PRODUCTS OF CONJUGACY CLASSES IN FINITE UNITARY GROUPS $GU(3,q^2)$ AND $SU(3,q^2)$

S.Yu. Orevkov

1. Introduction and statement of main results

1.1. Introduction. We study here the following problem (the *Class Product Problem*). Let $c_1,\dots,c_m$ be conjugacy classes in a given group. Does the unity of the group belong to their product? For the usual unitary group $SU(n)$, this problem is completely solved in [2] and [3]. Various partial cases of the class product problem (in particular, estimates for the covering number) for many groups were studied by many authors see, e. g., [1, 9, 11, 12] and numerous references therein.

In this paper we give a complete solution to the class product problem for the finite unitary groups $GU(3,q^2)$ and $SU(3,q^2)$, see §1.7 for precise statements. Due to Ennola duality (see §1.3), as a by-product, we obtain a solution for the groups $GL(3,q)$, $SL(3,q)$. For the sake of completeness, we also give in §5 a solution for the groups $GL(2,q)$, $GU(2,q^2)$ and $SU(2,q^2)\cong SL(2,q)$. A solution for corresponding projective groups PGL, PSL, PGU and PSU easily follows.

Our interest to the class product problem in all kinds of unitary groups is motivated by the study of braid monodromy of plane algebraic curves (see [14]).

As in [1, 12], the main tool used here for solving the class product problem is Burnside's formula for the *structure constants* via the character table. Namely, for a finite group Γ and its elements $x_1,\dots,x_m$, we denote the number of m-tuples $(y_1,\dots,y_m)$ such that y_i is a conjugate of x_i in Γ and $y_1\dots y_m = e$ by $N_\Gamma(x_1,\dots,x_m)$. Then Burnside's formula (see, e. g., [13; Th. I-5.8] or [1; Ch. 1, 10.1]) reads as

$$N_\Gamma(x_1,\dots,x_m) = \frac{|x_1^\Gamma|\cdot\ldots\cdot|x_m^\Gamma|}{|\Gamma|}\sum_{\chi\in\operatorname{Irr}(\Gamma)}\frac{\chi(x_1)\dots\chi(x_m)}{\chi(1)^{m-2}} \tag{1}$$

where $\operatorname{Irr}(\Gamma)$ is the set of irreducible characters of Γ and x^Γ denotes the conjugacy class of x in Γ. We denote the sum in the right hand side of (1) by $\bar N_\Gamma(x_1,\dots,x_m)$.

We use the character tables from [6] (GU/GL) and [15, 7] (SU/SL).

Acknowledgment. I am grateful to M. Geck, A. A. Klyachko, N. A. Vavilov and I. A. Vedenova for useful advises and discussions.

1.2. Determinant Relation and Rank Condition. If Γ is a subgroup of $GL(n,K)$ over any commutative field K and $A_1,\dots,A_m\in\Gamma$ are such that $I\in A_1^\Gamma\dots A_m^\Gamma$, then an evident restriction is the *determinant relation*

$$\det(A_1)\cdot\ldots\cdot\det(A_m) = 1 \tag{2}$$

Another evident restriction which takes place for any field, is the *rank condition*: if $\lambda_1 \dots \lambda_m = 1$, then

$$\operatorname{rk}(A_j - \lambda_j I) \le \sum_{i \ne j} \operatorname{rk}(A_i - \lambda_i I) \qquad \text{for any } j = 1, \dots, m \tag{3}$$

(I is the identity matrix). Indeed, if we denote the λ_i-eigenspace of A_i by V_i, then $\bigcap_{i \ne j} V_i \subset V_j$, thus $\operatorname{codim} V_j \le \operatorname{codim} \bigcap_{i \ne j} V_i \le \sum_{i \ne j} \operatorname{codim} V_i$. When $m > n$, this condition is always satisfied for any m-tuple of non-scalar matrices.

One more general restriction (see Case $(viii)$ in Theorem 1.3(a)) is

Proposition 1.1. *Let K be a perfect field and $A \sim B \in GL(3, K)$. If A does not have eigenvalues in K, then $A^{-1}B \ne \left(\begin{smallmatrix} 1&0&0\\ 1&1&0\\ 0&0&1 \end{smallmatrix}\right)$.*

Proof. Suppose the contrary. Let V be the eigenspace of $A^{-1}B$. Then $A|_V = B|_V$. Since A has no eigenvalues in K, we have $A(V) \ne V$. Let $e_2 \in V \cap A(V)$, $e_1 = A^{-1}(e_2)$, and $e_3 = A(e_2)$. Then $B(e_1) = A(e_1) = e_2$ and $B(e_2) = A(e_2) = e_3$. Thus, A and B take the canonical form in the same basis (e_1, e_2, e_3). Since $A \sim B$, this implies $A = B$. Contradiction. $\square$

It happens (see Theorem 1.3 in §1.7) that in the case of $GL(3, q)$, $q \ne 2$, there are no other restrictions on $A_1, \dots, A_m$. In the case of $GU(3, q^2)$, there are much more restrictions (see the lines in Table 2 not marked by the asterisk). An interesting question is to generalize them for any field and for any dimension.

1.3. Ennola duality and the sign convention. Throughout the paper, q is a prime power and GU (resp. SU, GL, SL) is an abbreviation of $GU(3, q^2)$ (resp. $SU(3, q^2)$, $GL(3, q)$, $SL(3, q)$) except §5 where the same convention is used with 3 replaced by 2.

Ennola [6] observed that the character tables of groups $GU(n, q^2)$ and $GL(n, q)$ are obtained from each other by changing the sign of q. The same is true for $SU(n, q^2)$ and $SL(n, q)$. Since the character table is our main tool, it is not surprising that all computations are almost the same for GU/SU and GL/SL. So, throughout the paper (except §4 and §5.3), we use the following *sign convention*: if a symbol $\pm$ or $\mp$ occurs in a formula, then the upper sign corresponds to the case of GU (resp. SU, PSU) and the lower sign corresponds to the case of GL (resp. SL, PSL). Throughout the paper (except §4 and §5.3), G (resp. S; PG; PS) stands for GU or GL (resp SU or SL; PGU or PGL; PSU or PSL) and we set

$$\delta_L = \frac{1 \mp 1}{2} = \begin{cases} 1, & G = GL, \\ 0, & G = GU. \end{cases} \tag{4}$$

1.4. Conjugacy classes in $GU(3, q^2)$ and $GL(3, q)$. Recall that $GU(3, q^2)$ is the group of 3×3 matrices A with coefficients in the finite field $\mathbb{F}_{q^2}$ such that $A^* A = I$ where $A^* = \overline{A^t}$ and $z \mapsto \bar{z}$ is the Frobenius automorphism of $\mathbb{F}_{q^2}$ defined by $z \mapsto z^q$.

We set $\Omega = \{z \in \mathbb{F}_{q^2} \mid z^{q \pm 1} = 1\}$, i. e., Ω is the multiplicative group $\mathbb{F}_q^*$ when $G = GL$ and Ω is "the unit circle" $\Omega = \{\, z \in \mathbb{F}_{q^2} \mid z\bar{z} = 1\}$ when $G = GU$.

We fix a multiplicative generator τ of $\mathbb{F}_{q^6}^*$ and we set $\rho = \tau^{q^4 + q^2 + 1}$ (a generator of $\mathbb{F}_{q^2}^*$), $\omega = \rho^{q \mp 1}$ (a generator of Ω), and $\theta = \tau^{q^3 \mp 1}$.

The conjugacy classes in $GL(n,q)$ are determined by the Jordan normal form (JNF). The conjugacy classes in $GU(n,q^2)$ have been computed in [5] and [16]. Each conjugacy class of $GU(n,q^2)$ is the intersection of $GU(n,q^2)$ with a conjugacy class of $GL(n,q^2)$, so, it is determined by JNF. The classes of GL and those of GU (represented by JNF in $GL(3,q^6)$) are listed in Table 1 which, for the reader's convenience, we reproduce from [6]. For an integer k, we denote the set $\{1,\dots,k\}$ by $[k]$. We set $R_{q^2-1} = \{k \in [q^2-1] \mid k \not\equiv 0 \mod q \mp 1\}$ and $R_{q^3\pm1} = \{k \in [q^3 \pm 1] \mid k \not\equiv 0 \mod q^2 \mp q + 1\}$.

Table 1. Conjugacy classes in G

Class	JNF over $\mathbb{F}_{q^6}$	det	class size	range of the parameters
$C_1^{(k)}$	$\begin{pmatrix}\omega^k & 0 & 0\\ 0 & \omega^k & 0\\ 0 & 0 & \omega^k\end{pmatrix}$	ω^{3k}	1	$k \in [q\pm1]$
$C_2^{(k)}$	$\begin{pmatrix}\omega^k & 0 & 0\\ 1 & \omega^k & 0\\ 0 & 0 & \omega^k\end{pmatrix}$	ω^{3k}	$(q\mp1)(q^3\pm1)$	$k \in [q\pm1]$
$C_3^{(k)}$	$\begin{pmatrix}\omega^k & 0 & 0\\ 1 & \omega^k & 0\\ 0 & 1 & \omega^k\end{pmatrix}$	ω^{3k}	$q(q^2-1)(q^3\pm1)$	$k \in [q\pm1]$
$C_4^{(k,l)}$	$\begin{pmatrix}\omega^k & 0 & 0\\ 0 & \omega^k & 0\\ 0 & 0 & \omega^l\end{pmatrix}$	ω^{2k+l}	$q^2(q^2\mp q+1)$	$(k,l)\in[q\pm1]^2$, $k\neq l$
$C_5^{(k,l)}$	$\begin{pmatrix}\omega^k & 0 & 0\\ 1 & \omega^k & 0\\ 0 & 0 & \omega^l\end{pmatrix}$	ω^{2k+l}	$q^2(q\mp1)(q^3\pm1)$	$(k,l)\in[q\pm1]^2$, $k\neq l$
$C_6^{(k,l,m)}$	$\begin{pmatrix}\omega^k & 0 & 0\\ 0 & \omega^l & 0\\ 0 & 0 & \omega^m\end{pmatrix}$	ω^{k+l+m}	$q^3(q\mp1)(q^2\mp q+1)$	$1\le k<l<m\le q\pm1$
$C_7^{(k,l)}$	$\begin{pmatrix}\omega^k & 0 & 0\\ 0 & \rho^l & 0\\ 0 & 0 & \rho^{\mp ql}\end{pmatrix}$	$\omega^{k\mp l}$	$q^3(q^3\pm1)$	$(k,l)\in[q\pm1]\times R_{q^2-1}$ $C_7^{(k,l)}=C_7^{(k,\mp ql)}$
$C_8^{(k)}$	$\begin{pmatrix}\theta^k & 0 & 0\\ 0 & \theta^{q^2k} & 0\\ 0 & 0 & \theta^{q^4k}\end{pmatrix}$	ω^k	$q^3(q\pm1)^2(q\mp1)$	$k\in R_{q^3\pm1}$ $C_8^{(k)}=C_8^{(q^2k)}=C_8^{(q^4k)}$

1.5. Conjugacy classes in $SU(3,q^2)$ and $SL(3,q)$. If 3 does not divide $q\pm1$, then $G = S\times Z(G)$ where $Z(G)=C_1\cong\Omega$ is the center of G, and hence, the classes of S are just those classes of G which are contained in S.

Let $q=3r\mp1$. In this case, the splitting of conjugacy classes in SL is described in [4; Ch. 11, §224] (see also [17]). As stated in [15], "it can be shown that the same splitting takes place in the unitary case". Each of $C_3^{(k)}$, $k=0,r,2r$, splits into three classes which we denote by $C_3^{(k,l)}$, $l=0,1,2$. The class $C_3^{(k,l)}$ in $SU(3,q^2)$ (resp. in $SL(3,q)$) consists of matrices which are conjugate in $SL(3,q^2)$ (resp. in $SL(3,q)$) to[1]

$$\begin{pmatrix}\omega^k & 0 & 0\\ z^l & \omega^k & 0\\ 0 & 1 & \omega^k\end{pmatrix}, \qquad z=\begin{cases}\rho, & S=SU(3,q^2),\\ \omega, & S=SL(3,q).\end{cases}$$

Other conjugacy classes of G contained in S are conjugacy classes of S.

[1]there is a misprint here in [15].

Proposition 1.2. *If $A \in C_3^{(k,l)}$, then $A^{-1} \in C_3^{(-k,l)}$ and $\omega^{k'} A \in C_3^{(k+k',l)}$.*

Remark. Each conjugacy class of SU is the intersection of SU with a conjugacy class of $SL(3,q^2)$. The situation is quite different for $SU(2,q^2)$, see §5.3.

1.6. Notation for eigenvalues. We denote the union of the conjugacy classes $C_i^{(\dots)}$ by C_i, $i = 1,\dots,8$. We denote the number of distinct eigenvalues of matrices from C_i by n_i and the number of distinct eigenvalues belonging to Ω by n'_i. So, we have

$$\begin{aligned} &n_1 = n_2 = n_3 = 1, \qquad n_4 = n_5 = 2, \qquad n_6 = n_7 = n_8 = 3; \\ &n'_i = n_i \quad (i = 1,\dots,6); \qquad n'_7 = 1, \qquad n'_8 = 0. \end{aligned}$$

We denote the multiplicity of an eigenvalue λ of a matrix A by $m_A(\lambda)$. Let $A \in C_i$. We denote the eigenvalues of A by $\lambda_1 = \lambda_1(A),\dots,\lambda_{n_i} = \lambda_{n_i}(A)$. We number them so that

$$m_A(\lambda_1) \ge \dots \ge m_A(\lambda_{n_i}) \quad \text{and} \quad \lambda_1,\dots,\lambda_{n'_i} \in \Omega. \tag{5}$$

For an m-tuple of matrices $\vec{A} = (A_1,\dots,A_m)$, $A_\nu \in C_{i_\nu}$, $\nu = 1,\dots,m$, we use the multi-index notation:

$$\vec{a} = (a_1,\dots,a_m), \quad [\vec{n}] = [n_{i_1}] \times \dots \times [n_{i_m}], \quad [\vec{n'}] = [n'_{i_1}] \times \dots \times [n'_{i_m}],$$

(recall that $[k]$ stands for $\{1,\dots,k\}$) and for $\vec{a} \in [\vec{n}]$ we set

$$\lambda_{\vec{a}} = \lambda_{a_1}(A_1)\dots\lambda_{a_m}(A_m), \qquad \delta_{\vec{a}} = \delta_{\vec{a}}(\vec{A}) = \begin{cases} 1, & \lambda_{\vec{a}} = 1, \\ 0, & \lambda_{\vec{a}} \ne 1. \end{cases}$$

In this notation, the rank condition (3) for $\vec{A} = (A_1,A_2,A_3)$ takes the form

$$\sum_{a=1}^{n'_{i_3}} \delta_{1,1,a} > 0 \qquad \text{if } \{i_1,i_2\} \subset \{2,4\} \tag{3$'$}$$

1.7. Statement of main results. In Theorems 1.3 and 1.6, we restrict ourselves by the case when $A_1,\dots,A_m$ are non-scalar and $m \ge 3$. To reduce the general case to this one, it is enough to know the class of the inverse of a given matrix and the class of its multiple by a scalar. For G, this is clear from JNF; for S, the answer is given in Proposition 1.2 in §1.5.

