ASYMPTOTICS OF THE NUMBER OF LATTICE TRIANGULATIONS OF RECTANGLES OF WIDTH 4 AND 5

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ABSTRACT. Let $f(m,n)$ be the number of primitive lattice triangulations of an $m \times n$ rectangle. We express the limits $\lim_n f(m,n)^{1/n}$ for $m = 4$ and $m = 5$ in terms of certain systems of Fredholm integral equations on generating functions (the case $m \le 3$ was treated in a previous paper). Solving these equations numerically, we compute approximate values of these limits with a rather high precision.

1. INTRODUCTION. MAIN RESULTS

This paper is a sequel of [7]. A *lattice triangulation* of a polygon in $\mathbb{R}^2$ is a triangulation with all vertices in $\mathbb{Z}^2$. A lattice triangulation is called *primitive* (or *unimodular*) if each triangle is primitive, i.e., has the minimal possible area 1/2 (a translate of $[(0,0),(x_1,y_1),(x_2,y_2)]$ with $x_1y_2 - x_2y_1 = 1$). We denote the number of primitive lattice triangulations of an $m \times n$ rectangle by $f(m,n)$. Let

$$c_{m,n} = \frac{\log_2 f(m,n)}{mn}, \qquad c_m = \lim_{n\to\infty} c_{m,n}, \qquad c = \lim_{m\to\infty} c_m = \lim_{n\to\infty} c_{n,n}.$$

The existence of the limits is proven in [3, Proposition 3.6]. In [3] the number $c(m,n)$ is called the *capacity* of an $m \times n$ rectangle. Some estimates of c_m and c were obtained in [1, 3, 5–7, 11] (see more details in [7]). In particular, the best known upper bound for c is $c \le 4\log_2 \frac{1+\sqrt{5}}{2} \approx 2.777$; see [11].

It is evident that $f(1,n) = \binom{2n}{n}$ and hence $c_1 = 2$. The exact value $c_2 = \frac{1}{2}\log_2 \frac{611+\sqrt{73}}{36}$ and 360 decimal digits of c_3 were computed in [7]. Here we compute 65 digits of c_4 and 15 digits of c_5 using the same approach, namely,

$$\lim_n f(4,n)^{\frac{1}{4n}} = 4.2987667575009691116179591311174699815749217822498628476374561525 1,$$

$$\lim_{n\to\infty} f(5,n)^{\frac{1}{5n}} = 4.340961619318,$$

and hence

$$c_4 = 2.1039228346930779088550919476503529059916330199147508947027581798 0,$$

$$c \ge c_5 = 2.11801466703561.$$

So far this is the best proven lower bound for c.

As in [7], we express c_4 (see §2) and c_5 (see §3) in terms of solutions of systems of integral equations, and we compute the approximate values of c_4 and c_5 by solving systems of usual linear equations obtained by replacing the integrals with Riemann sums. However, the computations here are more complicated than in [7].

Another difference from [7] is the following. In [7] we reduced the problem to a single classical Fredholm equation of the first kind with an analytic kernel of integration. Since the integral operator was compact, it was easy to prove the uniqueness of its solution and the exponential rate of convergence of its discretizations. In the present paper, the integral operators are no longer compact: the unknown functions are functions of two (for c_4) or three (for c_5) variables but the operators involve the integration with respect to only one variable (i.e., the kernel of integration is a kind of delta-function). In §2.6 we discuss the uniqueness of solutions and the convergence of discretizations for such operators.

In §4 we expose two simple observations which allowed us to improve the convergence. They were crucial for the computation of c_4 and c_5 with so high accuracy. In §5 we report about newly computed exact values of $f(m,n)$ and about an empirical estimate for c_6, c_7, and for the subexponential factor of the asymptotics, which we computed via known values of $f(m,n)$ by a method proposed by Lando and Zvonkin in [4, §6]. In §6 we give an asymptotic lower bound for the number of all (not necessarily primitive) lattice triangulations.

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2. Strips of width 4

In this section we express c_4 via solution of a certain system of integral equations on the generating functions, whose numerical solution allows us to compute c_4 to 65 decimal digits.

