the secondary polytope. According to a result due to Gelfand-Kapranov-Zelevinski [1], Q_n is combinatorially equivalent to the m-cube (recall that $m = n - 1$), and its vertices are the points $\{q_\alpha\}_{\alpha \subset \bar{m}}$, where the coordinates $(q^\alpha_0, \dots, q^\alpha_n)$ of q_α are defined as follows. If $\alpha = \{k_1, \dots, k_a\}$, $0 = k_0 < k_1 < \dots < k_a < k_{a+1} = n$, $k_{-1} = 1$, $k_{a+2} = n - 1$, then

$$q^\alpha_k = \begin{cases} k_{i+1} - k_{i-1}, & k = k_i \in \alpha \cup \{0, n\} \\ 0, & k \notin \alpha \cup \{0, n\} \end{cases}$$

4. Triangulation of a skew cube. Let $p_\alpha = (p^\alpha_1, \dots, p^\alpha_N)$ be sets in $\mathbf{R}^N$, indexed by subsets $\alpha \subset \bar{N}$, such that $p^\alpha_i > 0$ for $i \in \alpha$ and $p^\alpha_i = 0$ for $i \notin \alpha$. For a $\sigma \in S_N$, we denote by s_σ the simplex spanned on the points $p_{\sigma(\bar{k})}$, $k = 0, \dots, N$.

Lemma 1. *a). $\{s_\sigma\}_{\sigma \in S_N}$ is a triangulation of some (not necessarily convex) polyhedron P, homeomorphic to a cube (hence, $\operatorname{vol} P = \sum \operatorname{vol} s_\sigma$).*

b). If the convex hull P' of the points p_α is combinatorially equivalent to a cube (i.e. for any i, all the points$\{p_\alpha\}_{i \in \alpha}$ lye on the same $(N-1)$-face of P'), then $P' = P$.

Proof. Projecting from $p_{\bar{N}}$, let us define P inductively as the union of the cones over the intersections with the coordinate hyperplanes. □

Example: if P is a cube then $s_\sigma = \{(x_1, \dots, x_N) \in P \,|\, x_{\sigma(1)} \geq x_{\sigma(2)} \geq \dots \geq x_{\sigma(N)}\}$.

5. Recurrent relation. Let $\{s_\sigma\}_{\sigma \in S_m}$ be a triangulation of Q_n from Sect. 4.

Lemma 2. $\operatorname{vol} \pi_n(s_\sigma) = \big(\prod_{k=1}^{n-1} q^{\sigma(\bar{k})}_{\sigma(k)}\big)/(n-1)!$.

Proof. $\pi_n(q_{\sigma(\bar{0})}) = \pi_n(q_\varnothing) = 0$. Hence, $m! \operatorname{vol} \pi_n(s_\sigma) = |\det A_\sigma|$ where A_σ is the matrix composed of the vectors $\{q_{\sigma(\bar{k})}\}_{k \in \bar{m}}$ written as columns. It remains to note that $q^{\sigma(\bar{k})}_{\sigma(k)}$ is the k-th entry in the $\sigma(k)$-th row and all the entries to the left of it vanish. □

Lemma 3. $v_n = n \sum_{k=1}^{n-1} C^{k-1}_{n-2} v_k v_{n-k}$*, where* $v_1 = 1$, $v_n = (n-1)! \operatorname{vol} \pi_n(Q_n)$.

Proof. This follows from Lemmas 1 and 2, if we presents $\sum_{\sigma \in S_m}$ as $\sum_{k \in \bar{m}} \sum_{\sigma \in S^k_m}$, where $S^k_m = \{\sigma \,|\, \sigma(1) = k\}$, and then replace the innermost sum with the triple sum corresponding to the bijection $C^{k-1}_{\bar{m} \setminus \{1\}} \times S_{k-1} \times S_{m-k} \to S^k_m$, $(\alpha, \sigma_1, \sigma_2) \mapsto \sigma$ where $\sigma(1) = k$, $\sigma(i) = \mu_\alpha(\sigma(i))$ for $i < k$, $\sigma(i) = \mu_{\bar{m} \setminus (\alpha \cup \{k\})}(\sigma_2(i - k))$ for $i > k$. □

6. Identity. The Abel binomial identity can be written in the form [2; Sect.1.2.7] $\alpha\beta \sum_{k=0}^n C^k_n (\alpha + k)^{k-1} (\beta + n - k)^{n-k-1} = (\alpha + \beta)(\alpha + \beta + n)^{n-1}$. Substituting $\beta = -\alpha$, dividing by α^2 and taking the limit as $\alpha \to 0$, we get

$$\sum_{k=1}^{n-1} C^k_n k^{k-1} (n-k)^{n-k-1} = 2(n-1) n^{n-2}. \tag{2}$$

7. Proof of the theorems. From (2) and Lemma 3, we get by induction $v_n = 2^{n-1}n^{n-2}$. Let V be the plane defined by (1). Solving (1) with respect to u_0, u_1, we get a bijection $j_n : \mathbf{Z}^{n-1} \to V \cap \mathbf{Z}^{n+1}$, moreover, $|\det(j_n\pi_n)| = n$. Hence, $n \operatorname{vol} Q_n = \operatorname{vol} \pi_n(Q_n) = v_n/(n-1)!$. Theorem 1 is proved. Theorem 2 is proved similarly: using the recurrent relation $\bar{v}_n = n\sum_{k=2}^{n-1} C_{n-3}^{k-2}\bar{v}_k v_{n-k}$, we find $\bar{v}_n = (n+6)\,2^{n-3}\,n^{n-4}$ where $\bar{v}_n/(n-2)!$ is the volume of the projection of $\Delta(\bar{D}_n)$ onto the plane $y_n = 0$.

Theorem 3 follows from Lemma 1(a). According to [1], $\Delta(E_A)$ is the convex hull of the points in $\mathbf{R}^A$, corresponding to all triangulations of P_A. Let V be the set of the vertices of P_A ($i_0 = \#V$). For each $\alpha \subset A \setminus V$, let us consider any triangulation whose set of vertices is $\alpha \cup V$. The corresponding points $\{q_\alpha\} \subset \mathbf{R}^A$ lye on an M-plane ($M = N - c = \#A \setminus V$) and satisfy the hypothesis of Lemma 1(a). Hence, one can span $M!$ simplices on them so that the volume of each one is $\geq \prod (k+1)^{i_k}/M!$ (this follows from Lemma 2 and the description of Q_A, given in [1]). The points $\{q_\alpha\}$ lye on an M-dimensional section of Q_A ($\dim Q_A = N$), and this gives $M!/N!$.

References

1. I.M. Gelfand, M.M. Kapranov, A.V. Zelevinskii, *Discriminants, resultants and multidimensional determinants*, Birkhäuser, Boston, 1994.
2. I.P. Goulden, D.M. Jackson, *Combinatirial enumeration*, John Wiley and Sons, N.Y., 1983.

Steklov Math. Inst. of Russ. Acad. Sci. ul. Gubkina 8, Moscow, Russia