Theorem 1.3. *Let $A_1,\dots,A_m \in G \setminus C_1$, $m \ge 3$, satisfy* (2) *and* (3). *Let $A_\nu \in C_{i_\nu}$, $\nu = 1,\dots,m$.*

(a). *If $G = GU$, we suppose that one of the following conditions* (*i*)–(*vii*) *holds:*

(*i*) *$m = 3$, $i_1 \in \{6,7\}$, $i_2 \in \{3,5\}$, $i_3 \in \{2,4\}$, and $\delta_{111} = 1$;*

(*ii*) *$m = 3$, $i_1 = 5$, $i_2 \in \{3,5\}$, $i_3 \in \{2,4\}$, and $\delta_{211} = 1$;*

(*iii*) *$m = 3$, $i_1 = i_2 \in \{6,8\}$, $i_3 = 2$, and $\delta_{111}\delta_{221}\delta_{331} = 1$* (when $i_1 = i_2 = 8$, the last condition is equivalent to $\delta_{111} = 1$)*;*

(*iv*) *$m = 3$, $(i_1,i_2,i_3) = (3,2,2)$ or $(4,4,2)$;*

(*v*) *$m = 3$, $(i_1,i_2,i_3) = (5,4,4)$, and $\delta_{112}\delta_{121}\delta_{211} = 1$* (*see Remark 1.5*)*;*

(*vi*) *$m = 4$, $(i_1,i_2,i_3,i_4) = (3,2,2,2)$, and $\delta_{1111} = 1$;*

(*vii*) *$m = 4$, $(i_1,i_2,i_3,i_4) = (4,4,4,2)$, and $\delta_{1121}\delta_{1211}\delta_{2111} = 1$* (*see Remark 1.5*).

If $G = GL$, we suppose that one of the following conditions (viii)–(ix) holds:

(viii) $m = 3$, $(i_1, i_2, i_3) = (8, 8, 2)$ *and* $\delta_{111} = 1$;

(ix) $q = 2$, $m = 3$, $(i_1, i_2, i_3) = (8, 8, 3)$ *and* $A_1^G = A_2^G$.

Then $I \notin A_1^G \dots A_m^G$.

(b) *Suppose that none of the conditions of Part* (a) *holds for any permutation of* $A_1, \dots, A_m$ *and for any renumbering of the eigenvalues of the matrices under the restrictions* (5). *In the case* $G = GU$, *we suppose also that* $q \neq 2$. *Then* $I \in A_1^G \dots A_m^G$.

Remark 1.4. In Table 2 we present the list of all the cases when (2) can be satisfied for non-constant matrices $A_1, \dots, A_m \in G$, $m \geq 3$, $A_i \in C_{i_\nu}$, $i_\nu \geq 2$, but $I \notin A_1^G \dots A_m^G$ for $q > 2$. The cases marked by asterisk concern the both groups GU and GL; as stated in §1.2, in all of them except $(8, 8, 2)$ the rank condition (3) is not satisfied. The cases not marked by asterisk concern only GU.

Remark 1.5. Conditions (v) and (vii) in Theorem 1.3 mean that $(\mu_1 A_1^G, \dots, \mu_m A_m^G)$ is $\big(C_5^{(-k,0)}, C_4^{(0,k)}, C_4^{(0,k)}\big)$ or $\big(C_4^{(0,k)}, C_4^{(0,k)}, C_4^{(0,k)}, C_2^{(-k)}\big)$ for some $k \in \{1, \dots, q\}$ and for some $\mu_i \in \Omega$ with $\mu_1 \dots \mu_m = 1$.

Table 2. Cases when $\det A_1 \dots A_m = 1$, $I \notin A_1^G \dots A_m^G$ for $q > 2$ (see Remark 1.4)

$(i_1, \dots, i_m)$		$(i_1, \dots, i_m)$	
$(2,2,2)^*$	$\delta_{111} = 0$	$(6,4,2)^*$	$\delta_{111} + \delta_{211} + \delta_{311} = 0$
$(3,2,2)^*$	$\delta_{111} = 0$	$(6,4,3)$	$\delta_{111} + \delta_{211} + \delta_{311} = 1$
$(3,2,2)$	$\delta_{111} = 1$	$(6,4,4)^*$	$\delta_{111} + \delta_{211} + \delta_{311} = 0$
$(4,2,2)^*$		$(6,5,2)$	$\delta_{111} + \delta_{211} + \delta_{311} = 1$
$(4,3,2)^*$		$(6,5,4)$	$\delta_{111} + \delta_{211} + \delta_{311} = 1$
$(4,4,2)^*$	$\delta_{111} = 0$	$(6,6,2)$	$\sum_{\alpha \in S_3} \delta_{11^\alpha 1} \delta_{22^\alpha 1} \delta_{33^\alpha 1} = 1$
$(4,4,2)$	$\delta_{111} = 1$	$(7,2,2)^*$	$\delta_{111} = 0$
$(4,4,3)^*$	$\delta_{111} = 0$	$(7,3,2)$	$\delta_{111} = 1$
$(4,4,4)^*$	$\delta_{111} + \delta_{112}\delta_{121}\delta_{211} = 0$	$(7,4,2)^*$	$\delta_{111} = 0$
$(5,2,2)^*$	$\delta_{211} = 0$	$(7,4,3)$	$\delta_{111} = 1$
$(5,3,2)$	$\delta_{211} = 1$	$(7,4,4)^*$	$\delta_{111} = 0$
$(5,4,2)^*$	$\delta_{111} + \delta_{211} = 0$	$(7,5,2)$	$\delta_{111} = 1$
$(5,4,3)$	$\delta_{211} = 1$	$(7,5,4)$	$\delta_{111} = 1$
$(5,4,4)^*$	$\delta_{111} + \delta_{211} = 0$	$(8,2,2)^*$	
$(5,4,4)$	$\delta_{211}\delta_{121}\delta_{112} = 1$	$(8,4,2)^*$	
$(5,5,2)$	$\delta_{211} = 1$	$(8,4,4)^*$	
$(5,5,4)$	$\delta_{211} = 1$	$(8,8,2)^*$	$\delta_{111} + \delta_{121} + \delta_{131} = 1$
$(6,2,2)^*$	$\delta_{111} + \delta_{211} + \delta_{311} = 0$	$(3,2,2,2)$	$\delta_{1111} = 1$
$(6,3,2)$	$\delta_{111} + \delta_{211} + \delta_{311} = 1$	$(4,4,4,2)$	$\delta_{1121}\delta_{1211}\delta_{2111} = 1$

The case of $q = 2$ also is treated completely in Propositions 4.2 and 4.3 (for GU) and in Corollary 4.5 (for SU).

If 3 does not divide $q \pm 1$, then $G \cong S \times \Omega$, thus the class product problem in S reduces to that in G. Otherwise (when $3 | q \pm 1$) the solution is as follows.

Theorem 1.6. *Let* $A_1, \dots, A_m$, $m \geq 3$, *be as in Theorem 1.3. We suppose in addition that* $q = 3r \mp 1$ *and* $A_1, \dots, A_m \in S$, *recall that* S *is* $SU(3, q^2)$ *or* $SL(3, q)$.

(a). *Suppose that $m=3$.*
If $S=SU$, we suppose that

(*i*) $i_1=i_2=3$, $i_3\in\{2,4\}$, $A_1\in C_3^{(k_1,l_1)}$, $A_2\in C_3^{(k_2,l_2)}$, $l_1\ne l_2$.

If $S=SL$, we suppose that one of the following conditions (ii)–(v) holds:

(*ii*) $(i_1,i_2,i_3)=(3,3,2)$, $A_1\in C_3^{(k_1,l_1)}$, $A_2\in C_3^{(k_2,l_2)}$, $l_1\ne l_2$, *and* $\delta_{111}=0$;

(*iii*) $(i_1,i_2,i_3)=(3,3,4)$, $A_1\in C_3^{(k_1,l_1)}$, $A_2\in C_3^{(k_2,l_2)}$, $l_1\ne l_2$;

(*iv*) $q=4$, $(i_1,i_2,i_3)=(3,3,3)$, $A_\nu\in C_3^{(k_\nu,l_\nu)}$, $\nu=1,2,3$, $l_1=l_2\ne l_3$, *and* $\delta_{111}=1$;

(*v*) $q=4$, $(i_1,i_2,i_3)=(3,3,3)$, $A_\nu\in C_3^{(k_\nu,l_\nu)}$, $\nu=1,2,3$, $l_1=l_2=l_3$, *and* $\delta_{111}=0$.

Then $I\notin A_1^SA_2^SA_3^S$.

(b). *Suppose that $q>2$ and $I\in A_1^G\dots A_m^G$. Suppose that for any permutation of $A_1,\dots,A_m$, the hypothesis of Part* (a) *is not satisfied. Then $I\in A_1^S\dots A_m^S$.*

If 3 does not divide $q\pm1$, then $PG=PS=S$. If 3 divides $q\pm1$, the solution of the class product problem for PG and PS is as follows. Let $\tilde C_i^{(\dots)}$ be the conjugacy class of PG or PS corresponding to $C_i^{(\dots)}$.

Corollary 1.7. *Let $q=3r\mp1$, $q\ne2$. If $m\ge4$ (resp. $m\ge3$), then the product of any m-tuple of nontrivial conjugacy classes of PGU (resp. PGL) contains the identity matrix. All triples of nontrivial conjugacy classes of PGU which have representatives in GU satisfying* (2) *and* (3)*, but whose product does not contain the identity matrix, are*

(*i*)	$\tilde C_3^{(0)}$	$\tilde C_2^{(0)}$	$\tilde C_2^{(0)}$	
(*ii*)	$\tilde C_2^{(0)}$	$\tilde C_4^{(0,k)}$	$\tilde C_4^{(0,-k)}$	$k=1,\dots q$;
(*iii*)	$\tilde C_5^{(0,k)}$	$\tilde C_4^{(0,k)}$	$\tilde C_4^{(0,k)}$	$k=1,\dots,q,\quad k\notin\{r,2r\}$;
(*iv*)	$\tilde C_6^{(0,r,2r)}$	$\tilde C_3^{(0)}$	$\tilde C_2^{(0)}$	
(*v*)	$\tilde C_6^{(0,r,2r)}$	$\tilde C_5^{(0,k)}$	$\tilde C_4^{(0,-k)}$	$k=1,\dots,q$;
(*vi*)	$\tilde C_6^{(0,r,2r)}$	$\tilde C_6^{(0,r,2r)}$	$\tilde C_2^{(0)}$	

Corollary 1.8. *Let $q=3r\mp1$, $q\ne2$. If $m\ge4$, then the product of any m-tuple of nontrivial conjugacy classes of PS contains the identity matrix. All triples of nontrivial conjugacy classes which have representatives in S satisfying* (3)*, but whose product does not contain the identity matrix, are*

(*i*)	$\tilde C_3^{(0,l)}$	$\tilde C_2^{(0)}$	$\tilde C_2^{(0)}$	$l=0,1,2$;
(*ii*)	$\tilde C_2^{(0)}$	$\tilde C_4^{(k,-2k)}$	$\tilde C_4^{(-k,2k)}$	$k=1,\dots r-1$;
(*iii*)	$\tilde C_5^{(k,-2k)}$	$\tilde C_4^{(k,-2k)}$	$\tilde C_4^{(k,-2k)}$	$k=1,\dots,r-1,\quad 3k\notin\{r,2r\}$;
(*iv*)	$\tilde C_6^{(0,r,2r)}$	$\tilde C_3^{(0,l)}$	$\tilde C_2^{(0)}$	$l=0,1,2$;
(*v*)	$\tilde C_6^{(0,r,2r)}$	$\tilde C_5^{(k,-2k)}$	$\tilde C_4^{(k,-2k)}$	$k=1,\dots,r-1$;
(*vi*)	$\tilde C_6^{(0,r,2r)}$	$\tilde C_6^{(0,r,2r)}$	$\tilde C_2^{(0)}$	

(vii)	$\tilde C_3^{(0,l_1)}$	$\tilde C_3^{(0,l_2)}$	$\tilde C_2^{(0)}$	$0\le l_1<l_2\le 2;$
$(viii)$	$\tilde C_3^{(0,l_1)}$	$\tilde C_3^{(0,l_2)}$	$\tilde C_4^{(k,-2k)}$	$0\le l_1<l_2\le 2,\ \ k=1,\dots,r-1.$

in the case $PS=PSU$, and only the triples $(viii)$ in the case $PS=PSL$.

1.8. Covering number and extended covering number. Let Γ be a group. The *covering number* of Γ is the minimal integer m such that for any nontrivial conjugacy class c, we have $c^m=\Gamma$. It is denoted by $\mathrm{cn}(\Gamma)$. The *extended covering number* of Γ is the minimal integer m such that for any nontrivial conjugacy classes $c_1,\dots,c_m$ we have $c_1\dots c_m=\Gamma$. Covering numbers were studied in [1, 12].

Corollary 1.9.
$\mathrm{cn}(PSL)=3$ *and* $\mathrm{ecn}(PSL)=4$*;*
$\mathrm{cn}(PSU)=3$ *and* $\mathrm{ecn}(PSU)=4$ *if* $\gcd(q+1,3)=3$ *and* $q\ne 2$*;*
$\mathrm{cn}(PSU)=4$ *and* $\mathrm{ecn}(PSU)=5$ *if* $\gcd(q+1,3)=1$.

Remark 1.10. Karni [10] computed the numbers $\mathrm{cn}(PS)$ and $\mathrm{ecn}(PS)$ for $q=3,4,5$; Lev [11] proved that $\mathrm{cn}(PSL(n,K))=n$ for any $n\ge 3$ and for any field K which has more than 3 elements.