2.1. **Reduction to trapezoids.** (Cf. [7, §4.1].) For $a,e\ge 0$ such that $a+e$ is even, let $j^*_{a,e}$ be the number of primitive triangulations of the trapezoid $T_4(a,e)$ spanned by $(0,0)$, $(1,4)$, $(1+e,4)$, $(a,0)$. We set $j^*_{0,0}=1$ and $j^*_{a,e}=0$ when $a+e$ is odd. Consider the generating function

$$J^*(x)=\sum_n j^*_n x^n=\sum_{a,e\ge 0} j^*_{a,e}x^{a+e}=1+6x^2+750x^4+189121x^6+\dots$$

Let $j_{a,e}$ be the number of the primitive triangulations of $T_4(a,e)$ which do not have interior edges of the form $[(k,0),(l,4)]$ (by convention, $j_{0,0}=0$) and let

$$J(x)=\sum_n j_n x^n=\sum_{a,e\ge 0} j_{a,e}x^{a+e}=6x^2+714x^4+180337x^6+\dots$$

Then we have

$$J^*(x)=\frac{1}{1-J(x)}\qquad\text{and}\qquad \lim_{n\to\infty} f(4,n)^{1/n}=\lim_{n\to\infty}(j^*_{2n})^{1/n}$$

(see [7, §4.1] and [7, Lemma 3.1] respectively). Thus

$$c_4=-\frac{1}{2}\log_2\beta_4, \tag{2.1}$$

where β_4 is the first real root of the equation $J(x)=1$.

2.2. **Recurrence relations.** We introduce a notation similar to that in [7, §4]. For integers a,b,c,d,e such that $a\ge -1$ and $b,c,d,e\ge 0$, let $f_{a,b,c,d,e}$ denote the number of primitive lattice triangulations of the polygon $[(0,a),(1,b),(2,c),(3,d),(4,e),(4,0),(0,-1)]$ (see Figure 1) without interior edges of the form $[(0,y_1),(4,y_2)]$ (side-to-side edges). Following [3], we call these polygons *shapes*. Similarly we define g,h,j with corresponding subscripts and superscripts according to Figure 1. For example, if $a\ge -1$, $\min(c,d,e)\ge 0$, and $c-a\equiv 1 \bmod 2$, then we define $g^{(1)}_{a,c,d,e}$ to be the number of primitive lattice triangulations of the shape $[(0,a),(2,c),(3,d),(4,e),(4,0),(0,-1)]$ without interior side-to-side edges. If the inequalities or congruences are not satisfied, we set the corresponding numbers to zero. By convention, $j^{(1)}_{-1,0}=0$; in this case the shape degenerates to the segment $[(0,-1),(4,0)]$.

