# QUASIPOSITIVITY TEST VIA UNITARY REPRESENTATIONS OF BRAID GROUPS AND ITS APPLICATIONS TO REAL ALGEBRAIC CURVES

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## 1. Introduction

Let $B_m = \langle \sigma_1, \dots, \sigma_{m-1} \mid \sigma_j\sigma_{j+1}\sigma_j = \sigma_{j+1}\sigma_j\sigma_{j+1},\, \sigma_j\sigma_k = \sigma_k\sigma_j \text{ for } |j-k|>1\rangle$. We call *m-braids* or *braids* the elements of $B_m$. An $m$-braid $b$ is called *quasipositive* if $b = \prod_{j=1}^k a_j\sigma_1 a_j^{-1}$ for some $a_j \in B_m$.

In a series of previouse papers [12–16] we exploited the observation that the quasipositivity of a certain braid provides a necessary condition for the reailizability of a given isotopy type by a plane real algebraic curve of a given degree. As a test for the quasipositivity we used Murasugi-Tristram signature inequality, elementary argumets based on linking numbers, or Garside normal form for braid with three strings. Here we propose a new simple test for the quasipositivity and give an example when it gives some new restrictions for real algebraic curves of 7th degree – Theorem 2.3 (see Section 2).

The test is based on the following elementary observation. Suppose we are interested if a given braid $b$ is quasipositive. If it is then the number $k$ of the factors in any quasipositive presentation is just the image of $b$ under the abelianization $B_m \to \mathbf{Z}$. Let $\rho : B_m \to SU(n)$ be any unitary representation. Then the matrix $\rho(b)$ is a product of $k$ matrices each of which is conjugated to $\rho(\sigma_1)$. A neseccary and sufficient condition for a given matrix to be presented as a product of matrices from given conjugacy classes was obtained by Agnihotri and Woodward [1] (see Section 4).

In fact, to prove Theorem 2.3, we need two refinements of this idea: we work with a mixed braid group (see Sections 3 and 5.2) and we reduce the number of strings using the following result.

**Theorem 1.4.** *Let $b' \in B_{m+1}$ be obtained from $b \in B_m$ by a positive Markov move (i.e. $b' = b\sigma_m$ after the identification of $B_m$ with the subgroup of $B_{m+1}$ generated by $\sigma_1, \dots, \sigma_{m-1}$). If $b'$ is quasipositive then $b$ is also quasipositive.*

This theorem was proved in [17]. The proof is based on the results of Gromov [9] about pseudo holomorphic curves.

In the proof of Theorem 2.3 we show that certain three 4-braids are not quasipositive in the mixed braid group $B_{3,1}$. However, to complete the classification of complex $M$-schemes of degree 7 (see Section 2), one needs to prove that two more braids are not quasipositive in $B_{3,1}$.

In Appendix we construct a one-parameter deformation the Birman-Wenzl representation [4] of $B_{3,1}$ and show that it is unitarizable for some values of its parameters. Unfortunately, these representations did not help us to prove that the remaining two braids are not quasipositive.

I am grateful to A. Stoimenov and M. Finkelberg for useful discussions.

## 2. Statement of the results on 7-th degree complex $M$-schemes

**2.1. Complex schemes of real algebraic $M$-curves of degree 7.** A real algebraic non-singular curve $A$ in $\mathbf{RP}^2$ is called an *$M$-curve* if the set of its real points $\mathbf{R}A$ has the maximal possible number of connected components $(m-1)(m-2)/2+1$ where $m$ is the degree of $A$. The complexification of an $M$-curve $A$ is divided by $\mathbf{R}A$ into 2 halves. Each half induces the boundary orientation on $\mathbf{R}A$ which is called the *complex orientation*. Suppose $\deg A$ is odd. An oval $O$ of $\mathbf{R}A$ is called *positive* if $[O]=-2[J]\in H_1(\mathbf{RP}^2\setminus \operatorname{Int} O)$ where $J$ is the odd branch of $\mathbf{R}A$, otherwise $O$ is called *negative*.

The *real scheme of* $A$ is the isotopy type of $(\mathbf{RP}^2,\mathbf{R}A)$, the *complex scheme of* $A$ is the isotopy type of $(\mathbf{RP}^2,\mathbf{R}A)$ where $\mathbf{R}A$ is considered together with the complex orientation. We shall use the notation of real and complex schemes proposed by Viro (see [20]). For instance,

$$\langle J\sqcup\beta_+\sqcup\beta_-\sqcup 1_\varepsilon\langle\alpha_+\sqcup\alpha_-\rangle\rangle,\qquad \varepsilon=\pm, \tag{1}$$

denotes the complex scheme in Figure 1. It consists of the odd branch $J$, an oval $O$ (the *non-empty oval*) which is positive for $\varepsilon=+$ and negative for $\varepsilon=-$, there are $\alpha_+$ positive and $\alpha_-$ negative ovals inside $O$ (*interior ovals*), and $\beta_+$ positive and $\beta_-$ negative ovals outside $O$ (*exterior ovals*).

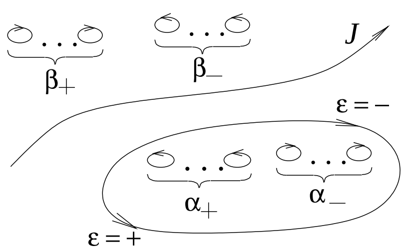


Fig. 1

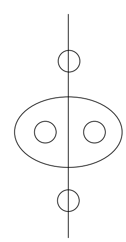

Fig. 2

The classification up to isotopy of non-singular real curves of degree 7 was obtained by Viro [19]. The complete list of all realizable real schemes is $\langle J\rangle$, $\langle J\sqcup 1\langle 1\langle 1\rangle\rangle\rangle$, and $\langle J\sqcup\beta\sqcup 1\langle\alpha\rangle\rangle$ where $\alpha+\beta\le 14$, $\alpha<14$. Here we study complex $M$-schemes of degree 7. All they are of the form (1) with $\alpha_++\alpha_-+\beta_++\beta_-=14$, $\alpha_++\alpha_-<14$ but not every such a scheme is realizable. The first restriction follows from the Rokhlin–Mishachev formula for complex orientations:

$$(\beta_+-\beta_-)+(1-2\varepsilon)(\alpha_+-\alpha_-)+\varepsilon=3. \tag{2}$$

Combining (2) with the orientation alternating rule, T. Fiedler [5] proved that

$$\begin{array}{lll} \varepsilon=(-1)^{\alpha+1}\quad\text{and}\quad \alpha_+-\alpha_-=(1-3\varepsilon)/2 & \text{(with a jump)} \\ |\alpha_+-\alpha_-|\le 1 & \text{(without jump)} \end{array} \tag{3}$$

where a curve of degree 7 is said to have a *jump* if it contains 5 ovals arranged as in Figure 2. Another formula for complex orientations [12; Sect. 1.5] yields

$$\beta_+ - \beta_- = \alpha_- - \alpha_+ = 1 \qquad \text{if} \quad \varepsilon = +1. \tag{4}$$

Using the method proposed in [12], we prove in Section 6.1 the following

**Theorem 2.1.** a). *The following complex schemes are not realizable by real algebraic $M$-curves of degree 7:*

$$\langle J \sqcup (10-k)_+ \sqcup (3-k)_- \sqcup 1_-\langle k_+ \sqcup (k+1)_-\rangle\rangle, \qquad 0 \le k \le 3, \tag{5}$$

$$\langle J \sqcup (9-k)_+ \sqcup (5-k)_- \sqcup 1_-\langle k_+ \sqcup k_-\rangle\rangle, \qquad k = 1,\ 3,4,5, \tag{6}$$

b). *Suppose that the complex scheme*

$$\langle J \sqcup 7_+ \sqcup 3_- \sqcup 1_-\langle 2_+ \sqcup 2_-\rangle\rangle \tag{7}$$

(*i.e. the scheme* (6) *with* $k=2$) *is realizable by a real algebraic curve $A$. Then there exists a point $p \in \mathbf{RP}^2$ such that $\mathbf{R}A$ is arranged as in Figure 3.* (*This arrangement is considered up to isotopies $\{h_t\}_{t\in[0,1]}$ of $\mathbf{RP}^2$ such that any $h_t(\mathbf{R}A)$ meets any line through $p$ at $\le 7$ points*). *Moreover,* $(\beta_1, \beta_2, \beta_3)$ *is one of*

$$(7,1,1), \quad (3,5,1), \quad (1,5,3), \quad (1,3,5), \quad (1,1,7). \tag{8}$$

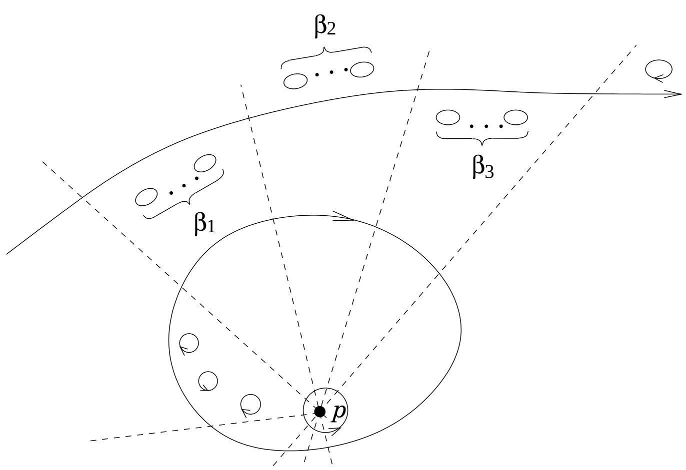


FIG. 3

Together with the prohibition [16] of (9) for $k = 5$, restrictions (2) – (4), and the constructions in [10,21,16,18], Theorem 2.1(a) shows that (7) is the only complex $M$-scheme of degree 7 whose realizability is open. Namely, one has