2. Class products in $GU(3,q^2)$ and $GL(3,q)$. Proof of Theorem 1.3

2.1. The character tables of $GU(3,q^2)$ and $GL(3,q)$. In this section we represent the character table of G (see [6]) in a form convenient to apply (1). The irreducible characters of G divide into 8 series parametrized by the same sets of parameters as the conjugacy classes. We denote the dimension of the irreducible representations corresponding to the j-th series by d_j. So,

$$\begin{array}{llll} d_1=1 & d_3=q^3 & d_5=q(q^2\mp q+1) & d_7=q^3\pm 1\\ d_2=q^2\mp q & d_4=q^2\mp q+1 & d_6=(q\mp 1)(q^2\mp q+1) & d_8=(q\pm 1)(q^2-1)\end{array}$$

The characters $\chi_{d_i}^{(t,\dots)}$, $i=1,\dots,8$, are irreducible and pairwise distinct only for some values of the parameters t, u, v, but we define them by the same formulas for any values of the parameters. Recall that for an integer n, we denote the set $\{1,\dots,n\}$ by $[n]$. Let

$$\begin{array}{ll} X_j=\{\chi_{d_j}^{(t)}\mid t\in[q\pm1]\},\ j=1,2,3, & X'_j=\{\chi_{d_j}^{(t,t)}\mid t\in[q\pm1]\},\ j=4,5,\\ X_j=\{\chi_{d_j}^{(t,u)}\mid (t,u)\in[q\pm1]^2\},\ j=4,5, & X'_6=\{\chi_{d_6}^{(t,u,u)}\mid (t,u)\in[q\pm1]^2\},\\ X_6=\{\chi_{d_6}^{(t,u,v)}\mid (t,u,v)\in[q\pm1]^3\}. & X'_7=\{\chi_{d_7}^{(t,(1\mp q)u)}\mid (t,u)\in[q\pm1]\},\\ X_7=\{\chi_{d_7}^{(t,u)}\mid (t,u)\in[q\pm1]\times[q^2-1]\}, & X'_8=\{\chi_{d_8}^{((q^2\mp q+1)t)}\mid t\in[q\pm1]\},\\ X_8=\{\chi_{d_8}^{(t)}\mid t\in[q^3\pm1]\}, & X''_6=\{\chi_{d_6}^{(t,t,t)}\mid t\in[q\pm1]\}\end{array}$$

and $\Xi_1=\{X_1,X_2,X_3,X'_4,X'_5,X''_6,X'_8\}$, $\Xi_2=\{X_4,X_5,X'_6,X'_7\}$, $\Xi_3=\{X_6\}$, $\Xi_4=\{X_7\}$, $\Xi_5=\{X_8\}$, $\Xi=\Xi_1\cup\dots\cup\Xi_5$. It is clear that if E is any expression depending on a character of G, then

$$\sum_{\chi\in\mathrm{Irr}(G)}E(\chi)=\sum_{X\in\Xi}s(X)\sum_{\chi\in X}E(\chi) \qquad (6)$$

where the symmetry factors $s(X)$ are given in Tables 3.1 and 3.2.

We fix a homomorphism of multiplicative groups $f:\mathbb{F}_{q^6}^*\to\mathbb{C}^*$ which takes τ to $\exp(2\pi i/(q^6-1))$, thus,

$$f(\omega)=e^{2\pi i/(q\pm1)},\quad f(\rho)=e^{2\pi i/(q^2-1)},\quad f(\theta)=e^{2\pi i/(q^3\pm1)}.$$

Let $A\in C_i$ and let $\lambda_1,\dots,\lambda_{n_i}$ be its eigenvalues numbered as in (5). Then

$$\chi^{(t)}(A)=c_i^X f(\det A)^t,\qquad \chi^{(t)}\in X\in\Xi_1$$

$$\chi^{(t,u)}(A)=\sum_{a=1}^{n_i'} c_{i,a}^X f(\lambda_a)^t f(\lambda_a^{-1}\det A)^u,\qquad \chi^{(t,u)}\in X\in\Xi_2,$$

$$\chi_{d_6}^{(t,u,v)}(A)=\sum_{\alpha\in\mathcal{A}_{6,i}} c_i^{X_6} f(\lambda_{\alpha(1)}^t\lambda_{\alpha(2)}^u\lambda_{\alpha(3)}^v),$$

$$\chi_{d_7}^{(t,u)}(A)=\sum_{\alpha\in\mathcal{A}_{7,i}} c_i^{X_7} f(\lambda_{\alpha(1)}^t\lambda_{\alpha(2)}^u),\qquad\qquad \chi_{d_8}^{(t)}(A)=\sum_{a=1}^{n_i} c_i^{X_8} f(\lambda_a^t).$$

where $\mathcal{A}_{6,i}$ and $\mathcal{A}_{7,i}$ are sets of triples $\alpha=(\alpha(1),\alpha(2),\alpha(3))$ and pairs $(\alpha(1),\alpha(2))$ respectively defined by

$$\begin{array}{lll}
\mathcal{A}_{6,i}=\{(1,1,1)\}, & \mathcal{A}_{7,i}=\{(1,1)\}, & i=1,2,3,\\
\mathcal{A}_{6,i}=\{(2,1,1),(1,2,1),(1,1,2)\}, & \mathcal{A}_{7,i}=\{(2,1)\}, & i=4,5,\\
\mathcal{A}_{6,6}=S_3, & \mathcal{A}_{7,7}=\{(1,2),(1,3)\}, & \\
\mathcal{A}_{6,7}=\mathcal{A}_{6,8}=\varnothing, & \mathcal{A}_{7,6}=\mathcal{A}_{7,8}=\varnothing. &
\end{array}$$

The coefficients c_i^X and $c_{i,a}^X$ (the latter denoted just by c_i^X in the cases when $n_i'=1$) are given in the Tables 3.1 and 3.2.

Table 3.1

X	c_1^X	c_2^X	c_3^X	c_4^X	c_5^X	c_6^X	c_7^X	c_8^X	$s(X)$
X_1	1	1	1	1	1	1	1	1	1
X_2	d_2	$\mp q$	0	$1\mp q$	1	2	0	-1	1
X_3	d_3	0	0	q	0	∓1	±1	∓1	1
X_4'	d_4	$1\mp q$	1	$2\mp q$	2	3	1	0	-1
X_5'	d_5	q	0	$2q\mp1$	-1	∓3	±1	0	-1
X_6''	d_6	$2q\mp1$	∓1	$3q\mp3$	∓3	∓6	0	0	$1/3$
X_8'	d_8	$-q\mp1$	∓1	0	0	0	0	∓3	$-1/3$
X_6	d_6	$2q\mp1$	∓1	$q\mp1$	∓1	∓1	0	0	$1/6$
X_7	d_7	±1	±1	$q\pm1$	±1	0	±1	0	$1/2$
X_8	d_8	$-q\mp1$	∓1	0	0	0	0	∓1	$1/3$

Table 3.2

X	c_2^X	c_2^X	c_3^X	$c_{4,1}^X$	$c_{4,2}^X$	$c_{5,1}^X$	$c_{5,2}^X$	$c_{6,a}^X$	c_7^X	$s(X)$
X_4	d_4	$1\mp q$	1	$1\mp q$	1	1	1	1	1	1
X_5	d_5	q	0	$q\mp1$	q	∓1	0	∓1	±1	1
X_6'	d_6	$2q\mp1$	∓1	$2(q\mp1)$	$q\mp1$	∓2	∓1	∓2	0	$-1/2$
X_7'	d_7	±1	±1	0	$q\pm1$	0	±1	0	±2	$-1/2$

2.2. Structure constant formula for $GU(3,q^2)$ and $GL(3,q)$. Let $A_1,\dots,A_m \in G$, $A_\nu \in C_{i_\nu}$, $\det A_1\dots A_m = 1$. We use the multi-index notation as explained in §1.6 and we set also

$$\vec{\mathcal{A}}_j = \mathcal{A}_{j,i_1} \times \dots \times \mathcal{A}_{j,i_m}, \quad j = 6,7.$$

Substituting the formulas from §2.1 into (1) and using (6), we obtain

$$\bar{N}_G(A_1,\dots,A_m) = \Sigma_1 + \dots + \Sigma_5$$

where Σ_i is the sum over Ξ_i:

$$\Sigma_1 = \sum_{X\in\Xi_1} s(X) \sum_{t=1}^{q\pm1} \frac{c^X_{i_1}\dots c^X_{i_m} f\big(\det A_1\dots A_m\big)^t}{(c^X_1)^{m-2}} = (q\pm1)\sum_{X\in\Xi_1}\frac{s(X)c^X_{i_1}\dots c^X_{i_m}}{(c^X_1)^{m-2}}$$

$$\begin{aligned}\Sigma_2 &= \sum_{X\in\Xi_2} s(X) \sum_{\vec a\in[\vec n']} \frac{c^X_{i_1,a_1}\dots c^X_{i_m,a_m}}{(c^X_1)^{m-2}} \sum_{t=1}^{q\pm1} f\big(\lambda_{\vec a}\big)^t \sum_{u=1}^{q\pm1} f\big(\lambda_{\vec a}^{-1}\det A_1\dots A_m\big)^u \\ &= (q\pm1)^2 \sum_{X\in\Xi_2} s(X) \sum_{\vec a\in[\vec n']} \frac{c^X_{i_1,a_1}\dots c^X_{i_m,a_m}\delta_{\vec a}}{(c^X_1)^{m-2}}\end{aligned}$$

$$\begin{aligned}\Sigma_3 &= \frac{1}{6}\sum_{\vec\alpha\in\vec{\mathcal{A}}_6} \frac{c^{X_6}_{i_1}\dots c^{X_6}_{i_m}}{d_6^{m-2}} \sum_{t=1}^{q\pm1} f(\lambda_{\vec\alpha(1)})^t \sum_{u=1}^{q\pm1} f(\lambda_{\vec\alpha(2)})^u \sum_{v=1}^{q\pm1} f(\lambda_{\vec\alpha(3)})^v \\ &= \frac{(q\pm1)^3}{6} \sum_{\vec\alpha\in\vec{\mathcal{A}}_6} \frac{c^{X_6}_{i_1}\dots c^{X_6}_{i_m}}{d_6^{m-2}} \delta_{\vec\alpha(1)}\delta_{\vec\alpha(2)}\delta_{\vec\alpha(3)}\end{aligned}$$

$$\begin{aligned}\Sigma_4 &= \frac{1}{2}\sum_{\vec\alpha\in\vec{\mathcal{A}}_7} \frac{c^{X_7}_{i_1}\dots c^{X_7}_{i_m}}{d_7^{m-2}} \sum_{t=1}^{q\pm1} f(\lambda_{\vec\alpha(1)})^t \sum_{u=1}^{q^2-1} f(\lambda_{\vec\alpha(2)})^u \\ &= \frac{(q\pm1)(q^2-1)}{2} \sum_{\vec\alpha\in\vec{\mathcal{A}}_7} \frac{c^{X_7}_{i_1}\dots c^{X_7}_{i_m}}{d_7^{m-2}} \delta_{\vec\alpha(1)}\delta_{\vec\alpha(2)}\end{aligned}$$

$$\Sigma_5 = \frac{1}{3}\sum_{\vec a\in[\vec n]} \frac{c^{X_8}_{i_1}\dots c^{X_8}_{i_m}}{d_8^{m-2}} \sum_{t=1}^{q^3\pm1} f(\lambda_{\vec a})^t = \frac{(q^3\pm1)}{3}\sum_{\vec a\in[\vec n]} \frac{c^{X_8}_{i_1}\dots c^{X_8}_{i_m}}{d_8^{m-2}}\delta_{\vec a}$$

2.3. Structure constants for triple products in $GU(3,q^2)$ and $GL(3,q)$. Using the formulas from §2.2, we computed the structure constants for all triples (i_1,i_2,i_3). To write down the result in a compact form, we introduce the following notation. We define $\vec{\mathcal{A}}^*_6$ as the quotient of $\vec{\mathcal{A}}_6$ by the action of the symmetric group S_3 defined by $\vec\alpha^\pi = (\alpha^\pi_1,\dots,\alpha^\pi_m)$ where $\alpha^\pi_\nu = (\alpha_\nu(1^\pi),\alpha_\nu(2^\pi),\alpha_\nu(3^\pi))$. Similarly, we define

$\vec{\mathcal{A}}_7^*$ as the quotient of $\vec{\mathcal{A}}_7$ by the action of $\mathbb{Z}_2$ which exchanges the elements of $\mathcal{A}_{7,7}$. Given $\vec{a} \in [\vec{n}']$, let $|\vec{a}|$ be the number of ν such that $a_\nu = 1$ and $i_\nu \in \{4,5\}$. We set

$$\Delta = \sum_{\vec{\alpha}\in\vec{\mathcal{A}}_6^*} \delta_{\vec{\alpha}(1)}\delta_{\vec{\alpha}(2)}\delta_{\vec{\alpha}(3)} + \sum_{\vec{\alpha}\in\vec{\mathcal{A}}_7^*} \delta_{\vec{\alpha}(1)}\delta_{\vec{\alpha}(2)}, \qquad \Delta_a = \sum_{\vec{a}\in[\vec{n}'],|\vec{a}|=a} \delta_{\vec{a}}.$$

We set also

$$\Delta' = \sum_{\vec{a}\in[\vec{n}]} \delta_{\vec{a}}.$$

We do the following substitutions (we may do them because of the determinant relation):

- (i) $\delta_{\vec{a}}^2 = \delta_{\vec{a}}$;
- (ii) $\delta_{\vec{a}}\delta_{\vec{b}} = 0$ if $\vec{a}$ and $\vec{b}$ differ at exactly one position, i. e., if there exists ν_0 such that $a_\nu = b_\nu$ if and only if $\nu = \nu_0$, for example, $\delta_{122}\delta_{132} = 0$;
- (iii) $\delta_{\vec{a}} = 0$ if there exists ν_0 such that $a_\nu \le n'_\nu$ if and only if $\nu = \nu_0$, for example, we set $\delta_{321} = 0$ if $(i_1,i_2,i_3) = (7,5,4)$;
- (iv) $\delta_{111}\delta_{n_{i_1},n_{i_2},n_{i_3}} = \delta_{111}$ if $i_1,i_2,i_3 \le 5$;
- (v) $\delta_{111} = 0$ if $i_1 \in \{4,5\}$ and $\{i_2,i_3\} \subset \{2,3\}$.