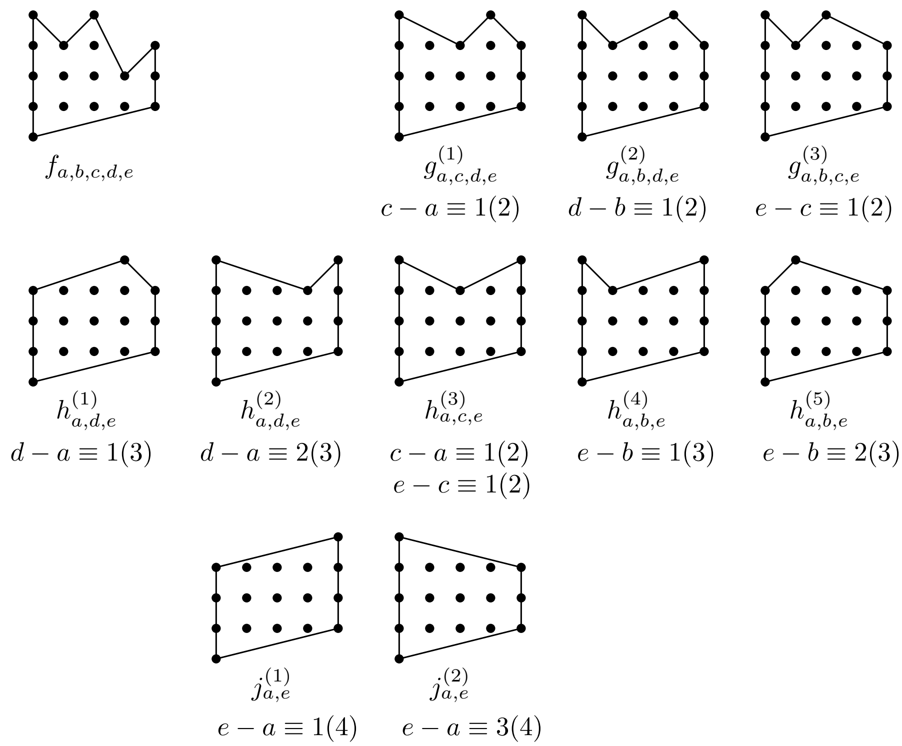

FIGURE 1

We introduce the generating functions for these numbers

$$F(x,y,z,w,v)=\sum_{a,b,c,d,e} f_{a,b,c,d,e}x^a y^b z^c w^d v^e, \qquad J_k(x,v)=\sum_{a,e} j^{(k)}_{a,e}x^a v^e \quad (k=1,2),$$

$$G_1(x,z,w,v)=\sum_{a,c,d,e} g^{(1)}_{a,c,d,e}x^a z^c w^d v^e, \qquad G_2(x,y,w,v)=\sum_{a,b,d,e} g^{(2)}_{a,b,d,e}x^a y^b w^d v^e,$$

$$G_3(x,y,z,v)=\sum_{a,b,c,e} g^{(3)}_{a,b,c,e}x^a y^b z^c v^e, \qquad H_3(x,z,v)=\sum_{a,c,e} h^{(3)}_{a,c,e}x^a z^c v^e,$$

$$H_k(x,w,v)=\sum_{a,d,e} h^{(k)}_{a,d,e}x^a w^d v^e \ (k=1,2), \quad H_k(x,y,v)=\sum_{a,b,e} h^{(k)}_{a,b,e}x^a y^b v^e \ (k=4,5),$$

and their symmetrizations

$$\begin{aligned}
\widetilde{F}(x,y,z,w,v)&=F(x,y,z,w,v)+F(v,w,z,y,x),\\
\widetilde{G}_1(x,z,w,v)&=G_1(x,z,w,v)+G_3(v,w,z,x),\\
\widetilde{G}_2(x,y,w,v)&=G_2(x,y,w,v)+G_2(v,w,y,x),\\
\widetilde{H}_k(x,w,v)&=H_k(x,w,v)+H_{6-k}(v,w,x), \qquad k=1,2,\\
\widetilde{H}_3(x,z,v)&=H_3(x,z,v)+H_3(v,z,x),\\
\widetilde{J}_1(x,v)&=J_1(x,v)+J_2(v,x).
\end{aligned}$$

Throughout the paper we use the notation $\mathfrak{cf}_{\mathbf{m}}\,\mathcal{F}$ for the coefficient of a monomial $\mathbf{m} = \mathbf{m}(x_1, x_2, \dots)$ in a Laurent series $\mathcal{F} = \mathcal{F}(x_1, x_2, \dots)$. In this notation, the recurrence relations in [7, Lemma 2.2] take the following form, cf. [7, §4.2]; here we omit the intermediate step consisting in finding the relations between the non-symmetrized generating functions.

$$\begin{aligned}\widetilde{F}(x,y,z,w,v)Q(x,y,z,w,v) = {}& y^{1/2}\widetilde{G}_1\big(xy^{1/2}, y^{1/2}z, w, v\big)(1-w-v)\\ &+ z^{1/2}\widetilde{G}_2\big(x, yz^{1/2}, z^{1/2}w, v\big)(1-x)(1-v)\\ &+ w^{1/2}\widetilde{G}_1\big(vw^{1/2}, w^{1/2}z, y, x\big)(1-y-x)\\ &- y^{1/2}w^{1/2}\widetilde{H}_3\big(xy^{1/2}, y^{1/2}zw^{1/2}, w^{1/2}v\big),\end{aligned}$$