**Corollary 2.2.** a). *All complex schemes realizable by real algebraic $M$-curves of degree 7 without jump are* $\langle J \sqcup 9_+ \sqcup 6_-\rangle$,

$$\langle J \sqcup (7-k)_+ \sqcup (6-k)_- \sqcup 1_-\langle (k+1)_+ \sqcup k_-\rangle\rangle, \qquad 0 \le k \le 6,\ k \ne 5, \tag{9}$$

$$\langle J \sqcup (7-k)_+ \sqcup (6-k)_- \sqcup 1_+\langle k_+ \sqcup (k+1)_-\rangle\rangle, \qquad 0 \le k \le 6, \tag{10}$$

*and, maybe,* (7).

b). *All complex schemes realizable by real algebraic $M$-curves of degree* 7 *with a jump are* (10) *for* $1 \le k \le 5$ *and*

$$\langle J \sqcup (5-k)_+ \sqcup (7-k)_- \sqcup 1_-\langle (k+2)_+ \sqcup k_-\rangle\rangle, \qquad 0 \le k \le 5. \tag{11}$$

As an example of application of the methods based on unitary representations of braid groups we prove in Section 6.3 the following

**Theorem 2.3.** *Under the hypothesis of Theorem 2.1*(b), $(\beta_1, \beta_2, \beta_3)$ is either $(7,1,1)$ or $(1,1,7)$.

*Remark 2.4.* Earlier and independently, S. Fiedler-LeTouzé [6] prohibited complex schemes (5) for $k = 1,2,3$ and (6) for $k = 5$ (as well as five complex schemes excluded by (4)) using auxiliary pencils of cubics.

*Remark 2.5.* The complex schemes (5) for $k = 0$, (6) for $k = 1$, and (9) for $k = 5$, were prohibited in [15,16]. We claimed erroneously in [16] that these prohibitions together with those from [6] completed the classification.

**2.2. Complex schemes of real pseudo-holomorphic $M$-curves of degree 7.** We say that a Riemann surface $C$, embedded (or immersed) in $\mathbf{CP}^2$, is a *real pseudo-holomorphic curve* if $C$ is a $J$-holomorphic curve in some tame almost complex structure $J$ (see [9]) such that $\operatorname{Conj}(C) = C$ and $\operatorname{Conj}_* \circ J = J^{-1} \circ \operatorname{Conj}_* : T_x \to T_{\bar{x}}$ for all $x \in \mathbf{CP}^2$ (here $\operatorname{Conj} : \mathbf{CP}^2 \to \mathbf{CP}^2$ denotes complex conjugation $x \mapsto \bar{x}$). The *degree* of a pseudo-holomorphic curve $C$ is the number $m$ such that $C \sim mL$ in $H_2(\mathbf{CP}^2)$ where $L$ is a line.

Real pseudo-holomorphic curves are flexible curves in the sense of Viro [20], hence they satisfy all topological restrictions: Harnack inequality (hence, one can define *pseudo-holomorphic $M$-curves*), Petrovsky inequality, Gudkov-Arnold-Rokhlin congruence, complex orientations formula etc. Due to a result of Gromov [9], there exists a unique pseudo-holomorphic line (resp. conic) through any two (resp. five) generic points in $\mathbf{CP}^2$. If the points are real then the uniqueness implies that the line (conic) is real because otherwise the conjugate would be another line (conic). Thus, all the arguments which use auxiliary lines, conics, and pencils of lines still work in pseudo-holomorphic context.

In particular, everything in Section 2.1 before Theorem 2.1 is valid for real pseudo-holomorphic curves. Theorems 2.1 and 2.3, (and hence, Corollary 2.2) also are valid for real pseudo-holomorphic curves.

**Theorem 2.1′.** a). *Complex schemes* (5) *and* (6) *are not realizable by by real pseudo-holomorphic $M$-curves of degree* 7.

b). *Suppose that the complex scheme* (6) *with* $k = 2$ *is realizable by a real pseudo-holomorphic curve $A$. Then there exists a point $p \in \mathbf{RP}^2$ such that $\mathbf{R}A$ is arranged as in Figure 3 where* $(\beta_1, \beta_2, \beta_3)$ *is one of* (8).

**Corollary 2.2$'$.** a). *All complex schemes realizable by real pseudo-holomorphic $M$-curves of degree 7 without jump are* $\langle J \sqcup 9_+ \sqcup 6_- \rangle$, (9), (10), *and, maybe,* (7).

b). *All complex schemes realizable by real pseudo-holomorphic $M$-curves of degree 7 with a jump are* (10) *for* $1 \le k \le 5$ *and* (11).

**Theorem 2.3$'$.** *Under the hypothesis of Theorem 2.1$'$*(b), $(\beta_1, \beta_2, \beta_3)$ *is either* $(7,1,1)$ *or* $(1,1,7)$.

We prove Theorems 2.1$'$ and 2.3$'$ in Section 6. Theorems 2.1 and 2.3 are immediate consequences from Theorems 2.1$'$ and 2.3$'$.

## 3. Mixed braid groups and their unitary representations

For reader's convenience we give in this section some facts about unitary Burau representations which we use below. We do not pretend that some results in this section are new.

**3.1. Mixed braid group.** Denote by $\varphi : B_m \to S_m$ the standard homomorphism to the symmetric group which takes $\sigma_j$ into the transposition $(j, j+1)$. Let us fix a partition $m = m_1 + \dots + m_h$, $m_k > 0$ and set $s_0 = 0$, $s_k = m_1 + \dots + m_k$, $k = 1, \dots, h$. Define the *mixed braid group* $B_{m_1,\dots,m_h}$ as the set of all $b \in B_m$ such that the sets $\{s_{k-1}+1, \dots, s_k\}$, $k = 1, \dots, h$, are invariant under $\varphi(b)$. In particular, $B_{1,\dots,1}$ is the *pure braid group.*

Put $\pi_{k,l} = \prod_{j=k}^{l} \sigma_j$, i.e. $\pi_{k,k} = \sigma_k$, $\pi_{k,l} = \sigma_k \sigma_{k+1} \dots \sigma_l$ for $k < l$, and $\pi_{k,l} = \sigma_k \sigma_{k-1} \dots \sigma_l$ for $k > l$.

Using the induction by $h$, it is easy to check that $B_{m_1,\dots,m_h}$ is generated by the following $m - h + h(h-1)/2$ elements of $B_m$:

$$\begin{aligned} &\sigma_j, && j \in \{1, \dots, m-1\} \setminus \{s_1, \dots, s_{h-1}\} \\ &\pi_{s_k,s_l}\pi_{s_l,s_k}, && 1 \le k \le l < h \end{aligned}$$

In particular, $B_{n,m-n}$ is generated by $\sigma_1, \dots, \sigma_{n-1}$, $\sigma_{n+1}, \dots, \sigma_{m-1}$, and $\sigma_n^2$.

*Remark.* One can show that $B_{n,1}$ can be defined by generators $\sigma_1, \dots, \sigma_{m-1}, \tau$ (where $\tau$ corresponds to $\sigma_m^2$) and relations $\sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1}$ $(j < n-1)$, $\sigma_{n-1} \tau \sigma_{n-1} \tau = \tau \sigma_{n-1} \tau \sigma_{n-1}$, $\sigma_j \sigma_k = \sigma_k \sigma_j$ $(j+1 < k < n)$, $\sigma_j \tau = \tau \sigma_j$ $(j < n-1)$. Thus, $B_{n,1}$ is the generalised braid group associated to the system of roots $C_n$. This is the group whose representations we use in the proof of Theorem 2.3$'$.

**3.2. Mixed braid groupoid.** Recall that a *groupoid* $G$ is a category all whose morphisms are invertible. We shall consider any group as a groupoid with a single object. *Homomorphisms of groupoids* are defined as covariant functors. A *representation* (over a ring $R$) of a groupoid $G$ is a functor to the category of $R$-modules; a *representation in an $R$-module $V$* is a homomorphism $G \to \mathrm{Aut}(V)$.