The result of computation is presented in Table 4. Recall that δ_L is defined by (4). In the third column, which is entitled "length of Δ", we give the number of monomials in Δ or in Δ' survived after the substitutions (i)–(v). If there are restrictions on $\delta_{\vec{a}}$ imposed by the rank condition, then we write them in the brackets in the second column (if the rank condition is never satisfied, then we write "[false]").

It is clear from Table 4 that $N_G(A_1,A_2,A_3) = 0$ in the cases (i)–(ix) of Theorem 1.3(a).

Also, when $G = GL$, it is clear from Table 4 that $N_G(A_1,A_2,A_3) \ne 0$ unless the cases $(viii)$ and (ix) of Theorem 1.3; maybe, it worth to note only that $\Delta \le \delta_{1,1,n_{i_3}}$ for $i_1 = i_2 = 7$, $i_3 \in \{2,3,4,5\}$ and that for $(i_1,i_2,i_3) = (8,8,2)$ the proof is the same as in the case $G = GU$.

In the last column we give a reference to a proof of Theorem 1.3(b) for $G = GU$ and $q \ge 5$ in the corresponding case ("ev." means "evident"). The case of $G = GU$, $q = 2,3,4$, is done in §4 and §2.4.

Table 5 serves to prove Theorem 1.3(b) for the triples (i_1,i_2,i_3) appearing in cases (i), (ii), (iii), (v) of Theorem 1.3(a). In the second column we write condition $(*)$ on $\delta_{\vec{a}}$. It is a condition which is equivalent to the fact that the hypothesis of Theorem 1.3(b) is satisfied, i. e., the conditions (i)–(v) are not satisfied for any permutation of (i_1,i_2,i_3) and for any renumbering of the eigenvalues under (5). As in Table 4, the rank condition is written in the brackets. In the third column we write the structure constant for $G = GU$ under condition $(*)$. In each case it is obviously nonzero for $q \ge 5$.

2.4. The cases of $GL(3,2)$ and $GU(3,q^2)$ for $q = 3,4$.

These cases are treated in [10]: p. 64 for $GL(3,2)$, pp. 69–71 for $GU(3,3^2)$ and pp. 89–93 for $GU(3,4^2)$. The correspondence between the notation of conjugacy classes in [6] (used in this paper) and the notation in [10] is given in Tables 6.1, 6.2 and 6.3. Note that in all these cases 3 does not divide $q \pm 1$, hence it is enough to consider the case of SU instead of GU.

2.5. Proof of Theorem 1.3 for $m = 3$. Here we complete the proof for triples (i_1,i_2,i_3) not covered by Table 5. In this section $G = GU$.

Table 4. Structure constants for $GU(3,q^2)$ $\binom{\delta_L=0}{\pm=+}$ and $GL(3,q)$ $\binom{\delta_L=1}{\pm=-}$

(i_1,i_2,i_3)	$N_G(A_1,A_2,A_3)/\lvert A_1^G\rvert$	length of Δ	proof of Th. 1.3
$(2,2,2)$	$(2q^2\delta_L \pm q - 2)\delta_{111}$ $[\delta_{111}=1]$		ev.
$(3,2,2)$	$2\delta_L\delta_{111}$ $[\delta_{111}=1]$		ev.
$(3,3,2)$	$q^2(1\mp\delta_{111}) + (q-1)\delta_{111} - 4q\delta_L\delta_{111}$		ev.
$(3,3,3)$	$q^2(q^2-2) + q(q^2\pm 2q-2)\delta_{111}$		ev.
$(4,2,2)$	0 [false]		ev.
$(4,3,2)$	0 [false]		ev.
$(4,3,3)$	$q(q\pm1)^2(q\mp1)$		ev.
$(4,4,2)$	$2(q^2-1)\delta_L\delta_{111}$ $[\delta_{111}=1]$		ev.
$(4,4,3)$	$(q\pm1)^2(q\mp1)\delta_{111}$ $[\delta_{111}=1]$		ev.
$(4,4,4)$	$(2q^2\delta_L\pm1)\delta_{111} + q(q\mp1)\delta_{112}\delta_{121}\delta_{211}$ $[\delta_{111}+\delta_{112}\delta_{121}\delta_{211}=1]$		ev.
$(5,2,2)$	$q\delta_{211}$ $[\delta_{211}=1]$		ev.
$(5,3,2)$	$q(q\pm1)(1\mp\delta_{211})$		tbl. 5
$(5,3,3)$	$q(q\mp1)^2\big((q\mp2)+\delta_{211}\big)$		ev.
$(5,4,2)$	$(q\mp q\pm1)\delta_{111} + q\delta_{211}$ $[\delta_{111}+\delta_{211}=1]$		ev.
$(5,4,3)$	$(q\pm1)\big(q+(2q\delta_L-1)\delta_{111}\mp q\delta_{211}\big)$		tbl. 5
$(5,4,4)$	$(q\pm1)\delta_{111} + q\delta_{211}(1\mp\delta_{112}\delta_{121})$ $[\delta_{111}+\delta_{211}=1]$		tbl. 5
$(5,5,2)$	$q^2\pm q + 2\big((q-1)^2\delta_L-1\big)\delta_{111} \mp (q^2\pm q)\Delta_1 \mp q^2\delta_{221}$		tbl. 5
$(5,5,3)$	$(q\pm1)\big(q(q^2\mp2q-2) + (q^2-4q\delta_L+1)\delta_{111}$ $+q(q\pm1)\Delta_1 + q^2\delta_{221}\big)$		ev.
$(5,5,4)$	$q(q\pm1)\big(\delta_{112}\delta_{121}\delta_{211}\mp\delta_{121}\mp\delta_{211}+1\big)$ $\mp q^2\delta_{221} + (2q^2\delta_L-2q\mp1)\delta_{111}$		tbl. 5
$(5,5,5)$	$q(q\pm1)(q^2\mp3q-2+q\Delta_1) + (q^3\pm3q^2-2q^2+3q$ $\pm1)\delta_{111} + q(q\pm1)^2\big(\Delta_2\mp\delta_{112}\delta_{121}\delta_{211}\big) + q^3\delta_{222}$		ev.
$(6,2,2)$	$(q\pm1)\Delta_0$ $[\Delta_0=1]$		ev.
$(6,3,2)$	$(q\pm1)^2(1\mp\Delta_0)$		tbl. 5
$(6,3,3)$	$(q\pm1)^2\big(q^2\mp2q-1+(q\pm1)\Delta_0\big)$		ev.
$(6,4,2)$	$(q\pm1)\Delta_1$ $[\Delta_1=1]$		ev.
$(6,4,3)$	$(q\pm1)^2(1\mp\Delta_1)$		tbl. 5
$(6,4,4)$	$(q\pm1)\Delta_2\mp q\Delta$ $[\Delta_2=1]$	6	ev.
$(6,5,2)$	$(q\pm1)\big((q\pm1)(1\mp\Delta_1)\mp q\Delta_0\big)$		tbl. 5
$(6,5,3)$	$(q\pm1)^2\big((q^2\mp3q-1)+(q\pm1)\Delta_1+q\Delta_0\big)$		ev.
$(6,5,4)$	$(q\pm1)\big((q\pm1)(1\mp\Delta_2)\mp q(\delta_{121}+\delta_{221}+\delta_{321})+q\Delta\big)$	6	tbl. 5
$(6,5,5)$	$(q\pm1)\big((q\pm1)(q^2\mp4q-1)+(q\pm1)^2\Delta_2$ $+q(q\pm1)(\Delta_1\mp\Delta)+q^2\Delta_0\big)$	6	§2.5
$(6,6,2)$	$(q\pm1)\big((q\pm1)\mp q\Delta_0+(2q\mp1)\Delta\big)$	6	tbl. 5
$(6,6,3)$	$q(q\pm1)^2\big(q\mp4+\Delta_0\mp\Delta\big)$	6	§2.5
$(6,6,4)$	$(q\pm1)\big(1+q(1\mp\Delta_1)\big)+q^2\Delta$	18	§2.5
$(6,6,5)$	$q(q\pm1)\big((q\pm1)(q\mp5)+(q\pm1)\Delta_1+q\Delta_0\mp q\Delta\big)$	18	§2.5
$(6,6,6)$	$(q\pm1)^2(q^2\mp6q+1)+q^2(q\pm1)\Delta_0\mp q^3\Delta$	36	§2.5

Table 4 (continued-1)

(i_1,i_2,i_3)	$N_G(A_1,A_2,A_3)/\lvert A_1^G\rvert$	length of Δ	proof of Th. 1.3
$(7,2,2)$	$(q\mp 1)\delta_{111} \quad [\delta_{111}=1]$		ev.
$(7,3,2)$	$(q^2-1)(1\mp\delta_{111})$		tbl. 5
$(7,3,3)$	$(q\pm 1)(q^2-1)\big(q\mp 1+\delta_{111}\big)$		ev.
$(7,4,2)$	$(q\mp 1)\delta_{111} \quad [\delta_{111}=1]$		ev.
$(7,4,3)$	$(q^2-1)(1\mp\delta_{111})$		tbl. 5
$(7,4,4)$	$(q\mp 1)\delta_{111} \quad [\delta_{111}=1]$		ev.
$(7,5,2)$	$(q\mp 1)\big((q\pm 1)(1\mp\delta_{111})\mp q\delta_{121}\big)$		tbl. 5
$(7,5,3)$	$(q^2-1)\big(q^2\mp q+1+(q\pm 1)\delta_{111}+q\delta_{121}\big)$		ev.
$(7,5,4)$	$(q\mp 1)\big((q\pm 1)(1\mp\delta_{111})\mp q\delta_{121}\big)$		tbl. 5
$(7,5,5)$	$(q\mp 1)\big((q\pm 1)(q^2\mp 2q-1)+(q\pm 1)^2\delta_{111}$ $+q(q\pm 1)\Delta_1+q^2\delta_{122}\big)$		ev.
$(7,6,2)$	$(q\mp 1)\big(q\pm 1\mp q\Delta_0\big)$		ev.
$(7,6,3)$	$q(q^2-1)\big(q\mp 2+\Delta_0\big)$		ev.
$(7,6,4)$	$(q\mp 1)\big(q\pm 1\mp q\Delta_1\big)$		ev.
$(7,6,5)$	$q(q\mp 1)\big((q\pm 1)(q\mp 3)+(q\pm 1)\Delta_1+q\Delta_0\big)$		ev.
$(7,6,6)$	$(q\mp 1)\big((q\pm 1)(q^2\mp 4q+1)+q^2\Delta_0\big)$		ev.
$(7,7,2)$	$(q\mp 1)\big(1+q(1\mp\delta_{111})\pm\Delta\big)$	2	ev.
$(7,7,3)$	$q(q^2-1)\big(q+\delta_{111}\pm\Delta\big)$	2	ev.
$(7,7,4)$	$(q\mp 1)\big(q\pm 1\mp q\delta_{111}\big)+q^2\Delta$	2	ev.
$(7,7,5)$	$q(q\mp 1)\big(q^2-1+(q\pm 1)\delta_{111}+q\delta_{112}\pm q\Delta\big)$	2	ev.
$(7,7,6)$	$(q\mp 1)\big((q^2-1)(q\mp 1)+q^2\Delta_0\big)$		ev.
$(7,7,7)$	$(q^4-1)+q^2(q\mp 1)\delta_{111}\pm q^3\Delta$	4	ev.
$(8,2,2)$	$0 \quad$ [false]		ev.
$(8,3,2)$	$q^2\mp q+1$		ev.
$(8,3,3)$	$(q^2\mp q+1)(q^2\pm q-1)$		ev.
$(8,4,2)$	$0 \quad$ [false]		ev.
$(8,4,3)$	$q^2\mp q+1$		ev.
$(8,4,4)$	$0 \quad$ [false]		ev.
$(8,5,2)$	$q^2\mp q+1$		ev.
$(8,5,3)$	$(q^2-1)(q^2\mp q+1)$		ev.
$(8,5,4)$	$q^2\mp q+1$		ev.
$(8,5,5)$	$(q^2\mp q+1)(q^2\pm q-1)$		ev.
$(8,6,2)$	$q^2\pm q+1$		ev.
$(8,6,3)$	$q(q\mp 1)(q^2\mp q+1)$		ev.
$(8,6,4)$	$q^2\mp q+1$		ev.
$(8,6,5)$	$q(q\mp 2)(q^2\mp q+1)$		ev.
$(8,6,6)$	$(q^2\mp q+1)(q^2\mp 3q+1)$		ev.
$(8,7,2)$	$q^2\mp q+1$		ev.
$(8,7,3)$	$q(q^3\pm 1)$		ev.
$(8,7,4)$	$q^2\mp q+1$		ev.
$(8,7,5)$	$q^2(q^2\mp q+1)$		ev.
$(8,7,6)$	$(q^2\mp q+1)^2$		ev.
$(8,7,7)$	q^4+q^2+1		ev.

Table 4 (continued-2)

(i_1,i_2,i_3)	$N_G(A_1,A_2,A_3)/\lvert A_1^G\rvert$	length of Δ	proof of Th. 1.3
$(8,8,2)$	$(q^2\mp q+1)(1-\Delta'/3)$	9	tbl. 5
$(8,8,3)$	$q(q^2\mp q+1)\big(q\pm 2\mp\Delta'/3)$	9	ev.
$(8,8,4)$	$q^2\mp q+1$		ev.
$(8,8,5)$	$q(q^3\pm 1)$		ev.
$(8,8,6)$	$(q^2+1)(q^2\mp q+1)$		ev.
$(8,8,7)$	$(q\pm 1)(q^3\pm 1)$		ev.
$(8,8,8)$	$(q^2\mp q+1)(q^2\pm 3q+1)\mp q^3\Delta'/3$	27	§2.5

Table 5.

