ASYMPTOTIC NUMBER OF TRIANGULATIONS WITH VERTICES IN $\mathbf{Z}^2$

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Abstract. Let $\mathcal{T}_n^2$ be the set of all triangulations of the square $[0,n]^2$ with all the vertices belonging to $\mathbf{Z}^2$. We show that $Cn^2 < \log\operatorname{Card}\mathcal{T}_n^2 < Dn^2$.

Triangulations with integral vertices appear in the algebraic geometry. They are used in Viro's method of construction of real algebraic varieties with controlled topological properties [2]. In [1], the discriminant of a polynomial $\sum_{a\in A} c_a x^a$ with a fixed finite set of multi-indices $A\subset\mathbf{Z}^d$ is described in terms of triangulations of the convex hull of A with vertices in A. Here we study the asymptotics of the number of triangulations with integral vertices when the size of the triangulated polytope tends to infinity.

Denote by I_n the segment $[0,n]\subset\mathbf{R}$ and let $I_n^d := I_n\times\dots\times I_n\subset\mathbf{R}^d$ be the d-dimensional cube with the side n. Denote by $\mathcal{T}_n^d$ the set of all triangulations of I_n^d whose vertices are integral points.

Question. *What are the asymptotics of* $\log\operatorname{Card}\mathcal{T}_n^d$ *when* $n\to\infty$ *?*

Only the evident estimates are known for an arbitrary $d\ge 2$:

$$A_d\,n^d < \log\operatorname{Card}\mathcal{T}_n^d < B_d\,n^d\log n, \qquad (A_d, B_d > 0).$$

To get the left inequality, divide I_n^d into n^d cubes; each of them can be subdivided into simplices at least in two ways. To obtain the right inequality, note that the number of all integral d-simplices contained in I_n^d is bounded by $c_1n^{c_2}$ and the number of d-simplices in each $T\in\mathcal{T}_n^d$ is bounded by c_3n^d, hence,

$$\operatorname{Card}\mathcal{T}_n^d \le \sum_{q=1}^{c_3n^d}\binom{c_1n^{c_2}}{q} \le c_4(c_1n^{c_2})^{c_3n^d}$$

(here $c_1,\dots,c_4$ depend on d but do not depend on n).

In this note we show that for $d=2$ the main term of the asymptotics is const$\cdot n^2$.

A triangulation $T\in\mathcal{T}_n^d$ is called *primitive* if the volume of each d-dimensional simplex equals $1/d!$. Let $\mathcal{PT}_n^d = \{T\in\mathcal{T}_n^d \mid T \text{ is primitive}\}$.

Theorem. *There exist positive constants A and B such that*

$$An^2 < \log\operatorname{Card}\mathcal{PT}_n^2 < Bn^2.$$

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Corollary. *There exist positive constants C and D such that*

$$Cn^2 < \log \operatorname{Card} \mathcal{T}_n^2 < Dn^2.$$

Denote by $\pi : I_n^2 \to I_n$ the projection $(x,y) \mapsto x$. Given a $T \in \mathcal{T}_n^2$, let us say that a subset $U \subset I_n^2$ is T-*univalent* if U is a closed simplicial subcomplex of T such that $I_n \times \{0\} \subset U$, and all the fibers $(\pi|_U)^{-1}(x)$, $x \in I_n$ are connected.

Lemma. *Let $T \in \mathcal{PT}_n^2$ and let U be a T-univalent subset of I_n^2. If $U \neq I_n^2$ then there exists a triangle $\sigma \in T$ such that $\sigma \not\subset U$ and $U \cup \sigma$ is T-univalent.*

Proof. Denote by T_U the set of all triangles $\sigma \in T$ such that $\sigma \not\subset U$ and $\sigma \cap U$ contains a segment. Let $\sigma \in T_U$. Denote the vertices of σ by a, b, c so that $\pi(a) \le \pi(b)$, $[ab] \subset U$. We shall say that σ *hangs to the left* (resp. *to the right*) if $\pi(c) < \pi(a)$ (resp. $\pi(b) < \pi(c)$) and $c \notin U$. Denote by $T_U^{(L)}$ (resp. $T_U^{(R)}$) the set of triangles $\sigma \in T_U$ hanging to the left (resp. to the right). Clearly that if $\sigma \in T_U$ and $U \cup \sigma$ is not T-univalent then $\sigma \in T_U^{(L)} \cup T_U^{(R)}$. Thus, in the case $T_U^{(L)} = T_U^{(R)} = \varnothing$ we can choose any $\sigma \in T_U$. Suppose $T_U^{(L)} \neq \varnothing$ (the case $T_U^{(R)} \neq \varnothing$ can be treated the same way). Say that $\sigma_1 \in T_U$ is to the left of $\sigma_2 \in T_U$ if $\sigma_1 \cap U$ is to the left of $\sigma_2 \cap U$. Let σ_0 be the most left triangle from $T_U^{(L)}$ and let $\sigma \in T_U$ be such that $\sigma \cap U$ is adjacent from the left to $\sigma_0 \cap U$. Then $\sigma \notin T_U^{(L)}$ because σ_0 was the most left. We have also $\sigma \notin T_U^{(R)}$. Indeed, otherwise σ would intersect σ_0 because σ hangs to the right and σ_0 hangs to the left. Hence, $U \cup \sigma$ is T-univalent. □