Let us fix a partition $m = m_1 + \dots + m_h$, $m_k > 0$ and independent indeterminates $t_1, \dots, t_h$. Let us define the *mixed braid groupoid* (or the *groupoid of colored braids*) $\hat{B}_{m_1,\dots,m_h}[t_1, \dots, t_h]$ (or just $\hat{B}_{m_1,\dots,m_h}$). The objects are all $m$-tuples $(t_{i_1}, \dots, t_{i_m})$ such that each $t_k$ appears $m_k$ times. The morphisms are braids with $m$ strings whose strings are labeled by $t_1, \dots, t_h$ in such a way that each $t_k$ appears $m_k$ times (*admissible labelings*). The source (resp. destination) object of such a morphism is the $m$-tuple of the labels ordered according to the beginnings (resp. ends) of the strings.

**Example.** The objects of $\hat{B}_{2,1}$ are $(t_1, t_1, t_2)$, $(t_1, t_2, t_1)$, and $(t_2, t_1, t_1)$. A morphism from $(t_1, t_1, t_2)$ to $(t_1, t_2, t_1)$ is shown in Figure 5.

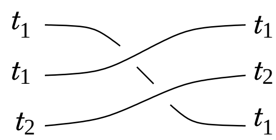


FIG. 5

The mixed braid groupoid is generated by the standard generators $\sigma_1, \dots, \sigma_{m-1}$ of the braid group supplied with all admissible labelings. The defining relations are all the defining relations of the braid group ($\sigma_j\sigma_k = \sigma_k\sigma_j$ for $|k-j| > 1$ and $\sigma_k\sigma_{k+1}\sigma_k = \sigma_{k+1}\sigma_k\sigma_{k+1}$) also supplied with all admissible labelings.

**3.3. Burau representation of a mixed braid groupoid.** Now we define the *Burau representation* $\rho : \hat{B}_{m_1,\dots,m_h}[t_1, \dots, t_h] \to GL_m(R)$ where $R$ is the ring of Laurent polynomials $\mathbf{Q}[t_1, \dots, t_h, t_1^{-1}, \dots, t_h^{-1}]$. It is sufficient to define $\rho$ on the standart generators and then to check the relations. Consider a standard generator $\sigma_k$ with a labelling $(t_{j_1}, \dots, t_{j_m})$ on the beginnings of the strings (and hence, with $(t_{j_1}, \dots, t_{j_{k+1}}, t_{j_k}, \dots, t_{j_h})$ on the ends). We associate to it the matrix

$$\begin{pmatrix} I_{k-1} & 0 & 0 & 0 \\ 0 & 1-t_{j_k} & t_{j_{k+1}} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{m-k-1} \end{pmatrix}$$

where $I_n$ denotes the identity $n \times n$ matrix. The relations (see Figure 6) are:

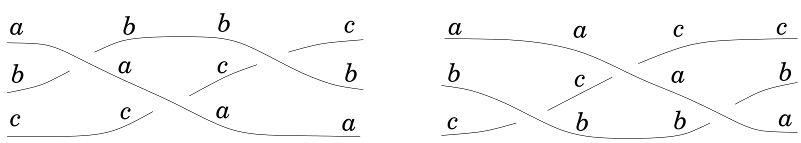


FIG. 6

$$\begin{pmatrix} 1-a & b & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-a & c \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1-b & c & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-a & c-ac & bc \\ 1-b & c & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-b & c \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1-a & c & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-a & b \\ 0 & 1 & 0 \end{pmatrix}$$

*Remark.* In essential, this representation was defined in [8; Section 6]. For any $b \in B_m$, consider its image $T$ under the standard embedding $B_m \to \mathrm{Aut}(F_m)$ where $F_m$ is the free group $\langle x_1, \dots, x_m \rangle$. Then the Burau matrix of $b$ is the image of the matrix of Fox derivatives $||\partial T(x_i)/\partial x_j||$ under the replacements $x_k \mapsto t_{j_k}$ (instead of $x_k \mapsto t$ used by Fox [8] in his interpretation of the Burau representation).

**3.4. Reduced Burau representation of a mixed braid group.** It is clear that $B_{m_1,\dots,m_h}$ coincides with the automorphism group of the object $(t_{j_1},\dots,t_{j_m})$ of $\hat{B}_{m_1,\dots,m_h}$ where

$$(j_1,\dots,j_m) = (\underbrace{1,\dots,1}_{m_1},\underbrace{2,\dots,2}_{m_2},\ \dots,\ \underbrace{h,\dots,h}_{m_h}) \tag{12}$$

Hence, the Burau representation of $\hat{B}_{m_1,\dots,m_h}$ induces a representation $\rho : B_{m_1,\dots,m_h} \to GL_m(R)$ which is also called the *Burau representation*. For the pure braid group $B_{1,\dots,1}$, this is the Gassner representation.

This representation is reducible. To extract a non-trivial irreducible factor, we set $\rho_1(b) = A^{-1}\rho(b)A$ where $A$ is the $m \times m$ matrix $||a_{ik}||$ whose the only non-zero entries are $a_{ii} = t_{j_i}$, $a_{i+1,i} = -1$ (here $j_1,\dots,j_m$ are as in (12)). Then one can delete the last row and column, and the obtained representation is called the *reduced Burau representation* of the mixed braid group.

The images of the generators of $B_{m_1,\dots,m_h}$ (see Section 3.1) are $(m-1)\times(m-1)$ matrices obtained by deleting the first and the last rows and columns from the following $(m+1)\times(m+1)$ matrices:

$$\sigma_k \overset{\rho_1}{\mapsto} \begin{pmatrix} I_{k-1} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -t_{j_k} & t_{j_k} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{m-k-1} \end{pmatrix}, \qquad k \neq s_1,\dots,s_{h-1};$$

$$\pi_{s_k,s_l}\pi_{s_l,s_k} \overset{\rho_1}{\mapsto} \begin{pmatrix} I_{s_k-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1-a_{u+1} & ta_{u+1} & 0 & (1-t)a_{u+1} & 0 \\ 0 & 1-a_u & t(a_u-1) & & (1-t)a_u & 0 \\ \vdots & \vdots & \vdots & tI_u & \vdots & \vdots \\ 0 & 1-a_1 & t(a_1-1) & & (1-t)a_1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{m-s_l-1} \end{pmatrix}$$

where $1 \le k \le l < h$, $t = t_k$, $u = s_l - s_k$, and $a_1 = t_{j_p}$, $a_2 = t_{j_p}t_{j_{p-1}}$, $\dots$, $a_{u+1} = t_{i_p}t_{j_{p-1}}\dots t_{j_{p-u}}$ for $p = s_l + 1$. In particular,

$$\pi_{s_k,s_k}\pi_{s_k,s_k} = \sigma_{s_k}^2 \overset{\rho_1}{\mapsto} \begin{pmatrix} I_{s_k-1} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1-t_{k+1} & t_k t_{k+1} & t_{k+1}(1-t_k) & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{m-s_k-1} \end{pmatrix}.$$

**Proposition 3.1.** *Let $e_1,\dots,e_{m-1}$ be the base of $R^{n-1}$ (in which the matrices $\rho_1(b)$ are written).*

*The eigenvalues of $\rho_1(\sigma_k)$ for $k \neq s_1,\dots,s_{h-1}$ are $-t_{j_k}$ (the corresponding eigenvector is $e_k$), $1,\dots,1$.*

*The eigenvalues of $\rho_1(\sigma_{s_k}^2)$ for $k = 1,\dots,h-1$ are $t_k t_{k+1}$ (the corresponding eigenvector is $e_{s_k}$), $1,\dots,1$.* □

**3.5. Invariant Hermitian form for the Burau representation.** Let us introduce new variables $x_1,\dots,x_h$ such that $x_k^2=t_k$. We shall construct an invariant form which is anti-Hermitian with respect to the conjugation defined by $x_k\mapsto x_k^{-1}$.