(i_1,i_2,i_3)	condition $(*)$	$N_G(A_1,A_2,A_3)/\lvert A_1^G\rvert$ under $(*)$ for $G=GU(3,q^2)$
$(5,3,2)$	$\delta_{211}=0$	$q(q+1)$
$(5,4,3)$	$\delta_{211}=0$	$(q+1)(q-\delta_{111})$
$(5,4,4)$	$\delta_{112}\delta_{121}\delta_{211}=0$	$(q+1)\delta_{111}+q\delta_{211}\quad[\delta_{111}+\delta_{211}=1]$
$(5,5,2)$	$\Delta_1=0$	$q(q+1)-2\delta_{111}-q^2\delta_{221}$
$(5,5,4)$	$\delta_{211}=\delta_{121}=0$	$q(q+1)-q^2\delta_{221}-(2q+1)\delta_{111}$
$(6,3,2)$	$\Delta=0$	$(q+1)^2$
$(6,4,3)$	$\Delta_1=0$	$(q+1)^2$
$(6,5,2)$	$\Delta_1=0$	$(q+1)\big(1+q(1-\Delta_0)\big)$
$(6,5,4)$	$\Delta_2=0$	$(q+1)\big(1+q(1+\Delta-\delta_{121}-\delta_{221}-\delta_{321})\big)$
$(6,6,2)$	$\Delta=0$	$(q+1)\big(1+q(1-\Delta_0)\big)$
$(7,3,2)$	$\delta_{111}=0$	q^2-1
$(7,4,3)$	$\delta_{111}=0$	q^2-1
$(7,5,2)$	$\delta_{111}=0$	$(q-1)\big(1+q(1-\delta_{121})\big)$
$(7,5,4)$	$\delta_{111}=0$	$(q-1)\big(1+q(1-\delta_{121})\big)$
$(8,8,2)$	$\Delta'=0$	q^2-q+1

Table 6.1. Notation correspondence for conjugacy classes in $SL(3,2)=GL(3,2)$

in [10]	in §1.4	in [10]	in §1.4	in [10]	in §1.4
1A	$C_1^{(0)}$	3B	$C_7^{(1)}=C_7^{(2)}$	7A	$C_8^{(1)}=C_8^{(2)}=C_8^{(4)}$
2A	$C_2^{(0)}$	4B	$C_3^{(0}$	7A	$C_8^{(3)}=C_8^{(5)}=C_8^{(6)}$

Table 6.2. Notation correspondence for conjugacy classes in $SU(3,3^2)$

in [10]	in [6]	in [10]	in [6]	in [10]	in [6]
1A	$C_1^{(0)}$	4B	$C_4^{(3,2)}$	8A	$C_7^{(1,1)}=C_7^{(1,5)}$
2A	$C_4^{(2,0)}$	4C	$C_6^{(0,1,3)}$	8B	$C_7^{(3,3)}=C_7^{(3,7)}$
3A	$C_2^{(0)}$	6A	$C_5^{(2,0)}$	12A	$C_5^{(1,2)}$
3B	$C_3^{(0)}$	7A	$C_8^{(4)}=C_8^{(8)}=C_8^{(16)}$	12B	$C_5^{(3,2)}$
4A	$C_4^{(1,2)}$	7B	$C_8^{(12)}=C_8^{(20)}=C_8^{(24)}$		

Table 6.3. Notation correspondence for conjugacy classes in $SU(3,4^2)$

in [10]	in [6]	in [10]	in [6]	in [10]	in [6]
1A	$C_1^{(0)}$	5E	$C_6^{(0,1,4)}$	13C	$C_8^{(20)}=C_8^{(50)}=C_8^{(60)}$
2A	$C_2^{(0)}$	5F	$C_6^{(0,2,3)}$	13D	$C_8^{(35)}=C_8^{(40)}=C_8^{(55)}$
3A	$C_7^{(0,5)}=C_7^{(0,10)}$	10A	$C_5^{(1,3)}$	13A	$C_8^{(5)}=C_8^{(15)}=C_8^{(45)}$
4A	$C_3^{(0)}$	10B	$C_5^{(2,1)}$	13B	$C_8^{(10)}=C_8^{(25)}=C_8^{(30)}$
5A	$C_4^{(1,3)}$	10C	$C_5^{(4,2)}$	15A	$C_7^{(3,8)}=C_7^{(3,13)}$
5B	$C_4^{(2,1)}$	10D	$C_5^{(3,4)}$	15B	$C_7^{(1,1)}=C_7^{(1,11)}$
5C	$C_4^{(4,2)}$			15C	$C_7^{(2,2)}=C_7^{(2,7)}$
5D	$C_4^{(3,4)}$			15D	$C_7^{(4,4)}=C_7^{(4,14)}$

The case $(i_1,i_2,i_3)=(6,5,5)$**.** We have

$$\begin{aligned}\Delta_1-\Delta=\delta_{112}(1-\delta_{211}\delta_{321})+\delta_{212}(1-\delta_{311}\delta_{121})+\delta_{312}(1-\delta_{111}\delta_{221})\\ +\delta_{121}(1-\delta_{211}\delta_{312})+\delta_{221}(1-\delta_{311}\delta_{112})+\delta_{321}(1-\delta_{111}\delta_{212})\geq 0.\end{aligned}$$

The case $(i_1,i_2,i_3)=(6,6,3)$**.** We have

$$\begin{aligned}\Delta_0-\Delta=\delta_{111}(1-\delta_{221}\delta_{331}-\delta_{231}\delta_{321})+\delta_{121}(1-\delta_{211}\delta_{331}-\delta_{231}\delta_{311})\\ +\delta_{131}(1-\delta_{211}\delta_{321}-\delta_{221}\delta_{311})+\sum_{\vec a\in\vec n;a_1>1}\delta_{\vec a}\geq 0\end{aligned}$$

The case $(i_1,i_2,i_3)=(6,6,4)$**.** If $\Delta>0$, then there exist permutations of the eigenvalues such that the product of corresponding diagonal matrices is the identity matrix. So, we consider only the case when $\Delta=0$. In this case $N_G(A_1,A_2,A_3)/|A_1^G|=(q+1)\big(1+q(1-\Delta_1)\big)$ which cannot be zero for any integers $q>1$ and Δ_1.

The case $(i_1,i_2,i_3)=(6,6,5)$**.** Here we write for shortness ν^α instead of $\alpha(\nu)$. We have $\Delta=\sum_{\alpha\in S_3}\sum_{\beta\in\mathcal{A}_{6,5}}\delta_{1,1^\alpha,1^\beta}\delta_{2,2^\alpha,2^\beta}\delta_{3,3^\alpha,3^\beta}=\sum_{\alpha\in S_3}E(\alpha)$ where

$$E(\alpha)=\delta_{1,1^\alpha,1}\delta_{2,2^\alpha,1}\delta_{3,3^\alpha,2}+\delta_{1,1^\alpha,1}\delta_{2,2^\alpha,2}\delta_{3,3^\alpha,1}+\delta_{1,1^\alpha,2}\delta_{2,2^\alpha,1}\delta_{3,3^\alpha,1}.$$

Summating $E(\alpha)$ separately over odd and even permutations α and estimating each triple product of the deltas by one of its factors, we obtain

$$\begin{aligned}\sum_{\text{odd }\alpha}E(\alpha)\leq\sum_{\text{odd }\alpha}\big(\delta_{3,3^\alpha,2}+\delta_{2,2^\alpha,2}+\delta_{1,1^\alpha,2}\big)=\Delta_0,\\ \sum_{\text{even }\alpha}E(\alpha)\leq\sum_{\text{even }\alpha}\big(\delta_{1,1^\alpha,1}+\delta_{3,3^\alpha,1}+\delta_{2,2^\alpha,1}\big)=\Delta_1\end{aligned}$$

which implies $\Delta_1+\Delta_0-\Delta\geq 0$ and the result follows for $q>5$.

Let $q=5$. The above considerations show that the structure constant is positive when $\Delta_1>0$. So, we suppose that $\Delta_1=0$. Then $\Delta=0$ because each triple product in Δ includes some $\delta_{\vec a}$ involved in Δ_1. If we have two triples of distinct residues mod 6 (the parameters (k,l,m) of $C_6^{(k,l,m)}$) not of the same parity, then their pairwise sums attain all values mod 6 except, maybe one, thus Δ_0 or Δ_1 is nonzero. So, it remains to consider the case $A_1,A_2\in C_6^{(0,2,4)}$. In this case, (2) implies $A_3\in C_5^{(k,l)}$ with l even, hence $\Delta_0>0$ and the result follows.

The case $(i_1,i_2,i_3) = (6,6,6)$. If $\Delta > 0$, then there exist permutations of the eigenvalues such that the product of corresponding diagonal matrices is the identity matrix. So, we consider only the case when $\Delta = 0$. In this case, the structure constant is positive for $q > 5$ and it is equal to $150\Delta_0 - 144 \neq 0$ for $q = 5$.

The case $(i_1,i_2,i_3) = (8,8,8)$.

Let the eigenvalues of A_ν be $(\lambda_\nu, \lambda_\nu^{q^2}, \lambda_\nu^{q^4})$, $\nu = 1,2,3$. Then we have

$$\Delta' = \sum_{0\le a,b,c\le 2} \delta_{a,b,c}, \qquad \delta_{a,b,c} = \begin{cases} 1, & \lambda_1^{q^{2a}}\lambda_2^{q^{2b}}\lambda_3^{q^{2c}} = 1, \\ 0, & \text{otherwise} \end{cases}$$

It is clear that $\delta_{a,b,c} = \delta_{a',b',c'}$ if $a - a' \equiv b - b' \equiv c - c' \mod 3$.

We are going to show that there is at most 9 triples (a,b,c) such that $\delta_{a,b,c} = 1$. Suppose that one of $\delta_{a,b,c}$ is nonzero. Without loss of generality we may assume that it is δ_{000} (otherwise we permute cyclically the eigenvalues of each matrix). So, we have $\lambda_1\lambda_2\lambda_3 = 1$.

Let us show that if $\delta_{a,b,c} = 1$, then either $a = b = c$ or a,b,c are pairwise distinct (there are only nine such triples). Suppose that this is not so, say, $a \neq b = c$. Then $\delta_{001}\delta_{112}\delta_{220} = 1$ or $\delta_{001}\delta_{112}\delta_{220} = 1$ (we consider only the first case). This means that $\lambda_1\lambda_2\lambda_3^{q^2} = 1$. Combined with $\lambda_1\lambda_2\lambda_3 = 1$ this yields $\lambda_3^{q^2} = 1$, i.e $\lambda_3 \in \mathbb{F}_{q^2}$. Contradiction.

Thus, we proved that $\Delta' \le 9$, hence

$$N_G(A_1,A_2,A_3)/|A_1^G| \ge (q^2-q+1)(q^2+3q+1) - 3q^3 = q^4 - q^3 - q^2 + 2q + 1 > 0.$$

2.6. End of proof of Theorem 1.3 (the case $m \ge 4$). Let us prove Theorem 1.3 for $m = 4$. So, let $m = 4$ and let $A_1,\dots,A_4$ be as in Theorem 1.3.

If $G = GL$ and $q \ge 3$, then for any $d, \lambda_1, \lambda_2 \in \Omega$ there exists $B \in C_3 \cup C_5 \cup C_6$ such that $d = \det B$ and λ_1, λ_2 are eigenvalues. Hence, we can choose B in $C_3 \cup C_5 \cup C_6$ such that the rank condition is satisfied for both triples (A_1,A_2,B) and (B^{-1},A_3,A_4). As we have already shown, there are no other restrictions for triple products in GL. This completes the proof of Theorem 1.3 for $G = GL$.

Lemma 2.1. *Let $G = GU$ and $q \ge 4$. Then for any $d, \mu \in \Omega$ there exists $B \in C_7$ such that $\det B = d$ and $\lambda_1(B) = \mu$.*

Proof. Obvious. □

Lemma 2.2. *Let $G = GU$ and $q \ge 5$. Suppose that one of the following conditions holds*

(*i*) $\{i_1,i_3\} \not\subset \{2,4\}$;
(*ii*) $\{i_1,i_2,i_3\} \subset \{2,4\}$ *and* $i_4 \in \{6,7,8\}$;
(*iii*) $i_1 = 4$, $\{i_2,i_3\} \subset \{2,4\}$, $i_4 \in \{3,5\}$;
(*iv*) $i_1 = i_2 = i_3 = 2$, $i_4 \in \{3,5\}$, *and* $\delta_{1111} = 0$;
(*v*) $\{i_1,i_2,i_3,i_4\} \subset \{2,4\}$ *and* $\delta_{1111} = 1$;
(*vi*) $i_1 = i_3 = 2$, $\{i_2,i_4\} \subset \{2,4\}$ *and* $\delta_{1111} = 0$;
(*vii*) $i_1 = i_2 = i_3 = i_4 = 4$ *and* $\delta_{1111} = 0$.

Then $I \in A_1^G \dots A_4^G$.

Proof. We set $d = \det(A_1A_2) = \det(A_3^{-1}A_4^{-1})$, $\mu_1 = \lambda_1(A_1)\lambda_1(A_2)$, and $\mu_2 = \lambda_1(A_3^{-1})\lambda_1(A_4^{-1})$. We consider the cases (*i*)–(*vii*) one by one and in each case we find B such that $B \in A_1^G A_2^G$ and $B^{-1} \in A_3^G A_4^G$. When we choose B in C_7, we use Lemma 2.1.

(i). We choose $B \in C_7$ such that $\det B = d$ and $\lambda_1(B) \notin \{\mu_1, \mu_2\}$.

(ii). We choose $B \in C_7$ such that $\det B = d$ and $\lambda_1(B) = \mu_1$.

(iii). We consider two cases.

Case 1. $\delta_{1111} = 1$, i. e., $\mu_1 = \mu_2$. We choose $B \in C_3 \cup C_5$ such that $\det B = d$ and $\lambda_1(B) = \mu_1 = \mu_2$.

Case 2. $\delta_{1111} = 0$, i. e., $\mu_1 \neq \mu_2$. Then we choose $B \in C_7$ such that $\det B = d$ and $\lambda_1(B) = \mu_1$.

(iv). The choice of B is the same as for *(iii)*, Case 2.

(v). Since $\delta_{1111} = 1$, we have $\mu_1 = \mu_2$. So, we choose $B \in C_7$ such that $\det B = d$ and $\lambda_1(B) = \mu_1 = \mu_2$.

(vi). Since $\delta_{1111} = 0$, we have $\mu_1 \neq \mu_2$. We choose $B \in C_5 \cup C_6$ such that $\det B = d$ and μ_1, μ_2 are eigenvalues of B.

(vii). Since $\delta_{1111} = 0$, we have $\mu_1 \neq \mu_2$. We choose $B \in C_4 \cup C_6$ such that $\det B = d$ and μ_1, μ_2 are eigenvalues of B. $\square$

For the cases not covered by Lemma 2.2 we compute the structure constant in $G = GU$:

(i_1, i_2, i_3, i_4)	δ_{1111}	$N_G(A_1, A_2, A_3, A_4)/\lvert A_1^G\rvert$
$(3,2,2,2)$	1	0
$(5,2,2,2)$	1	$(q+3)(q^2-1)$
$(4,4,4,2)$	0	$q(q^2-1)\big(q+1-q(\delta_{1121}+\delta_{1211}+\delta_{2111}) + (2q-1)\delta_{1121}\delta_{1211}\delta_{2111}\big)$

This completes the proof of Theorem 1.3 for $m = 4$.