Proof of Theorem. As we pointed out above, the estimate $An^2 < \log \operatorname{Card} \mathcal{PT}_n^2$ is evident. Let us prove that $\log \operatorname{Card} \mathcal{PT}_n^2 < Bn^2$.

To each $T \in \mathcal{PT}_n^2$ we associate the sequence of T-univalent subsets $I_n \times \{0\} = U_0 \subset U_1 \subset \cdots \subset U_N = I_n^2$, $N = 2n^2$, where $U_{j+1} = U_j \cup \sigma_j$ and σ_j is the most left among the triangles $\sigma \in T$ such that $\sigma \not\subset U_j$, $\sigma \cup U_j$ is T-univalent. Denote by E the set of all edges of T and put $E_j = \{e \in E \,|\, e \subset U_j\}$, $k_j = \operatorname{Card} E_j$. Let $e_1, \dots, e_{k_N}$ be all the edges of T numerated so that $E_0 = \{e_1, \dots, e_{k_0}\}$ and $E_{j+1} \setminus E_j = \{e_{k_j+1}, \dots, e_{k_{j+1}}\}$ $(j = 0, ..., N)$, the elements of each $E_{j+1} \setminus E_j$ being numerated from the left to the right. Let us define the vector $v_T = (v_1, \dots, v_{k_N})$ as follows. If e_j is vertical or $e_j \subset I_n \times \{1\}$ then we put $v_j = 1$. Otherwise, if $e_j = [ab]$, $\pi(a) < \pi(b)$, and $\sigma = [abc] \in T$ is the triangle adjacent to $[ab]$ from above, then put

$$v_j = \begin{cases} 1 & \text{if } \pi(c) < \pi(a), \\ 2 & \text{if } \pi(c) = \pi(a), \\ 3 & \text{if } \pi(a) < \pi(c) \le \pi(b), \\ 4 & \text{if } \pi(c) > \pi(b). \end{cases}$$

It is clear that that the number of all possible vectors v_T is bounded by 4^{3n^2}. Therefore, to complete the proof it suffices to show that T is uniquely determined by v_T. Indeed, if v_T is known then one can inductively reconstruct all the sets $E_0, E_1, \dots$ as follows. Suppose U_j is already reconstructed. Let $e_{j_1}, \dots, e_{j_m}$ be all the non-vertical edges (numerated from the left to the right) lying on $\partial U_j \setminus \partial I_n^2$. Then the triangle σ, adjacent to e_{j_i} from above, being attached to E_j yields a T-univalent set if and only if one of the following three cases holds: (i) $2 \le v_{j_i} \le 3$;

(ii) $v_{j_{i-1}} = 4$ and $v_{j_i} = 1$; (iii) $v_{j_i} = 4$ and $v_{j_{i+1}} = 1$. In all the cases σ is uniquely determined by $\sigma \cap E_j$ (due to the primitivity condition). $\square$

Proof of Corollary. Each $T \in \mathcal{T}_n^2$ can be subdivided to a $T' \in \mathcal{PT}_n^2$. Let E and E' be the sets of edges of T and T' not lying on ∂I_n^2. Clearly that T is uniquely determined by T' and E. Hence, $\operatorname{Card} \mathcal{T}_n^2 \le \sum_{T'} \operatorname{Card}\{E \subset E'\} < (\operatorname{Card} \mathcal{PT}_n^2) \cdot 2^{3n^2}$ since $\operatorname{Card} E' = 3n^2 - 2n < 3n^2$. $\square$

The properties of integral points were essential to our proof. However, the following generalization seems to be true. Given a finite set $A \subset \mathbf{R}^2$, denote by $\mathcal{T}(A)$ the set of triangulations of the convex hull of A with vertices belonging to A.

Conjecture. *There exists a constant* C_1 *such that* $\log \operatorname{Card} \mathcal{T}(A) \le C_1 \operatorname{Card} A$ *for any finite* $A \subset \mathbf{R}^2$.

This is well-known when A is the set of vertices of a convex polygon. In this case $\operatorname{Card} \mathcal{T}(A)$ is the Catalan number $\frac{1}{n-1}\binom{2n-4}{n-2}$ where $n = \operatorname{Card} A$. Note also, that an analogue of the Lemma is true for any finite $A \subset \mathbf{R}^2$.

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References

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