Let $j_1,\dots,j_m$ be as in (12). Define

$$\rho_2 : B_{m_1,\dots,m_h}\to GL_{m-1}\big(\mathbf{Q}[\,x_1,\dots,x_h,\ x_1^{-1},\dots,x_h^{-1}\,]\big)$$

as $\rho_2(b)=D\rho_1(b)D^{-1}$ where $D$ is the diagonal matrix with $(x_{j_1},\,x_{j_1}x_{j_2},\,\dots,\,x_{j_1}x_{j_2}\dots x_{j_{m-1}})$ on the diagonal. Then the images of the generators are as follows (again, we write $(m+1)\times(m+1)$ matrices whose the first and the last rows and columns should be deleted).

$$\sigma_k\overset{\rho_2}{\mapsto}\begin{pmatrix} I_{k-1} & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & x_{j_k} & -x_{j_k}^2 & x_{j_k} & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & I_{m-k-1}\end{pmatrix},\quad k\ne s_1,\dots,s_{h-1};\qquad \pi_{s_k,s_l}\pi_{s_l,s_k}\overset{\rho_2}{\mapsto}$$

$$\begin{pmatrix} I_{s_k-1} & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & \beta_1(1-a_{u+1}) & x\beta_1 a_{u+1} & 0 & (1-t)\alpha_{u+1} & 0\\ 0 & \beta_2(1-a_u) & x\beta_2(a_u-1) & & (1-t)\alpha_u & 0\\ \vdots & \vdots & \vdots & tI_u & \vdots & \vdots\\ 0 & \beta_{u+1}(1-a_1) & x\beta_{u+1}(a_1-1) & & (1-t)\alpha_1 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & I_{m-s_l-1}\end{pmatrix}$$

where $1\le k\le l<h,\ x=x_k, t=x^2=t_k,\ u=s_l-s_k;\ a_1=t_{j_p}, a_2=t_{j_p}t_{j_{p-1}},\dots,$ $a_{u+1}=t_{i_p}t_{j_{p-1}}\dots t_{j_{p-u}}, \alpha_1=x_{j_p}, \alpha_2=x_{j_p}x_{j_{p-1}},\dots,\ \alpha_{u+1}=x_{j_p}x_{j_{p-1}}\dots x_{j_{p-u}}$ for $p=s_l+1$ (i.e. $a_j=\alpha_j^2$); $\beta_1=x_{j_q}, \beta_2=x_{j_q}x_{j_{q+1}},\dots\ \beta_{u+1}=x_{j_q}x_{j_{q+1}}\dots x_{j_{q+u}}$ for $q=s_k$. In particular,

$$\pi_{s_k,s_k}\pi_{s_k,s_k}=\sigma_{s_k}^2\overset{\rho_2}{\mapsto}\begin{pmatrix} I_{s_k-1} & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & x_k(1-t_{k+1}) & t_kt_{k+1} & x_{k+1}(1-t_k) & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & I_{m-s_k-1}\end{pmatrix}.$$

The invariant Hermitian form is defined by the $(m-1)\times(m-1)$-matrix

$$Q=||q_{ik}||,\quad q_{ik}=\begin{cases}\frac{t_{j_i}t_{j_{i+1}}-1}{(t_{j_i}-1)(t_{j_{i+1}}-1)} & \text{if } k=i,\\ -\frac{x_{j_n}}{t_{j_n}-1} & \text{if } k=i\pm1,\ n=\max(k,i),\\ 0 & \text{otherwise}\end{cases}$$

For example, if $m=6$, $h=3$, $(m_1,m_2,m_3)=(2,3,1)$ then

$$Q=\begin{pmatrix} \frac{t_1+1}{t_1-1} & -\frac{x_1}{t_1-1} & 0 & 0 & 0\\ -\frac{x_1}{t_1-1} & \frac{t_1t_2-1}{(t_1-1)(t_2-1)} & -\frac{x_2}{t_2-1} & 0 & 0\\ 0 & -\frac{x_2}{t_2-1} & \frac{t_2+1}{(t_2-1)} & -\frac{x_2}{t_2-1} & 0\\ 0 & 0 & -\frac{x_2}{t_2-1} & \frac{t_2+1}{(t_2-1)} & -\frac{x_2}{t_2-1}\\ 0 & 0 & 0 & -\frac{x_2}{t_2-1} & \frac{t_2t_3-1}{(t_2-1)(t_3-1)}\end{pmatrix}.$$

For any $B = \rho_2(b)$, $b \in B_{m_1,\dots,m_h}$, one has $B^*QB = Q$ where $B^*$ denotes the image of the transpose of $B$ under the automorphism of conjugation defined by $x_k \to x_k^{-1}$, $k = 1,\dots,h$. Of course, it suffices to check this for the generators.

It is clear that $Q^* = -Q$.

**3.6. Unitary Burau representations $B_{m_1,\dots,m_h} \to U(m-1)$.** If $t_1,\dots,t_h$ are complex numbers such that $|t_j| = 1$, $t_j \neq 1$, $j = 1,\dots,h$ then the above representation preserves the Hermitian form $iQ$. Now we shall study when $iQ$ is definite using Sylvester's rule. Let $\Delta_0(Q) = 1, \Delta_1(Q), \dots, \Delta_{m-1}(Q) = \det Q$ be the sequence of principal minors of the matrix $Q$. One can easily show by induction that

$$\Delta_{k-1}(Q) = \frac{t_{j_1}t_{j_2}\dots t_{j_k} - 1}{(t_{j_1}-1)(t_{j_2}-1)\dots(t_{j_k}-1)}\,, \quad k = 1,\dots,m.$$

Hence, the principal minors of the matrix $iQ$ are

$$\Delta_{k-1}(iQ) = i^{k-1}\Delta_{k-1}(Q) = \frac{2\,\mathrm{Im}(x_{j_1}\dots x_{j_k})}{2^k\,\mathrm{Im}\,x_{j_1}\dots\mathrm{Im}\,x_{j_k}}\,, \quad k = 1,\dots,m.$$

Set $t_j = \exp(2\pi i\,\theta_j)$, $0 < \theta_j < 1$ for $j = 1,\dots,h$.

**Proposition 3.2.** *$iQ$ is positive definite if and only if $\theta_{j_1} + \theta_{j_2} + \dots + \theta_{j_m} < 1$.*
*$iQ$ is negative definite if and only if $(1-\theta_{j_1}) + \dots + (1-\theta_{j_m}) < 1$.*

*Proof.* Since all $\Delta_k(iQ)$'s are functions of $t_j$'s, the fact that $iQ$ is definite does not depend on the choice of signs of $x_j$'s. Hence, we may assume that $x_j = \exp(\pi i\theta_j)$. Then all $\mathrm{Im}\,x_j > 0$, hence $iQ$ is positive definite if and only if

$$\mathrm{Im}(x_{j_1}\dots x_{j_k}) = \sin\big(\pi(\theta_{j_1} + \dots + \theta_{j_k})\big) > 0 \quad \text{for all } k. \quad \square$$

## 4. When a matrix can be written as a product of unitary matrices with given eigenvalues (after S. Agnihotri and C. Woodward)

A complete answer to the question from the title of this section is obtained in [1] in terms of quantum Schubert calculus developed in [2, 3]. Ealier, the infinitesimal analogue of this question (about a sum of Hermitian matrices) was answered by Klyachko and Helmke-Rosenthal in terms of classical Schubert calculus. Here we collect from [1, 2] all information needed to formulate the answer in a closed form suitable for computations but we do not discuss the quantum cohomology nature of the involved objects.

**4.1. Quantum Schubert calculus.** Fix positive integers $k$ and $r$, and set $n = k + r$. Let $q, \sigma_1, \dots, \sigma_k$ be indeterminates,[1] with $\sigma_i$ of degree $i$ and $q$ of degree $n$. We also set $\sigma_0 = 1$ and $\sigma_i = 0$ for $i < 0$ or $i > k$. Define polynomials

$$Y_m = Y_m(\sigma_1,\dots,\sigma_k) = \det ||\sigma_{1+j-i}||_{1\le i,j\le m}\,.$$

Let $A = A(r, n)$ be the commutative algebra over $\mathbf{Z}[q]$ defined by

$$A = \mathbf{Z}[q,\sigma_1,\dots,\sigma_k]/(Y_{r+1},\dots,Y_{n-1},Y_n + (-1)^k q)$$

[1] No relation with the generators of the braid group.

(this is the quantum cohomology ring of the Grassmanian $G(r,n)$; see [2]).

Let $\mathcal{A}=\mathcal{A}(r,n)=\{\,(a_1,\dots,a_r)\mid k\ge a_1\ge\dots\ge a_r\ge 0\,\}$. For $\vec a\in\mathcal{A}$ we define the element $\sigma_{\vec a}\in A$ by the formula

$$\sigma_{\vec a}=\det||\sigma_{a_i+j-i}||_{1,\le i,j\le r}\,. \tag{13}$$

In particular, $\sigma_i=\sigma_{(i,0,\dots,0)}$, so we may abbriviate $\sigma_{(a_1,\dots,a_j,0,\dots,0)}$ to $\sigma_{a_1\dots a_j}$.