Let $m = 5$, $q \geq 5$. Easy to see that there exists $B \in (A_1^G A_2^G) \cap (C_3 \cup C_5 \cup C_6 \cup C_7 \cup C_8)$. Then $I \in B^G A_3^G A_4^G A_5^G$. Theorem 1.3 is proven.

3. Products of conjugacy classes in $SU(3, q^2)$ and $SL(3, q)$. Proof of Theorem 1.6

3.1. The character table of $SU(3, q^2)$ and $SL(3, q)$. Let G be $GU(3, q^2)$ or $GL(3, q)$ and let $S = \{A \in G \mid \det A = 1\}$. So, S is $SU(3, q^2)$ or $SL(3, q)$. The character table of S is computed in [15]. It has some mistakes which are corrected in [7] (it is written in the comments in [7] that the character table for $SU(3, q^2)$ is taken from [8]). Since $G = S \times \Omega$ when 3 does not divide $q \pm 1$, we consider only the case when $q = 3r \mp 1$.

The conjugacy classes of S are as follows. Each of $C_3^{(k)}$, $k = 0, r, 2r$, splits into three classes $C_3^{(k,l)}$, $l = 0, 1, 2$. The class $C_3^{(k,l)}$ in $SU(3, q^2)$ (resp. in $SL(3, q)$) consists of matrices which are conjugate in $SL(3, q^2)$ (resp. in $SL(3, q)$) to[2]

$$\begin{pmatrix} \omega^k & 0 & 0 \\ z^l & \omega^k & 0 \\ 0 & 1 & \omega^k \end{pmatrix}, \qquad z = \begin{cases} \rho, & S = SU(3, q^2), \\ \omega, & S = SL(3, q). \end{cases}$$

[2]there is a misprint here in [15].

Other conjugacy classes of G contained in S are conjugacy classes of S.

The irreducible characters of S can be described as follows. We consider the action of the cyclic group of order $q\pm1$ on $\mathrm{Irr}(G)$ such that the action of the generator is

$$\chi_{d_j}^{(t)} \mapsto \chi_{d_j}^{(t+1)}\ (j=1,2,3);\quad \chi_{d_j}^{(t,u)} \mapsto \chi_{d_j}^{(t+1,u+1)}\ (j=4,5);$$

$$\chi_{d_6}^{(t,u,v)} \mapsto \chi_{d_6}^{(t+1,u+1,v+1)};\quad \chi_{d_7}^{(t,u)} \mapsto \chi_{d_7}^{(t+1,u\mp q+1)};\quad \chi_{d_8}^{(t)} \mapsto \chi_{d_8}^{(t+q^2\mp q+1)}.$$

Then the restriction of all characters to S are constant on each orbit of this action. All orbits but three are of length $q\pm1$ and their representatives restricted to S are irreducible. There are three orbits of length r, namely the orbits of $\chi_{d_6}^{(0,r,2r)}$ and $\chi_{d_8}^{(u(q^2\mp q+1)/3)}$, $u=1,2$. Being restricted to S, each of these three characters splits into three irreducible characters. This yields irreducible characters $\chi_{d_6/3}^{(t)}$, $\chi_{d_8/3}^{(t,u)}$, $t=0,1,2$, $u=1,2$, such that $\chi_{d_6/3}^{(t)}(A)=\frac13\chi_{d_6}^{(0,r,2r)}(A)$ and $\chi_{d_8/3}^{(t,u)}(A)=\frac13\chi_{d_8}^{(u(q^2\mp q+1)/3)}(A)$ when $A\notin C_3$. For $A\in C_3^{(k,l)}$, $k,lr\in\{0,r,2r\}$, we have

$$\chi_{d_6/3}^{(t)}(A)=\begin{cases} q-r, & l=t,\\ -r, & l\ne t,\end{cases}\qquad \chi_{d_8/3}^{(t,u)}(A)=\varepsilon^{uk}\chi_{d_6/3}^{(t)}(A).$$

where $\varepsilon=f(\omega)=\exp(2\pi i/(q\pm1))$.

Thus, for any function E on $\mathrm{Irr}(S)$, we have

$$\sum_{\chi\in\mathrm{Irr}(S)}E(\chi)=\frac{1}{q\pm1}\Big(\sum_{\chi\in\mathrm{Irr}(G)}E(\chi|_S)\Big)-\frac13\Big(E\big(\chi_{d_6}^{(0,r,2r)}|_S\big)+\sum_{u=1}^{2}E\big(\chi_{d_8}^{u(q^2\mp q+1)/3}|_S\big)\Big)$$
$$+\sum_{t=0}^{2}\Big(E\big(\chi_{d_6/3}^{(t)}\big)+\sum_{u=1}^{2}E\big(\chi_{d_8/3}^{(t,u)}\big)\Big)$$

3.2. Structure constants for $SU(3,q^2)$ and $SL(3,q)$. Let $A_1,\dots,A_m\in S$, $A_\nu\in C_{i_\nu}$, $\nu=1,\dots,m$. We suppose that $i_1=\dots=i_n=3$ and $i_\nu\ne3$ for $\nu>n$. Let $A_\nu\in C_3^{(k_\nu,l_\nu)}$ for $\nu=1,\dots,n$.

We denote $E_1(\chi)=\chi(A_1)\dots\chi(A_n)$, $E_2(\chi)=\chi(A_{n+1})\dots\chi(A_m)$, and $E(\chi)=E_1(\chi)E_2(\chi)/\chi(I)^{m-2}$. Combining the formulas from the previous section with the fact that $\chi_{d_6}^{(0,r,2r)}(A_\nu)=\mp1$ and $\chi_{d_8}^{u(q^2\mp q+1)/3)}(A_\nu)=\mp\varepsilon^{k_\nu}$ for $\nu\le n$, we obtain

$$E_1\big(\chi_{d_6}^{(0,r,2r)}\big)=(\mp1)^n,\quad E_1\big(\chi_{d_8}^{(u(q^2\mp q+1)/3)}\big)=(\mp1)^n\varepsilon^{(k_1+\dots+k_n)u},$$

$$E_2\big(\chi_{d_6/3}^{(t)}\big)=3^{n-m}E_2\big(\chi_{d_6}^{(0,r,2r)}\big),\quad E_2\big(\chi_{d_8/3}^{(t,u)}\big)=3^{n-m}E_2\big(\chi_{d_8}^{(u(q^2\mp q+1)/3)}\big),$$

$$E_1\big(\chi_{d_8/3}^{(t,u)}\big)=\varepsilon^{(k_1+\dots+k_n)u}E_1\big(\chi_{d_6/3}^{(t)}\big),\quad \chi_{d_6/3}^{(t)}(I)=d_6/3,\quad \chi_{d_8/3}^{(t,u)}(I)=d_8/3,$$

and finally,

$$\bar N_S(A_1,\dots,A_m)=\frac{\bar N_G(A_1,\dots,A_m)}{q\pm1}+\Big(-\frac{(\mp1)^n}{3}+3^{n-2}\sum_{t=0}^{2}E_1\big(\chi_{d_6/3}^{(t)}\big)\Big)$$
$$\times\left(\frac{E_2\big(\chi_{d_6}^{(0,r,2r)}\big)}{d_6^{m-2}}+\sum_{u=1}^{2}\frac{\varepsilon^{(k_1+\dots+k_n)u}E_2\big(\chi_{d_8}^{(u(q^2\mp q+1)/3)}\big)}{d_8^{m-2}}\right)$$

In particular, we see from this formula that if $n=0$ or $n=1$, then $\bar N_G=(q\pm1)\bar N_S$, i. e., we have $(I\in A_1^G\dots A_m^G)\Leftrightarrow(I\in A_1^S\dots A_n^S)$. Indeed, if $n=0$, then the factor $\bigl(-\frac{(\mp1)^n}{3}+\dots\bigr)$ is equal to $-1/3+1/9\,(1+1+1)=0$, and if $n=1$, then it is equal to $\pm1/3+1/3\,\bigl((q-r)-r-r\bigr)=0$. This equivalence also follows immediately from the fact that $C_3^{(k)}$ are the only classes that split in S.

3.3. Triple products in $SU(3,q^2)$ and $SL(3,q)$. Proof of Theorem 1.6. Let $m=3$. It is enough to consider the cases $n=2$ and $n=3$. We use the following notation in Table 7. If $n=2$, then we set

$$\delta^*=\delta^*(A_1,A_2)=\begin{cases}1, & l_1=l_2,\\ 0, & l_1\ne l_2.\end{cases}.$$

If $A_3\in C_8^{((q\pm1)k')}$ (the last line of the table), then we set

$$\delta^*_{111}=\begin{cases}1, & k_1+k_2+k'\equiv 0 \mod q\pm1,\\ 0, & \text{otherwise}.\end{cases}$$

Table 7. Structure constants: $S=SU(3,q^2)$ or $SL(3,q)$, $q=3r\mp1$, $A_\nu\in C_{i_\nu}$

(i_1,i_2,i_3)		$N_S(A_1,A_2,A_3)/\lvert A_1^S\rvert$
$(3,3,3)$	distinct l_1,l_2,l_3	$qr\bigl(qr+(2qr\mp q+r)\delta_{111}\bigr)$
$(3,3,3)$	$l_1=l_2\ne l_3$	$qr\bigl(q(r\mp1)-(qr\mp q-r+1)\delta_{111}\bigr)$
$(3,3,3)$	$l_1=l_2=l_3$	$q\bigl(q(r^2-1)+(2q(r\mp1)^2+r^2\mp1)\delta_{111}\bigr)$
$(3,3,2)$		$\bigl(q^2-(q^2\mp q+1)\delta_{111}\bigr)\delta^*+2qr\delta_L\delta_{111}$
$(3,3,4)$		$q^2\delta^*$
$(3,3,5)$		$q^2r(q\mp1\mp3\delta^*+\delta_{211})$
$(3,3,6)$	$\lambda_1(A_3)^r=\lambda_2(A_3)^r$	$q^2\bigl((q-1)r\mp2q\delta^*+r\Delta_0\bigr)$
$(3,3,6)$	$\lambda_1(A_3)^r\ne\lambda_2(A_3)^r$	$q^2\bigl((q-1)r\mp q(1-\delta^*)+r\Delta_0\bigr)$
$(3,3,7)$		$q^2r(q\mp1+\delta_{111})$
$(3,3,8)$		$q^2\bigl((q-1)r\pm q(\delta^*+\delta^*_{111}-3\delta^*\delta^*_{111})\bigr)$

It is clear that if $r>1$, then the structure constants are positive except the case when $i_3\in\{2,4\}$ and $\delta^*=0$ (note that the case $i_3=6$, $q=5$, $\lambda_1(A_3)^r=\lambda_2(A_3)^r$ is impossible). This completes the proof of Theorem 1.6 for $m=3$.

For $m=4$, the proof is the same as in 2.6. Moreover, since at least two of $A_1,\dots,A_4$ belong to C_3, then only Case (i) of Lemma 2.2 is to be considered.

4. The case $q = 2$

4.1. Class products in $GU(3,q^2)$ for $q=2$. Let $G=GU(3,2^2)$, $S=SU(3,2^2)$. Then $|G|=648$, $|S|=216$. We have the following conjugacy classes in G:

$$\begin{aligned}
\det(A)=1:&\qquad C_1^{(k)},C_2^{(k)},C_3^{(k)}(k=0,1,2),\ C_6^{(0,1,2)},\\
\det(A)=\rho:&\qquad C_4^{(0,1)},C_4^{(2,0)},C_4^{(1,2)},C_5^{(0,1)},C_5^{(2,0)},C_5^{(1,2)},C_8^{(1)},\\
\det(A)=\rho^2:&\qquad C_4^{(0,2)},C_4^{(1,0)},C_4^{(2,1)},C_5^{(0,2)},C_5^{(1,0)},C_5^{(2,1)},C_8^{(2)}
\end{aligned}$$

We see from Table 4 that $C_6 \cdot C_6 = C_1 \cup C_6$, hence $H = C_6 \cup C_1$ is a normal subgroup of G of order 27. We have $|G/H| = 24$ and $S/H = 8$. The sizes of classes and the orders of their representatives in G/H are:

Class:	${C_1}^{(k)}$	${C_2}^{(k)}$	${C_3}^{(k)}$	${C_4}^{(k,l)}$	${C_5}^{(k,l)}$	${C_6}^{(0,1,2)}$	${C_8}^{(k)}$
Size:	1	9	54	12	36	24	72
Order in G/H:	1	2	4	3	6	1	3

The elements of C_2 (resp. C_3) represent elements of order 2 (resp. 4) in S/H. Since $|H| = |C_2| = 27$ and $|C_3| = 162$, it follows that S/H has one element of order 2 and six elements of order 4. Therefore, S/H is isomorphic to the unit quaternionic group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$. Since the exact sequence $1 \to S/H \to G/H \xrightarrow{\det} \{1, \rho, \rho^2\}$ splits, it follows that G/H is isomorphic to a semi-direct product of Q and $\mathbb{Z}_3$. We denote it by F. Since G/H has no element of order 12, this product is not direct, hence F can be identified with the group whose elements are $\pm a^m$, $\pm ia^m$, $\pm ja^m$, $\pm ka^m$, $m = 0, 1, 2$, subject to relations $ia = aj$, $ja = ak$, $ka = ai$, $a^3 = 1$. We denote a^2 by b. The conjugacy classes in F are: $\{1\}$, $\{-1\}$, $i^F = \{\pm i, \pm j, \pm k\}$, $a^F = \{a, ia, ja, ka\}$, $-a^F = \{-a, -ia, -ja, -ka\}$, $b^F = \{b, -ib, -jb, -kb\}$, $-b^F = \{-b, ib, jb, kb\}$. Their pairwise products are:

$$\begin{array}{ccccccc}
\{1\} & \{-1\} & i^F & a^F & -a^F & b^F & -b^F \\
\{-1\} & \{1\} & i^F & -a^F & a^F & -b^F & b^F \\
i^F & i^F & Q & Qa & Qa & Qb & Qb \\
\\
a^F & -a^F & Qa & Qb & Qb & \{1\} \cup i^F & \{-1\} \cup i^F \\
-a^F & a^F & Qa & Qb & Qb & \{-1\} \cup i^F & \{1\} \cup i^F \\
\\
b^F & -b^F & Qb & \{1\} \cup i^F & \{-1\} \cup i^F & Qa & Qa \\
-b^F & b^F & Qb & \{-1\} \cup i^F & \{1\} \cup i^F & Qa & Qa
\end{array}$$

Comparing the class sizes and the orders of their representatives, we easily see that the correspondence between the classes under the projection $G \to F$ is

$$\begin{array}{lll}
C_1 \cup C_6 \to \{1\} & C_{48}^{(1)} \to a^F & C_{48}^{(2)} \to b^F \\
C_2 \to \{-1\} & C_5^{(1)} \to -a^F & C_5^{(2)} \to -b^F \\
C_3 \to i^F & &
\end{array}$$

where $C_{48}^{(k)} = (C_4 \cup C_8) \cap G^{(k)}$, $C_5^{(k)} = C_5 \cap G^{(k)}$, and $G^{(k)} = \{A \in G \mid \det A = \rho^k\}$, $k = 1, 2$. Thus, the multiplication table for the preimages in G of the conjugacy classes of F is

$$\begin{array}{ccccccc}
H & C_2 & C_3 & C_{48}^{(1)} & C_5^{(1)} & C_{48}^{(2)} & C_5^{(2)} \\
C_2 & H & C_3 & C_5^{(1)} & C_{48}^{(1)} & C_5^{(2)} & C_{48}^{(2)} \\
C_3 & C_3 & S & G^{(1)} & G^{(1)} & G^{(2)} & G^{(2)} \\
\\
C_{48}^{(1)} & C_5^{(1)} & G^{(1)} & G^{(2)} & G^{(2)} & H \cup C_3 & C_2 \cup C_3 \\
C_5^{(1)} & C_{48}^{(1)} & G^{(1)} & G^{(2)} & G^{(2)} & C_2 \cup C_3 & H \cup C_3 \\
\\
C_{48}^{(2)} & C_5^{(2)} & G^{(2)} & H \cup C_3 & C_2 \cup C_3 & G^{(1)} & G^{(1)} \\
C_5^{(2)} & C_{48}^{(2)} & G^{(2)} & C_2 \cup C_3 & H \cup C_3 & G^{(1)} & G^{(1)}
\end{array}$$

The above discussion can be summarized as follows

Proposition 4.1. *Let $c = (c_1, \dots, c_m)$ is an unordered m-tuple of non-trivial conjugacy classes in F such that* $\deg_a c_1 + \dots + \deg_a c_m = 0$*. We suppose that $c_1 = \dots = c_{2n} = \{-1\}$ and $(c_{2n+1}, \dots, c_m)$ contains at most one occurrence of $\{-1\}$. Then $1 \notin c_1 \dots c_m$ if and only if $(c_{2n+1}, \dots, c_m)$ is one of $(\{-1\})$, (i^F), $(\{-1\}, i^F)$, $(a^F, -b^F)$, $(-a^F, b^F)$, $(\{-1\}, a^F, b^F)$, $(\{-1\}, -a^F, -b^F)$.*

Proof. It is enough to check that the product of any three non-trivial conjugacy classes different from $\{-1\}$ is a coset of Q in F. $\square$

Proposition 4.2. *Let $A_1, \dots, A_m \in G \setminus C_1$ be such that* $\det(A_1 \dots A_n) = 1$*. Let $A_1 \in C_{i_1}, \dots, A_m \in C_{i_m}$. Suppose that after removing any number of 6's and an even number of 2's from $(i_1, \dots, i_m)$, we obtain one of* (2), (3), $(2,3)$, $(5,4)$, $(8,5)$, $(4,4,2)$, $(5,5,2)$, $(8,4,2)$, $(8,8,2)$*. Then $I \notin A_1^G \dots A_n^G$.*

Proposition 4.3. *Let $A_1, \dots, A_m \in G \setminus C_1$, $m \geq 3$, be such that* $\det(A_1 \dots A_m) = 1$*. Let $A_1 \in C_{i_1}, \dots, A_m \in C_{i_m}$. Suppose that the conditions of Proposition 4.2 are not satisfied. Suppose also that the rank condition* (3) *holds and the conditions (i)–(vii) of Theorem 1.3(a) are not satisfied for any permutation of $A_1, \dots, A_m$ and for any renumbering of the eigenvalues under restrictions* (5)*.*

Then $I \notin A_1^G \dots A_m^G$ if and only if one of the following cases occurs up to changing the order of A_j's, multiplication them by scalar or simultaneous replacing of $A_1, \dots, A_m$ by $A_1^{-1}, \dots, A_m^{-1}$.

(i) $m = 4$, $A_1, A_2, A_3 \in C_4^{(0,1)}$ and $A_4 \in C_3^{(1)}$;
(ii) $m = 4$, $A_1, A_2 \in C_4^{(0,1)}$, $A_3 \in C_4^{(0,2)}$, and $A_4 \in C_5^{(1,0)}$.

Proof. Using the structure constants, we computed the products of all m-tuples of conjugacy classes for $m \leq 5$. So we check that the statement is true for $m \leq 5$. The general case easily follows from the following facts.

- $C_6^{(0,1,2)} C_6^{(0,1,2)} = H$;
- $C_2^{(k_1)} C_2^{(k_2)} = C_1^{(k_1+k_2)} \cup C_6^{(0,1,2)}$ for any k_1, k_2;
- $C_2^{(k)} C_6^{(0,1,2)} = C_2$ for any k;
- Let $m = 4$ or 5. If $(i_1, \dots, i_m)$ is not as in Proposition 4.2 and $\{i_1, \dots, i_m\} \not\subset \{2, 6\}$, then $A_1^G \dots A_m^G$ is a coset of S in G for any $A_1 \in C_{i_1}, \dots, A_m \in C_{i_m}$;

$\square$

4.2. Class products in $SU(3, q^2)$ for $q = 2$. There are 16 conjugacy classes in S. These are:

$$C_1^{(k)}, C_2^{(k)}, C_3^{(k,l)}, C_6^{(0,1,2)}, \qquad k, l = 0, 1, 2.$$

We have $S/H = Q$ and $S/(H \cup C_2) = Q/\{\pm 1\} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The cosets of $H \cup C_2$ in S are: $H \cup C_2$, $C_3^{(*,0)}$, $C_3^{(*,1)}$, $C_3^{(*,2)}$ where $C_3^{(*,l)}$ stands for $C_3^{(0,l)} \cup C_3^{(1,l)} \cup C_3^{(2,l)}$.

Proposition 4.4. *Let $A_1, \dots, A_m \in S \setminus C_1$, $m \geq 3$, $A_\nu \in C_{i_\nu}$, $\nu = 1, \dots, m$. If $3 \in \{i_1 \dots, i_m\}$, then $A_1^S \dots A_m^S$ is a coset of $H \cup C_2$ in S. Otherwise $A_1^S \dots A_m^S$ is a coset of H in $H \cup C_2$.*

Proof. It is enough to compute the structure constants for all triples $A_1, A_2, A_3 \in S$.

Corollary 4.5. *Let $A_1, \dots, A_m \in S \setminus C_1$, $m \geq 3$, $A_\nu \in C_{i_\nu}$, $\nu = 1, \dots, m$. Then $I \in A_1 \dots A_m$ if and only if none of the following conditions holds:*

(i) for some $l \in \{0, 1, 2\}$, the number of matrices among $A_1, \dots, A_m$ belonging to $C_3^{(*,l)}$ is odd;

(ii) $i_1,\dots,i_m \in \{2,6\}$ *and the number of* 2*'s in the sequence* $(i_1,\dots,i_m)$ *is odd.*

5. Products of conjugacy classes in $GU(2,q^2)$ and $SU(2,q^2)$.

Let G (resp. S; PS) be $GU(2,q^2)$ or $GL(2,q)$ (resp. $SU(2,q^2)$ or $SL(2,q)$; $PSU(2,q^2)$ or $PSL(2,q)$). We follow the sign convention from §1.3.

5.1. Class products in $GU(2,q^2)$ and $GL(2,q)$. We use the notation from [6] for conjugacy classes in G. The classes (and the respective Jordan normal forms) are:

$$C_1^{(k)} : \begin{pmatrix} \omega^k & 0 \\ 0 & \omega^k \end{pmatrix}, \quad C_2^{(k)} : \begin{pmatrix} \omega^k & 0 \\ 1 & \omega^k \end{pmatrix}, \quad C_3^{(k,l)} : \begin{pmatrix} \omega^k & 0 \\ 0 & \omega^l \end{pmatrix}, \quad C_4^{(k)} : \begin{pmatrix} \rho^k & 0 \\ 0 & \rho^{\mp qk} \end{pmatrix}.$$

In the last two cases we have $C_3^{(k,l)} = C_3^{(l,k)}$, $C_4^{(k)} = C_4^{(\mp qk)}$ and we claim that the matrix is non-scalar, i. e., that $k \neq l$ and $k \not\equiv \mp qk \mod q^2-1$ respectively. There are four families of irreducible characters: $\chi_1^{(t)}$, $\chi_q^{(t)}$ $(0 \le t \le q)$, $\chi_{q\mp1}^{(t,u)}$ $(1 \le t < u \le q \pm 1)$, $\chi_{q\pm1}^{(t)}$ $(1 \le t \le q^2,\ t \not\equiv 0 \mod q \mp 1,\ \chi_{q\pm1}^{(t)} = \chi_{t+1}^{(\mp qt)})$; see details in [6]. We denote the union of all $C_i^{(\dots)}$ by C_i. We define $\delta_{a_1,\dots,a_m}$ in the same way as in §1.6.

Theorem 5.1. *Let* $A_1,\dots,A_m \in G \setminus C_1$, $m \ge 3$, *be matrices which satisfy* (2). *Let* $A_\nu \in C_{i_\nu}$, $\nu = 1,\dots,m$. *Let*

$$i_0 = \begin{cases} 3, & G = GU, \\ 4, & G = GL, \end{cases} \qquad \text{and} \qquad C = \begin{cases} C_3^{(0,2)} \cup C_3^{(1,3)}, & G = GU(2,q^2), \\ C_4^{(2)}, & G = GL(2,q). \end{cases}$$

Then $I \notin A_1^G \dots A_m^G$ *if and only if one of the following conditions holds up to permutation of* $A_1,\dots,A_m$*:*

(i) $(i_1,\dots,i_m) = (i_0,i_0,2)$ *and* $\delta_{111}+\delta_{121} = 1$*;*
(ii) $q=3$, $A_1,\dots,A_{m-1} \in C$, *and* $A_m \in C_2$.
(iii) $q=2$, *and* 2 *occurs an odd number of times in* $(i_1,\dots,i_m)$.

Proof. Case $m=3$. It is enough to compute the structure constants. They are listed in Table 8.

Table 8. Structure constants for $G = GU(2,q^2)$ or $GL(2,q)$, $A_\nu \in C_{i_\nu}$

(i_1,i_2,i_3)	$N_G(A_1,A_2,A_3)/\lvert A_1^G\rvert$	(i_1,i_2,i_3)	$N_G(A_1,A_2,A_3)/\lvert A_1^G\rvert$
$(2,2,2)$	$q-2\delta_{111}$	$(4,3,2)$	$q\mp1$
$(3,2,2)$	$q\pm1$	$(4,3,3)$	$q\mp1$
$(3,3,2)$	$(q\pm1)(1\mp(\delta_{111}+\delta_{121}))$	$(4,4,2)$	$(q\mp1)(1\pm(\delta_{111}+\delta_{121}))$
$(3,3,3)$	$q\pm1\mp q\Delta$	$(4,4,3)$	$q\mp1$
$(4,2,2)$	$q\mp1$	$(4,4,4)$	$q\mp1\pm q\Delta$

$\Delta = \delta_{111}+\delta_{112}+\delta_{121}+\delta_{211}$

Case $m=4$. Suppose that $q \ge 4$. Let $C' = C_4$ if $G = GU$ and $C' = C_3$ if $G = GL$. Then for any $d \in \Omega$ there exists $B \in C'$ such that $\det B = d$. Hence we can choose $B \in C'$ such that $\det B = \det(A_1A_2)$. Then it follows from the above computations for $m=3$ that $B \in A_1^GA_2^G$ and $B^{-1} \in A_3^GA_4^G$.

When $q=3$, the result easily follows from the following fact. If (A_1,A_2,A_3) is a triple of non-scalar matrices which does not satisfy (ii), then $A_1^G A_2^G A_3^G$ is a coset of S in G, maybe, with one scalar matrix missing. If $q=2$, then G is isomorphic to $S_3\times\Omega$. $\square$

5.2. Conjugacy classes in $SU(2,q^2)\cong SL(2,q)$. In this section we do not apply the convention of §1.3. We use here "SU-language" but, using Table 9, everything can be easily translated to "SL-language". So, we set $S=SU(2,q^2)$ and $G=GU(3,q^2)$ and the notation $C_i^{(\dots)}$ is used for conjugacy classes of G and S (except the second column of Table 9).

It is known that S is isomorphic to $SL(2,q)$. In fact, these groups are conjugated in $GL(2,q^2)$ (but not in $SL(2,q^2)$!). Indeed, let $z\in\mathbb{F}_{q^2}$ be such that $\bar z=-z$. Then the Hermitian form $\left(\begin{smallmatrix}0&z\\ \bar z&0\end{smallmatrix}\right)$ is preserved by any element of $SL(2,q)$. We fix an isomorphism $\Phi: SU(2,q^2)\to SL(2,q)$.

If q is even, then $G=S\times\Omega$, so the class product problem for S is reduced to that for G (see §5.4 for more details). So, we suppose that $q=p^m=2r-1$. We set also $r'=r-1$ (so, $q=2r'+1$). In this case we can choose $z=\rho^r$.

The conjugacy classes of S are as follows. Each of $C_2^{(k)}$, $k=0,r$, splits into two classes $C_2^{(k,l)}$, $l=0,1$ so that $\Phi\big(C_2^{(k,l)}\big)$ is the conjugacy class in $SL(2,q)$ of $(-1)^{k/r}\left(\begin{smallmatrix}1&0\\ \sigma^l&1\end{smallmatrix}\right)$ where $\sigma=\rho^{q+1}$ is a generator of $\mathbb{F}_q^*$. This notation of conjugacy classes in S depends on the choice of Φ.

Other conjugacy classes of G contained in S are conjugacy classes of S. The list of all conjugacy classes of the both groups and the correspondence between them under the isomorphism Φ is given in Table 9.