The elements $\{\sigma_{\vec a}\}_{\vec a\in\mathcal{A}}$ form a base of $A$ as a $\mathbf{Z}[q]$-module. For $\vec a\in\mathcal{A}$ let $|\vec a|=a_1+\dots+a_r$. The multiplication in $A$ can be computed using the *quantum Pieri formula* due to Bertrand [2]:

$$\sigma_{\vec a}\cdot\sigma_i=\sum_{\vec b}\sigma_{\vec b}+q\sum_{\vec c}\sigma_{\vec c}, \tag{14}$$

the first sum over $\vec b=(b_1,\dots,b_r)\in\mathcal{A}$ with

$$|\vec b|=|\vec a|+i \qquad\text{and}\quad k\ge b_1\ge a_1\ge\dots\ge b_r\ge a_r\ge 0$$

and the second sum over $\vec c=(c_1,\dots,c_r)\in\mathcal{A}$ with

$$|\vec b|=|\vec a|+i-n \qquad\text{and}\quad a_1-1\ge c_1\ge a_2-1\ge\dots\ge a_r-1\ge c_r\ge 0.$$

Given $\vec a,\vec b\in\mathcal{A}$, one can expand $\sigma_{\vec a}\cdot\sigma_{\vec b}$ over the base $\{\sigma_{\vec c}\}_{\vec c\in\mathcal{A}}$ using (13) and (14). Indeed, one replaces $\sigma_{\vec b}$ by a polynomial in $\sigma_1,\dots,\sigma_k$ using (13) and then apply successively (14). (One can also use a formula from [3] for the coefficient of $\sigma_{\vec c}$ in $\sigma_{\vec a}\cdot\sigma_{\vec b}$ in terms of rim hooks of Young tableaux.) The quantum cohomology nature of the multiplication ensures the fact (non-evident a priori) that all the coefficients are non-negative.

**Example 4.1.** If $k=r=2$ then $\sigma_{21}^2=\sigma_{21}\sigma_1\sigma_2=(q+\sigma_{22})\sigma_2=q\sigma_2+q\sigma_{11}$.

**Example 4.2.** If $r=1$ then $A=\mathbf{Z}[\sigma_1]$, $q=\sigma_1^n$, $\sigma_j=\sigma_1^j$ for $j=0,\dots,k$.

**4.2. Agnihotri-Woodward inequalities.** Let the notation be as in Section 4.1. For $\vec a=(a_1,\dots,a_r)\in\mathcal{A}$ let $I(\vec a)=\{i_1,\dots,i_r\}\subset\{1,\dots,n\}$ where $i_j=k+j-a_j$ (it is clear that $0<i_1<i_2<\dots<i_r\le n$). Let

$$\mathfrak{t}_+=\{\,(\lambda_1,\dots,\lambda_n)\in\mathbf{R}^n\mid\lambda_1\ge\dots\ge\lambda_n,\ \lambda_1+\dots+\lambda_n=0,\ \text{ and }\ \lambda_1-\lambda_n\le 1\,\}$$

For any $B\in SU(n)$ there exist a unique vector $\lambda(B)=(\lambda_1(B),\dots,\lambda_n(B))\in\mathfrak{t}_+$ such that the eigenvalues of $B$ are $\exp(2\pi i\lambda_1(B)),\dots,\exp(2\pi i\lambda_n(B))$. This is a bijection between conjugacy classes of $SU(n)$ and points of $\mathfrak{t}_+$.

**Theorem 4.3.** (Agnihotri-Woodward [1]). *Let $B_1,\dots,B_l\in SU(n)$. A matrix $B\in SU(n)$ can be written in the form $B=\prod_{j=1}^{l}A_jB_jA_j^{-1}$ for some $A_1,\dots,A_l\in SU(n)$ if and only if for any $r=1,\dots,n-1$, for any $\vec a_1,\dots,\vec a_l\in\mathcal{A}(r,n)$, and for any $\vec a\in\mathcal{A}(r,n)$ such that $q^d\sigma_{\vec a}$ appears in $\sigma_{\vec a_1}\cdot\dots\cdot\sigma_{\vec a_l}$ with a non-zero coefficient one has*

$$\sum_{i\in I(\vec a_1)}\lambda_i(B_1)+\sum_{i\in I(\vec a_2)}\lambda_i(B_2)+\dots+\sum_{i\in I(\vec a_l)}\lambda_i(B_l)\ \le\ d+\sum_{i\in I(\vec a)}\lambda_i(B)\,. \tag{15}$$

*Remark 4.4.* It is clear that $\lambda(B^{-1})=\big(-\lambda_n(B),\dots,-\lambda_1(B)\big)$. A symmetry of the quantum multiplication implies that every inequality (15) written for $B_1^{-1},\dots,B_l^{-1}$, and $B^{-1}$ coincides with a certain inequality of the same form (with $n-r$ instead of $r$) written for $B_1,\dots,B_l$, and $B$ (see [1]).

**Example 4.5.** (See Example 4.2 and Remark 4.4). If $n = 3$ then (15) takes form

$$\lambda_{3-a_1}(B_1) + \cdots + \lambda_{3-a_l}(B_l) \le \lambda_{3-a}(B) + d,$$
$$\lambda_{1+a_1}(B_1) + \cdots + \lambda_{1+a_l}(B_l) \ge \lambda_{1+a}(B) - d$$

for all $0 \le a_1, \dots, a_l \le 2$ where $a, d$ are defined by $a_1 + \cdots + a_l = 3d + a$, $0 \le a \le 2$.

**Example 4.6.** Let $B_1 \in SU(3)$, $\lambda(B_1) = (2\lambda, -\lambda, -\lambda)$. A matrix $B \in SU(3)$ can be written in the form $B = \prod_{j=1}^4 A_j B_1 A_j^{-1}$ for some $A_1, \dots, A_4 \in SU(3)$ if and only if $(\lambda_1, \lambda_2, \lambda_3) = \lambda(B)$ satisfies the inequalities

$$\begin{array}{rrr}
-1-\lambda \le \lambda_1 \le 2-4\lambda, & -1-4\lambda \le \lambda_2 \le 1-4\lambda, & -1-4\lambda \le \lambda_3 \le 1-4\lambda,\\
-4\lambda \le \lambda_1 \le 1-\lambda, & -1-\lambda \le \lambda_2 \le 2-4\lambda, & -1-\lambda \le \lambda_3 \le 2-4\lambda,\\
-\lambda \le \lambda_1 \le 2-\lambda, & -4\lambda \le \lambda_2 \le 1-\lambda, & -4\lambda \le \lambda_3 \le 1-\lambda,\\
-1-2\lambda \le \lambda_1 \le 8\lambda, & -1+2\lambda \le \lambda_2 \le 5\lambda, & -2+2\lambda \le \lambda_3 \le 2\lambda,\\
-2+8\lambda \le \lambda_1 \le 1+2\lambda, & -2+5\lambda \le \lambda_2 \le 1+2\lambda, & -2+5\lambda \le \lambda_3 \le 5\lambda.
\end{array}$$

Since $0 \le 3\lambda = \lambda_1(B_1) - \lambda_3(B_1) \le 1$, some of these inequalities follow from others. Removing them we obtain

$$\max(-\lambda, -2+8\lambda) \le \lambda_1 \le \min(8\lambda, 1-\lambda), \tag{16}$$

$$-1+2\lambda \le \lambda_2 \le \min(1-4\lambda, 5\lambda), \tag{17}$$

$$\max(-4\lambda, -2+5\lambda) \le \lambda_3 \le 2\lambda. \tag{18}$$

## 5. Methods to get restrictions for real curves

In this section we

(1) recall the method of restriction for the topology of real algebraic (and real pseudo-holomorphic) curves proposed in [12];
(2) introduce a new method based on Agnihotri-Woodward inequalities (Section 4.2) for unitary representations of mixed braid groups.

These methods will be used in Section 6 for the proof of Theorems 2.1′ and 2.3′ respectively.

**5.1. Pseudo-holomorphic curves and quasipositive braids.** Let us recall the main construction from [12]. Let $\mathbf{R}A = A \cap \mathbf{RP}^2$ where $A$ is a nodal real pseudo-holomorphic curve of degree $m$. Let $\mathcal{L}_p$ be the pencil of pseudo-holomorphic lines through a generic point $p$. We shall suppose that any real line $l \in \mathcal{L}_p$ meets $A$ at least at $m-2$ real points (counting the multiplicities) and there exists a real line $l_\infty \in \mathcal{L}_p$ meeting $A$ transversally at $m$ distinct real points.

We encode the arrangement of $\mathbf{R}A$ with respect to $\mathcal{L}_p$ (the *$\mathcal{L}_p$-scheme of* $\mathbf{R}A$) as follows. Choose affine coordinates in $\mathbf{R}^2 = \mathbf{RP}^2 \setminus \mathbf{R}l_\infty$ so that the lines of $\mathcal{L}_p$ are vertical. Let us move a vertical ruler from left to the right. Each time it is tangent to $\mathbf{R}A$ at a point $x$ we write the symbol $\supset_{k+1}$ (if $\mathbf{R}A$ is to the left of $x$) or $\subset_{k+1}$ (if $\mathbf{R}A$ is to the right of $x$). When the ruler passes through a double point $x$, we

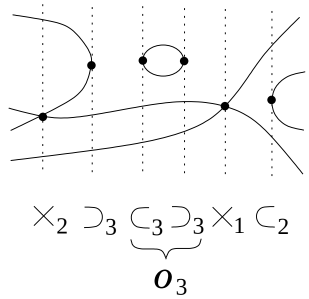


FIG. 4

write $\times_{k+1}$. In all the cases $k$ is the number of the intersection points below $x$ (see Figure 4). We abbreviate $\subset_k \supset_k$ to $o_k$ (an empty oval in the $k$-th horisontal band).

If $l_t$, $t \in [0,1]$, is a generic closed path in $\mathcal{L}_p$, it defines a braid on $m$ strings $b \in B_m$. Indeed, the $m$ points $\mathbf{C}l_t \cap \mathbf{C}A$ travel on the complex plane (we may identify all $l_t$ using a complexification of the chosen affine coordinates). Let $b$ be the braid corresponding to a simple closed path surrounding all the lines from the upper half-plane of $\mathcal{L}_p$ which are tangent to $A$ (when we say of the upper half-plane, we identify $\mathcal{L}_p$ with $\mathbf{CP}^1$).

Let $\mathrm{pr}_p : \mathbf{CP}^2 \setminus p \to \mathbf{CP}^1$ be the projection along the lines of $\mathcal{L}_p$. Since $p$ is a generic point, all the ramifications of $\mathrm{pr}_p|_A$ are simple. The fact that $A$ and $l_t$'s are pseudo-holomorphic implies that all the ramifications are positive, i.e. the braid associated to a small loop around any branch point is $\sigma_j$ for some $j = 1, \dots, m-1$. Hence, the obtained braid is quasipositive.

Under the assumptions about $\mathcal{L}_p$ formulated in the beginning of this section, the braid $b$ is determined by the $\mathcal{L}_p$-scheme of $\mathbf{R}A$ and it can be easily computed as follows. Put $\pi_{k,l} = \sigma_k \dots \sigma_l$ and $\Delta_m = \pi_{1,m-1}\pi_{1,m-2}\dots\pi_{1,1}$. Then $b = b_{\mathbf{R}}\Delta_m$ where $b_{\mathbf{R}}$ is obtained from the encoding word by the following algorithm:

$(i)$. Replace each subword $\supset_k \times_{i_1} \dots \times_{i_n} \subset_l$ with $\sigma_k^{-1}\delta_1 \dots \delta_n \tau_{k,l}$ where

$$\delta_j = \begin{cases} \sigma_{i_j}^{-1}, & i_j < k-1, \\ \sigma_{i_j+2}^{-1}, & i_j > k-1, \\ \tau_{k,k+1}\sigma_{k+1}^{-1}\tau_{k+1,k}, & i_j = k-1; \end{cases} \qquad \tau_{k,l} = \begin{cases} \pi_{l,k+1}^{-1}\pi_{k,l-1}, & k < l, \\ \pi_{l,k-1}^{-1}\pi_{k,l+1}, & k > l, \\ 1, & k = l. \end{cases}$$

$(ii)$. Replace each $\times_k$ (which was not replaced in the step $(i)$) with $\sigma_k^{-1}$.

As we already mentioned, the quasipositivity of the braid $b$ is a necessary condition for the realizability of a given $\mathcal{L}_p$-scheme as the set of real points of some real pseudo-holomorphic curve. It can be easily seen (see [7; Section 4] for details) that this condition is sufficient too. Namely,

**Proposition 5.1.** *Let $C \subset \mathbf{RP}^2$ be a union of immersed real circles with transversal intersections. Let $p$ be a generic point in $\mathbf{RP}^2$ such that almost any real line through $p$ meets $C$ at least at $m-2$ points. Denote by $\mathbf{R}\mathcal{L}_p$ the pencil of real lines through $p$. Let $b \in B_m$ be the braid constructed from $(C, \mathbf{R}\mathcal{L}_p)$ by the above procedure. Then $b$ is quasipositive if and only if there exist a tame* Conj*-invariant almost complex structure $J$ in $\mathbf{CP}^2$ and a $J$-holomorphic* Conj*-invariant curve $A \subset \mathbf{CP}^2$*

*such that $C = A \cap \mathbf{RP}^2$ and each line from $\mathbf{R}\mathcal{L}_p$ is the set of real points of some real $J$-holomorphic line.*

*Remark 5.2.* Proposition 5.1 can be reformulated without the "$(m-2)$-hypothesis". In general case, the $\mathcal{L}_p$-scheme of $C$ determines a subset $\mathcal{B}$ of the set of conjugacy classes of $B_m$ ($\operatorname{Card}\mathcal{B} = 1$ under the "$(m-2)$-hypothesis"; see [12]). $C$ is pseudo-holomorphically realizable if and only if $\mathcal{B}$ contains a quasipositive braid.

*Remark 5.3.* Any $\mathcal{L}_p$-flexible curve in the sence of [12] is a real pseudo-holomorphic curve and any real pseudo-holomorphic curve is an $\mathcal{L}_p$-flexible curve for any generic point $p$.

**Corollary 5.4.** *Let $C, C' \subset \mathbf{RP}^2$ be unions of immersed real circles with transversal intersections. Let $p \in \mathbf{RP}^2$ be a generic point.*

a). (see [12; Proposition 3.6]) *Suppose that the $\mathcal{L}_p$-scheme of $C'$ is obtained from the $\mathcal{L}_p$-scheme of $C$ by one of the following substitutions:*

$$\times_j \supset_{j\pm1} \;\longleftrightarrow\; \times_{j\pm1} \supset_j \qquad \subset_{j\pm1} \times_j \;\longleftrightarrow\; \subset_j \times_{j\pm1} \qquad \times_j u_k \;\longleftrightarrow\; u_k \times_j \quad (19)$$

$$\subset_j \supset_{j\pm1} \;\to\; \varnothing \qquad\qquad \subset_j \supset_k \;\to\; \supset_k \subset_j \qquad (20)$$

*where $|k-j| > 1$ and "$u$" stands for one of the symbols "$\times$", "$\subset$", or "$\supset$".*

*If $C$ is realizable by a real pseudo-holomorphic curve then $C'$ is also realizable.*

b). (cp. [16; Section 4]) *Suppose that the $\mathcal{L}_p$-scheme of $C'$ is obtained from the $\mathcal{L}_p$-scheme of $C$ by the substitution*

$$\supset_j \subset_j \;\to\; \times_j. \qquad (21)$$

*Let $[\ell_1\ell_2]$ be the segment of $\mathbf{R}\mathcal{L}_p$ where the modification (21) is performed. Suppose that $\ell_1$ meets $C$ at $m$ real points.*

*If $C$ is realizable by a real pseudo-holomorphic curve then $C'$ is also realizable.*

**5.2. Quasipositivity tests.** As we have seen in Section 5.1, to prove that an $\mathcal{L}_p$-scheme is pseudo-holomorphically (and hence, algebraically) unrealizable, it suffices to show that the corresponding braid is not quasipositive. Some quasipositivity tests are described in [12, 16]. One of them (Murasugi-Tristram inequality) we shall use below. For the reader's convenience, let us formulate it.

For an oriented link $L$ in the 3-sphere and $\zeta \in \mathbf{C}$, $|\zeta| = 1$, $\zeta \neq 1$, we denote $\sigma_\zeta(L) = \operatorname{signature}(V_\zeta)$, $n_\zeta(L) = 1 + \operatorname{nullity}(V_\zeta)$ where $V_\zeta = (1-\zeta)V + (1-\bar\zeta)V^T$ for the Seifert matrix $V$ of a connected Seifert surface of $L$.

For a braid $b$, let us denote by $\hat b$ the closure of $b$ in the 3-sphere endowed with the orientation induced by the natural projection of $\hat b$ onto the circle. For a braid $b = \prod_j \sigma_{i_j}^{k_j}$ we set

$$e(b) = \sum_j k_j. \qquad (22)$$

**Proposition 5.5.** (Murasugi-Tristram inequality). *If $b \in B_m$ is quasipositive then*

$$n_\zeta(\hat b) \geq |\sigma_\zeta(\hat b)| + m - e(b) \qquad (23)$$

*for any $\zeta \in \mathbf{C}$ such that $|\zeta| = 1$ and $\zeta \neq 1$.*

Another quasipositivity test is provided by unitary representations. If a braid $b \in B_m$ is quasipositive then for any unitary representation $\rho : B_m \to SU(n)$, the image $\rho(b)$ of $b$ is the product of $e(b)$ matrices conjugated to $\rho(\sigma_1)$. Thus, Agnihotri-Woodward inequalities (see Section 5) must be satisfied. This method can be refined in two ways. The first refinement is the use of Theorem 1.4 which allows one to reduce the number of braids. The second refinement provided by passing to mixed braid groups is as follows.

Denote by $H$ the open upper half-plane of $\mathcal{L}_p$. Suppose that

$$\mathbf{C}A \cap \mathrm{pr}_p^{-1}(H) = A_1 \sqcup \cdots \sqcup A_h \tag{24}$$

is not connected. Denote the degree of $\mathrm{pr}_p|_{A_i}$ by $m_i$. Then to each generic path in $H$ we may associate a colored braid (an element of the groupoid $\hat{B}_{m_1,\dots,m_h}$) whose stings are labeled by variables $t_1,\dots,t_h$ corresponding to the components $A_1,\dots,A_h$. Let $b$ be the braid considered in Section 5.1 endowed with this coloring. Then the fact that $b$ is quasipositive, can be precised as follows:

$$b = \prod_j^{e(b)} a_j x_j a_j^{-1} \tag{25}$$

where each $x_j$ is a standard generator of $\hat{B}_{m_1,\dots,m_h}$ such that the crossing strings are of the same color.