Table 9. Correspondence of classes in $SU(2,q^2)$ and $SL(2,q)$, $q=3r-1=3r'+1$

Class in SU	Class in SL	Range of the parameters	Order
$C_1^{(rk)}$	$C_1^{(r'k)}$	$k=0,1$	$k+1$
$C_2^{(rk,l)}$	$C_2^{(r'k,l)}$	$k=0,1;\quad l=0,1$	$(k+1)p$
$C_3^{(k,-k)}$	$C_4^{((q-1)k)}$	$k=1,\dots,r-1$	$(q+1)/\gcd(q+1,k)$
$C_4^{((q+1)k)}$	$C_3^{(k,-k)}$	$k=1,\dots,r'-1$	$(q-1)/\gcd(q-1,k)$

The class product problem for pairs of matrices (to determine the class of the inverse matrix) has an evident solution for C_1,C_3,C_4. The answer for C_2 is:

Proposition 5.2. *Let $A\in SU(2,q^2)$, $q=2r-1$. Let $A\in C_2^{(k,l)}$, $k,rl\in\{0,r\}$. Then $A^{-1}\in C_2^{(k,l)}$ when r is odd and $A^{-1}\in C_2^{(k,1-l)}$ when r is even.*

Proof. This follows from the fact that two matrices $\left(\begin{smallmatrix}1&0\\ a&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&0\\ b&1\end{smallmatrix}\right)$, $ab\ne0$ are conjugated in $SL(2,K)$ if and only if ab is a square in K. $\square$

Remark 5.3. (cp. Remark in §1.5). Let C be the conjugacy class of $\left(\begin{smallmatrix}1&0\\ 1&1\end{smallmatrix}\right)$ in $GL(2,q^2)$. Then $C\cap SL(2,q^2)$ splits into two classes, let us denote them by $C^{(0)}$ and $C^{(1)}$ However, the splitting of $C_2^{(0)}$ in SU does not follow the splitting of C. We have $C^{(1)}\cap SU=\varnothing$ and $C^{(0)}\cap SU=C_2$. This is why there is no any canonical form of these classes in $SL(2,q^2)$.

5.3. Class products in $SU(2,q^2) \cong SL(2,q)$.

Theorem 5.4. *Let $G = GU(2,q^2)$, $S = SU(2,q^2)$, $q = 2r-1$. Let $A_1,\dots,A_m \in S \setminus C_1$, $m \geq 3$, be such that $I \in A_1^G \dots A_m^G$. Then $I \notin A_1^S \dots A_m^S$ if and only if $m = 3$ and one of the following conditions holds up to change of the order of $A_1,\dots,A_m$:*

(*i*) $m = 3$, $A_\nu \in C_2^{(rk_\nu,l_\nu)}$ $(\nu = 1,2,3)$, $l_1 \neq l_2$, *and* $\delta_{111} = 0$ (*i. e.*, $k_1+k_2+k_3$ *is odd*);

(*ii*) $m = 3$, $A_\nu \in C_2^{(rk_\nu,l_\nu)}$ $(\nu = 1,2)$, $A_3 \in C_3^{(k_3,-k_3)}$, *and* $r(k_1+k_2+1)+k_3+l_1+l_2$ *is odd* (see Table 10);

(*iii*) $m = 3$, $A_\nu \in C_2^{(k_\nu,l_\nu)}$ $(\nu = 1,2)$, $A_3 \in C_4^{((q+1)k_3)}$, *and* $(r-1)(k_1+k_2+1)+k_3+l_1+l_2$ *is odd* (see Table 10);

(*iv*) $q = 3$ *and* $\varphi(A_1)+\dots+\varphi(A_m) \not\equiv 0 \mod 3$ *where* $\varphi(A) = l+1$ *if* $A \in C_2^{(2k,l)}$ *and* $\varphi(A) = 0$ *if* $A \notin C_2$ (see Remark 5.5);

(*v*) $q = 5$, $m = 3$, $A_\nu \in C_2^{(3k_\nu,l_\nu)}$ $(\nu = 1,2,3)$, $l_1 = l_2 = l_3$, *and* $\delta_{111} = 1$ (*i. e.*, $k_1+k_2+k_3$ *is even*);

(*vi*) $q = 5$, $m = 4$, $A_\nu \in C_2^{(3k_\nu,l_\nu)}$ $(\nu = 1,2,3,4)$, $l_1 = \dots = l_4$, *and* $\delta_{1111} = 0$ (*i. e.*, $k_1+\dots+k_4$ *is odd*);

(*vii*) $q = 5$, $m = 4$, $A_\nu \in C_2^{(3k_\nu,l_\nu)}$ $(\nu = 1,2,3)$, $A_4 \in C_3^{(k_4,-k_4)}$, $l_1 = l_2 = l_3$, *and* $k_1+\dots+k_4$ *is odd.*

Remark 5.5. The mapping $a \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $i \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ defines an isomorphism $F \cong SL(2,3)$ where F is the group discussed in §4.1. The class products in F are described in Proposition 4.1. The correspondence of classes between F and $SU(2,3^2)$ is: $\{(-1)^k\} \to C_1^{(2k)}$, $i^F \to C_3^{(1,-1)}$, $(-1)^k a^F \to C_2^{(2k,0)}$, $(-1)^k b^F \to C_2^{(2k,1)}$.

Table 10. $\frac{N_{SU(2,q^2)}(A_1,A_2,A_3)}{qr(q-1)}$ for $q = 2r-1$, $A_1 \in C_2^{(rk_1,l_1)}$, $A_2 \in C_2^{(rk_2,l_2)}$, $A_3 \in C_3 \cup C_4$

r	r even								r odd							
l_1, l_2	$l_1 = l_2$				$l_1 \neq l_2$				$l_1 = l_2$				$l_1 \neq l_2$			
$k_1+k_2 \mod 2$	0		1		0		1		0		1		0		1	
$k_3 \mod 2$	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
$A_3 \in C_3^{(k_3,-k_3)}$	1	0	1	0	0	1	0	1	0	1	1	0	1	0	0	1
$A_4 \in C_4^{((q+1)k_3)}$	0	1	1	0	1	0	0	1	1	0	1	0	0	1	0	1

Proof. Case $m = 3$. It is enough to consider only the triples (A_1,A_2,A_3) containing at least two matrices from C_2 (otherwise $N_S(A_1,A_2,A_3) = N_G(A_1,A_2,A_3)$). We compute $N_S(A_1,A_2,A_3)$ for all such triples. If $A_\nu \in C_2^{(k_\nu,l_\nu)}$, $\nu = 1,2,3$, then we have

$$N_S(A_1,A_2,A_3) = \begin{cases} r(r-1)(2q-(3r-3e_r+1)\delta_{111}), & l_1 = l_2 = l_3, \\ r(r-1)(r-e_r-1)\delta_{111}, & l_1 = l_2 \neq l_3, \end{cases}$$

where $e_r = \frac{1+(-1)^r}{2}$.

If $A_1 \in C_2^{(rk_1,l_1)}$, $A_2 \in C_2^{(rk_2,l_2)}$, and $A_3 \in C_3 \cup C_4$, we have $N_S(A_1, A_2, A_3) = qr(q-1)\delta^*$ where the values of δ^* are given in Table 10.

Case $m \geq 4$. The result for $m > 4$ follows from the result for $m = 4$. So we assume that $m = 4$. If $q = 3$, then S is isomorphic to the group F discussed in §4 and the result follows from Proposition 4.1. If $q = 5$, then S then it is enough to compute explicitly the structure constants for all triples and quadruples. So, we assume that $q \geq 7$.

If one of A_ν does not belong to C_2, then we can choose $B \in C_4$ such that $B \in A_1^S A_2^S$ and $B^{-1} \in A_3^S A_4^S$.

If $A_\nu \in C_2^{(rk_\nu,l_\nu)}$, $k = 1, \dots, 4$, then without loss of generality we may assume that $l_3 = l_4 = l$. Let $B \in C_2^{(rk,l)}$ where $k + k_1 + k_2$ is even. Then $B \in A_1^S A_2^S$ and $B^{-1} \in A_3^S A_4^S$. $\square$

5.4. Class products in $PSU(2, q^2) \cong PSL(2, q)$.

Let $PS = PSU(2, q^2) \cong PSL(2, q)$, $q \geq 4$. Like in Corollary 1.8, we denote the projection of a class $C_i^{(\dots)}$ by $\tilde{C}_i^{(\dots)}$. Products of conjugacy classes in PS are partially computed in [1; Ch. 4, Th. 4.2]. For reader conenience we give the correspondence of notation in Tables 11.1 – 11.2.

Table 11.1. Conjugacy classes in $SU(2, q^2) = PSU(2, q^2) \cong SL(2, q) = PSL(2, q)$ for even q.

In [1]	Class in SU	Class in SL	Parameters	Order
C_1	$C_1^{(0)}$	$C_1^{(0)}$		1
C_2	$C_2^{(0)}$	$C_2^{(0)}$		2
R_k	$C_3^{(k,-k)}$	$C_4^{((q-1)k)}$	$k = 1, \dots, \frac{q}{2}$	$(q+1)/\gcd(q+1, k)$
K_k	$C_4^{((q+1)k)}$	$C_3^{(k,-k)}$	$k = 1, \dots, \frac{q-2}{2}$	$(q-1)/\gcd(q-1, k)$

Table 11.2. Conjugacy classes in $PSU(2, q^2) \cong PSL(2, q)$ for $q = p^m = 2r - 1$ (for a prime p), $r' = r - 1$.

In [1]	Class in PSU	Class in PSL	Parameters	Order
C_1	$\tilde{C}_1^{(0)}$	$\tilde{C}_1^{(0)}$		1
C_2	$\tilde{C}_2^{(0,0)}$	$\tilde{C}_2^{(0,0)}$		p
C_3	$\tilde{C}_2^{(0,1)}$	$\tilde{C}_2^{(0,1)}$		p
R_k	$\tilde{C}_3^{(k,-k)}$	$\tilde{C}_4^{((q-1)k)}$	$k = 1, \dots, [\frac{r}{2}]$	$r/\gcd(r, k)$
K_k	$\tilde{C}_4^{((q+1)k)}$	$\tilde{C}_3^{(k,-k)}$	$k = 1, \dots, [\frac{r'}{2}]$	$r'/\gcd(r', k)$

As in the previous section, we use here the "SU-notation" for conjugacy classes in PS (the second column in Tables 11.1 – 11.2).

Corollary 5.6. *Let $m \geq 3$, $q \geq 4$, and $c_1, \dots, c_m$ are non-identity conjugacy classes in PS. Then $I \notin c_1 \dots c_m$ if and only if $m = 3$ and one of the following cases occurs up to permutation:*

(*i*) *q is even and $(c_1, c_2, c_3) = \big(C_3^{(k,-k)}, C_3^{(k,-k)}, C_2^{(0)}\big)$, $k = 1, \dots, q/2$;*

(ii) $q = 2r-1 \equiv 1 \mod 4$ (*so, r is odd*) *and* (c_1,c_2,c_3) *is*
$\big(\tilde C_2^{(0,l_1)}, \tilde C_2^{(0,l_2)}, \tilde C_4^{((q+1)k)}\big)$, $k+l_1+l_2$ *is odd*, $k=1,\dots,\frac{r-1}{2}$;

(iii) $q = 2r-1 \equiv 3 \mod 4$ (*so, r is even*) *and* (c_1,c_2,c_3) *is one of:*
$\big(\tilde C_2^{(0,l_1)}, \tilde C_2^{(0,l_2)}, \tilde C_3^{(k,-k)}\big)$, $k+l_1+l_2$ *is odd*, $k=1,\dots,\frac{r}{2}$;
$\big(\tilde C_2^{(0,l)}, \tilde C_3^{(r/2,-r/2)}, \tilde C_3^{(r/2,-r/2)}\big)$, $l=0,1$.

In particular, we see that $\mathrm{cn}(PS)=3$, $\mathrm{ecn}(PS)=4$ (see §1.8). This fact was already proved in [1; Ch. 4].

References

1. Z. Arad, M. Herzog (eds.), *Products of conjugacy classes in groups*, Lecture Notes in Math. 1112, Springer-Verlag, Berlin, Heidelberg, N.Y., Tokyo, 1985.
2. S. Agnihotri, C. Woodward, *Eigenvalues of products of unitary matrices and quantum Schubert calculus*, Math. Research Letters **5** (1998), 817–836.
3. P. Belkale, *Local systems on $\mathbb{P}^1 - S$ for S a finite set*, Compos. Math. **129** (2001), 67–86.
4. L. E. Dickson, *Linear groups with an exposition of the Galois field theory*, Teubner, Leipzig, 1901.
5. V. Ennola, *On the conjugacy classes of the finite unitary groups*, Ann. Acad. Sci. Fennicae, Ser. A I. **313** (1962), 3–13.
6. V. Ennola, *On the characters of the finite unitary groups*, Ann. Acad. Sci. Fennicae, Ser. A I. **323** (1963), 3–35.
7. *GAP software; the file* `ctgeneri.tbl`.
8. M. Geck, *Diploma thesis.*
9. N. L. Gordeev, *Products of conjugacy classes in perfect linear groups. Extended covering number*, Zapiski Nauchn. Semin. POMI **321** (2005), 67–89 (Russian); English transl., J. Math. Sci. **136** (2006), 3867–3879.
10. S. Karni, *Covering number of groups of small order and sporadic groups*, Ch. 3 in [1], pp. 52–196.
11. A. Lev, *The covering number of the group* $\mathrm{PSL}_n(F)$, J. Algebra **182** (1996), 60–84.
12. M. W. Liebeck, A. Shalev, *Diameter of finite simple groups: sharp bounds and applications*, Ann. of Math. **154** (2001), 383–406.
13. G. Malle, B. H. Matzat, *Inverse Galois Theory*, Springer-Verlag, Berlin, N.Y., 1999.
14. S. Yu. Orevkov, *Quasipositivity test via unitary representations of braid groups and its applications to real algebraic curves*, J. of Knot Theory and Ramifications **10** (2001), 1005–1023.
15. W.A. Simpson, J.S. Frame, *The character tables for* $\mathrm{SL}(3,q)$, $\mathrm{SU}(3,q^2)$, $\mathrm{PSL}(3,q)$, $\mathrm{PSU}(3,q^2)$, Canad. J. Math. **25** (1973), 486–494.
16. G. E. Wall, *On the conjugacy classes in the unitary, symplectic and orthogonal groups*, J. Austral. Math. Soc. **3** (1963), 1–62.
17. G. E. Wall, *Conjugacy classes in projective and special linear groups*, Bull. Austral. Math. Soc. **22** (1980), 339-364.

Institut des Mathématiques de Toulouse, UPS, 118 route de Narbonne, 31062 Toulouse, France