Let $b \in \mathrm{Aut}(q)$ and let let $q_0 = (t_{j_1},\dots,t_{j_m})$ with $(j_1,\dots,j_m)$ from (12), i.e. $B_{m_1,\dots,m_h} = \mathrm{Aut}(q_0)$. Let us choose any $b_0 \in \mathrm{Mor}(q_0, q)$ and let $\iota : \mathrm{Aut}(q) \to \mathrm{Aut}(q_0) = B_{m_1,\dots,m_h}$ be the isomorphism $a \mapsto b_0 a b_0^{-1}$.

An easy exercise is to check that (25) implies that

$$\iota(b) = \Big(\prod_{j=1}^{e_1}[[\sigma_1]]\Big)\cdot\Big(\prod_{j=1}^{e_2}[[\sigma_{1+s_1}]]\Big)\cdot\dots\cdot\Big(\prod_{j=1}^{e_h}[[\sigma_{1+s_{h-1}}]]\Big)$$

where $s_i = m_1 + \dots + m_i$ (as in Section 3.1), $e_i$ is the number of branching points of $\mathrm{pr}_p|_{A_i}$, and $[[x]]$ denotes some conjugate of $x$ in $B_{m_1,\dots,m_h}$. Note, that $e_i$ is "visible" in $b$: this is the subsum in (22) corresponding to the crossings where the both strings are of the color $i$.

Thus, for any linear representation of $\rho$ of $\hat{B}_{m_1,\dots,m_h}$ whose restriction to $B_{m_1,\dots,m_h}$ is unitarisable, the eigenvalues of $\rho(\sigma_j)$, $j \notin \{s_1,\dots,s_{h-1}\}$ and those of $\rho(b)$ must satisfy the Agnihotri-Woodward inequalities.

When we have a decomposition (24)? For instance, if $A$ is *dividing* (or *of type I*), i.e. $\mathbf{C}A \setminus \mathbf{R}A = A_+ \sqcup A_-$ is not connected, then

$$\mathbf{C}A \cap \mathrm{pr}_p^{-1}(H) = \big(A_+ \cap \mathrm{pr}_p^{-1}(H)\big) \sqcup \big(A_- \cap \mathrm{pr}_p^{-1}(H)\big). \tag{26}$$

In this case, the partition of the strings of $b$ can be seen from a complex orientation of $\mathbf{R}A$ (which naturally transferts to the strings of $b$). Other examples can be found in [12].

## 6. Proof of Theorems 2.1′ and 2.3′

In this section we prove Theorems 2.1′ and 2.3′ by the methods described in Section 5. In particular, we shall use the quasipositivity test provided by Agnihotri-Woodward inequalities (Section 4.2) for the unitary Burau representation of the mixed braid group $B_{3,1}$ constructed in Section 3.6.

**6.1. Application of Murasugi-Tristram inequality (proof of Theorem 2.1′).** Note, that (3) yeilds the following

**Lemma 6.1.** *There does not exist a real pseudo-holomorphic curve of degree 7 with a jump whose complex scheme is* (5) *or* (6). □

The complex scheme (5) with $k = 0$ was prohibited in [15] (see also ?.?) and the complex scheme (6) with $k = 1$ was prohibited in [16]. So, let $A$ be a real pseudo-holomorphic $M$-curve of degree 7 without jump whose complex scheme is (5) with $k > 0$ or (6) with $k > 1$. Let $V$ be the non-empty oval of $A$ and $p_0$ a point in an empty exterior oval $v_0$. Let $v_1, \dots, v_\alpha$ be the interior ovals numbered in the natural order when viewed from $p_0$ (this means that $v_1, \dots, v_{i-1}$ are separated from $v_{i+1}, \dots, v_\alpha$ by $V \cup \ell$ where $\ell$ is a line through $p_0$ and $v_i$).

Since $A$ has no jump, Corollary 5.4b applied to the pencil of lines through an exterior oval allows us to replace $v_1$ and $v_2$ by a "figure eight" (we shall still denote its "halves" by $v_1$ and $v_2$). The orientation alternating implies that both $v_1$ and $v_\alpha$ are negative if the complex scheme is (5); $v_1$ and $v_\alpha$ are of the opposite signs if the complex scheme is (6). Since the numbering of ovals is defined up to the reversing of the order, in the latter case (the scheme (6)) we may assume that $v_1$ is positive.

Let us choose a point $p$ inside $v_1$ and let $l_\infty$ pass through $p$ and $v_2$. Using the fact that one can draw a conic through any 5 empty ovals and this conic meets the curve not more than in 14 points, one can show (one uses here that the curve has no jump) that the lines from $\mathcal{L}_p$ passing through the interior ovals $v_2, \dots, v_\alpha$ are not separated by those passing through the extarior ovals. Thus, using if necessary reductions (19) and (20), we may assume that the $\mathcal{L}_p$-scheme of $A$ has the form $[\supset_2 o_2^{\alpha-2} o_{i_1} o_{i_2} \dots o_{i_\beta} \subset_5 \times_6]$ where $i_j \in \{3,4\}$. The orientation alternating implies that the sign of the $j$-th exterior oval is $(-1)^{j+i_j}$, i.e. the 1st exterior oval is positive if $j_1 = 3$ and negative if $j_1 = 4$, the 2nd exterior oval is positive if $j_2 = 4$ and negative if $j_2 = 3$, etc.

For all the sequences $[i_1 \dots i_\beta]$ providing the complex schemes under the consideration, we check the Murasugi-Tristram inequality (23) with $\zeta = -1$. The computation shows that it is satisfied only for the $\mathcal{L}_p$-schemes

$$[\supset_2 o_2^2 o_3^{\beta_1} o_4^{\beta_2} o_3^{\beta_3} o_4 \subset_5 \times_6]$$

where $(\beta_1, \beta_2, \beta_3)$ is either one of (8) or one of

$$\begin{array}{lll} (1,1,1)\ [\theta = 6/17] & (3,1,1)\ [\theta = 6/17] & (1,3,3)\ [\theta = 5/17] \\ (1,1,3)\ [\theta = 6/17] & (5,1,1)\ [\theta = 4/17] & (1,1,5)\ [\theta = 5/17] \\ (1,3,1)\ [\theta = 6/17] & (1,5,1)\ [\theta = 4/17] & (1,7,1)\ [\theta = 2/9] \end{array}$$

The latter nine cases are excluded using (23) with $\zeta = \exp(2\pi i \theta)$ with the values of $\theta$ indicated in the brackets. Theorem 2.1′ is proved.

*Remark.* The latter five cases (8) verify (23) for all values of $\zeta$. The method of the double covering used in [16] also does not work for (8).

**6.2. Reduction of the number of strings.** Let $\Delta_m$, $\pi_{k,l}$ and $\tau_{k,l}$ be as in Section 5.1. The braids in the five remained cases are

$$c\,\sigma_4^{-1}\tau_{4,5}\sigma_6^{-1}\Delta_7 \in B_7, \qquad c = \sigma_2^{-3}\tau_{2,3}\sigma_3^{-\beta_1}\tau_{3,4}\sigma_4^{-\beta_2}\tau_{4,3}\sigma_3^{-\beta_3}\tau_{3,4}, \qquad (27)$$

where $(\beta_1,\beta_2,\beta_3)$ is one of (8). Let $w$ and $w'$ be two word in the generators $\sigma_j^{\pm1}$. We say that $w'$ is obtained from $w$ by a *positive Markov move* if $w' = w_1\sigma_m w_2$ and $w = w_1w_2$ where $w_1$ and $w_2$ are words in $\sigma_1^{\pm1},\dots,\sigma_{m-1}^{\pm1}$. The following is a sequence of transformations, each of which being either an identity in the braid group, or a conjugation ($\xrightarrow{\text{conj}}$), or the inverse of a positive Markov move ($\xrightarrow{\text{Mm}}$); see Figure 7 (where the strings are numerated from the bottom to the top, in contrary to Figures 5 and 6).

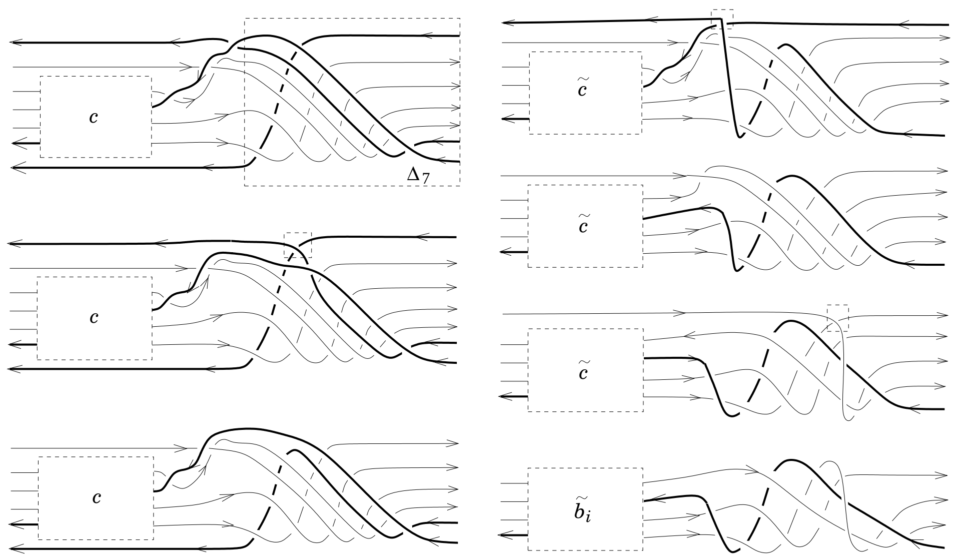


FIG. 7

$$\begin{aligned}c\,\sigma_4^{-1}\tau_{4,5}\sigma_6^{-1}\Delta_7 &= c\,\sigma_4^{-1}\tau_{4,5}\pi_{1,5}\underline{\sigma_6}\sigma_5^{-1}\Delta_6 \xrightarrow{\text{Mm}} c\,\sigma_4^{-1}\tau_{4,5}\pi_{1,4}\Delta_5\pi_{5,1}\\ &\xrightarrow{\text{conj}} \pi_{5,1}c\sigma_4^{-1}\tau_{4,5}\pi_{1,4}\Delta_5 = \tilde c\,\sigma_3^{-1}\tau_{3,4}\underline{\sigma_5}\pi_{4,1}\pi_{1,4}\Delta_5\\ &\xrightarrow{\text{Mm}} \tilde c\,\pi_{2,1}\Delta_4\pi_{3,2}\underline{\sigma_4}\pi_{3,1}\Delta_3 \xrightarrow{\text{Mm}} \tilde c\,\pi_{2,1}\Delta_4\pi_{3,2}\pi_{3,1}\Delta_3.\end{aligned}$$

This transforms the braids (27) into

$$\tilde c\,\pi_{2,1}\Delta_4\pi_{3,2}\pi_{3,1}\Delta_3 \in B_4, \qquad c = \sigma_1^{-3}\tau_{1,2}\sigma_2^{-\beta_1}\tau_{2,3}\sigma_3^{-\beta_2}\tau_{3,2}\sigma_2^{-\beta_3}\tau_{2,3}, \tag{28}$$

where $(\beta_1,\beta_2,\beta_3)$ is one of (8). Theorem 1.4 implies that if one of the braids (27) were quasipositive then the corresponding braid (28) would be quasipositive too.

**6.3. Application of Agnihotri-Woodward inequalities (proof of Theorem 2.3′).** We shall use the partition (26). The corresponding coloring of the strings is indicated in Figure 7 by arrows (a complex orientation). Let $b$ be one of the the braids (28) and let $\rho : \hat B_{3,1} \to GL_4(\mathbf{C})$ be the Burau representation from Section 3.3 with $t_j = \exp(2\pi i\theta_j)$, $j = 1,2$. We compute $\rho(b)$ for the values of $(\theta_1,\theta_2)$ given in Table 1.

For instance, all braids in (28) start by $b = \sigma_1^{-3}\sigma_2^{-1}\sigma_1\sigma_2^{-1}\dots$ and $b \in \operatorname{Aut}(q)$, $q = (t_2,t_1,t_1,t_1) \in \operatorname{Obj}(\hat B_{3,1})$, hence

$$\rho(b) = J_1(t_1,t_2)^{-1}J_1(t_2,t_1)^{-1}J_1(t_1,t_2)^{-1}J_2(t_1,t_2)^{-1}J_1(t_1,t_1)J_2(t_2,t_1)^{-1}\dots$$

where

$$J_1(a,b) = \begin{pmatrix} 1-a & b & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad J_2(a,b) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-a & b & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues of $\rho(b)$ are 1 and $\exp(2\pi i\Lambda_j)$, $j=1,2,3$ (the values of $\Lambda_j$ we computed numerically). Hence, the eigenvalues of $\rho_2(b)$ (see Section 3.5) are $\exp(2\pi i\Lambda_j)$, $j=1,2,3$. Since $3\theta_1+\theta_2<1$ (see Table 1), the representation $\rho_2$ is unitary with respect to a positive definite form by Proposition 3.2. To get the image of $\sigma_1$ in $SU(3)$, we multiply $\rho_2$ by $\exp(-2\pi i\lambda)$ where $\lambda = 1/6-\theta_1/3$

As it was explained in Section 5.2, the existence of the flexible curves would implies that a conjugate of $b$ is the product of four braids from $B_{3,1}$ conjugated to $\sigma_1$. Hence, the numbers $\lambda$ and $\lambda_j := \Lambda_j - 4\lambda$, $j=1,2,3$ must satisfy the Agnohotri-Woodward inequalities (16) – (18). One sees in Table 1 that the inequality $-4\lambda \le \lambda_3$ does not hold. This contradiction proves Theorem 2.3′.

Table 1.

| $\beta_1$ | $\beta_2$ | $\beta_3$ | $\theta_1$ | $\theta_2$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $4\lambda$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 5 | 1 | 5/54 | 91/324 | 0.43371 | 0.11348 | −0.54720 | 0.54320 |
| 1 | 5 | 3 | 2/9 | 1/108 | 0.44244 | −0.07143 | −0.37100 | 0.37037 |
| 1 | 3 | 5 | 2/9 | 1/36 | 0.41657 | −0.04018 | −0.37638 | 0.37037 |

## Appendix. Unitary Birman-Wenzl representations of $B_4$ and $B_{3,1}$

Birman and Wenzl [4] constructed a series of representations of braid groups. In particular, their constraction yeilds the the following representation $B_4 \to GL_6(\mathbf{Q}[a,a^{-1},l,l^{-1}])$ (see [4; p.271]). Set $m = a + a^{-1}$ and $\sigma_i \mapsto J_i$ where

$$J_1 = \begin{pmatrix} a^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & l^{-1} & m/l & am/l \\ 0 & 1 & 0 & 0 & m & 0 \\ 0 & 0 & 1 & 0 & 0 & m \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & l^{-1} & m/l & m & 0 & 0 \\ 1 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a^{-1} \end{pmatrix},$$

$$J_3 = \begin{pmatrix} l^{-1} & m & 0 & 0 & m/a & 0 \\ 0 & m & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

The mixed braid group $B_{3,1}$ has a presentation

$$B_{3,1} = \langle \sigma_1, \sigma_2, \tau \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2,\ \sigma_1\tau = \tau\sigma_1,\ \sigma_2\tau\sigma_2\tau = \tau\sigma_2\tau\sigma_2 \rangle$$

where the natural embedding $B_{3,1} \to B_4$ is defined by $\tau \mapsto \sigma_3^2$.

The restriction of Birman-Wenzl representation to $B_{3,1}$ has the following deformation: $\rho(\sigma_1)=J_1$, $\rho(\sigma_2)=J_2$,

$$\rho(\tau)=T=\begin{pmatrix} k^{-2} & p(k^{-1}+m) & m & 0 & p(k^{-1}+m)/a & m/a\\ 0 & ap+a^{-2} & m & 0 & 0 & 0\\ 0 & -p & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & a^{-2} & 0 & 0\\ 0 & 0 & 0 & 0 & ap+a^{-2} & m\\ 0 & 0 & 0 & 0 & -p & -1 \end{pmatrix}$$

where $p=alk^{-1}+a^{-1}$ and $k$ is a new independent parameter. Let

$$Q=\begin{pmatrix} -q_1 & 1 & q_2k & 0 & a^{-1} & q_2k/a\\ 1 & -q_3 & l^{-1} & 1 & l & 0\\ q_2/k & l & -q_1 & a^{-1} & m(a^{-1}+l) & q_2k\\ 0 & 1 & a & -q_3 & l^{-1} & a/l\\ a & l^{-1} & m(a+l^{-1}) & l & -q_3 & l^{-1}\\ q_2a/k & 0 & q_2/k & l/a & l & -q_1 \end{pmatrix},$$

$$q_1=\frac{(a-k)(a^2kl-1)}{akp(ak+1)},\quad q_2=\frac{m(al+1)}{p(ak+1)},\quad q_3=\frac{(a-l)(al-1)}{alm}.$$

One can check that $Q^*=Q$ and $B^*QB=Q$ for any $B=J_1,J_2,T$ (the involution of conjugation is defined by $a\mapsto a^{-1}$, $l\mapsto l^{-1}$, $k\mapsto k^{-1}$). The substitution $k=l$ in $T$ gives $J_3^2$. Thus, $\rho$ specializes (when $k=l$) to the above Birman-Wenzl representation of $B_4$ restricted to $B_{3,1}$.

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