where $Q(x,y,z,w,v) = 1 - x - y - z - w - v + xz + xw + xv + yw + yv + zv - xzv$,

$$\begin{aligned}\widetilde{G}_1(x,z,w,v)(1-w-v) = {}& \mathfrak{cf}_{\xi^{-1}}\,\widetilde{F}\Big(\frac{x}{\xi}, \xi^2, \frac{z}{\xi}, w, v\Big)(1-w-v)\\ &- w^{1/2}\,\mathfrak{cf}_{\xi^{-1}}\,\widetilde{G}_1\Big(vw^{1/2}, \frac{w^{1/2}z}{\xi}, \xi^2, \frac{x}{\xi}\Big)\\ &+ z^{1/3}(1-v)\widetilde{H}_1\big(xz^{1/3}, z^{2/3}w, v\big)\\ &+ w^{1/2}\widetilde{H}_3\big(x, zw^{1/2}, w^{1/2}v\big),\end{aligned}$$

$$\begin{aligned}\widetilde{G}_2(x,y,w,v)(1-x)(1-v) = {}& \mathfrak{cf}_{\xi^{-1}}\,\widetilde{F}\Big(x, \frac{y}{\xi}, \xi^2, \frac{w}{\xi}, v\Big)(1-x)(1-v)\\ &+ y^{1/3}(1-v)\widetilde{H}_2\big(xy^{2/3}, y^{1/3}w, v\big)\\ &+ w^{1/3}(1-x)\widetilde{H}_2\big(vw^{2/3}, w^{1/3}y, x\big),\end{aligned}$$

$$\widetilde{H}_1(x,w,v)(1-v) = \mathfrak{cf}_{\xi^{-1}}\,\widetilde{G}_2\Big(\frac{x}{\xi^2}, \xi^3, \frac{w}{\xi}, v\Big)(1-v) + \frac{1}{x},$$

$$\widetilde{H}_2(x,w,v) = \mathfrak{cf}_{\xi^{-1}}\,\widetilde{G}_1\Big(\frac{x}{\xi}, \xi^3, \frac{w}{\xi^2}, v\Big),$$

$$\widetilde{H}_3(x,z,v) = \mathfrak{cf}_{\xi^{-1}}\widetilde{G}_1\Big(x, \frac{z}{\xi}, \xi^2, \frac{v}{\xi}\Big) + \mathfrak{cf}_{\xi^{-1}}\widetilde{G}_1\Big(v, \frac{z}{\xi}, \xi^2, \frac{x}{\xi}\Big) - \mathfrak{cf}_{\xi_1^{-1}\xi_2^{-1}}\widetilde{F}\Big(\frac{x}{\xi_1}, \xi_1^2, \frac{z}{\xi_1\xi_2}, \xi_2^2, \frac{v}{\xi_2}\Big),$$

$$\widetilde{J}_1(x,v) = \mathfrak{cf}_{\xi^{-1}}\,\widetilde{H}_2\Big(\frac{v}{\xi}, \xi^4, \frac{x}{\xi^3}\Big).$$

Notice that (see §2.1 for the definition of $J(x)$)

$$J(x) = x\widetilde{J}_1(x,x). \tag{2.2}$$

2.3. **A change of variables.** We set

$$f(t,s,u) = \widetilde{F}\Big(\frac{x}{t}, \frac{x^2t^2}{s}, \frac{x^2s^2}{tu}, \frac{x^2u^2}{s}, \frac{x}{u}\Big), \qquad q(t,s,u) = Q\Big(\frac{x}{t}, \frac{x^2t^2}{s}, \frac{x^2s^2}{tu}, \frac{x^2u^2}{s}, \frac{x}{u}\Big),$$

$$g_1(s,u) = \frac{1}{s^{1/2}}\,\widetilde{G}_1\Big(\frac{x^2}{s^{1/2}}, \frac{x^3s^{3/2}}{u}, \frac{x^2u^2}{s}, \frac{x}{u}\Big), \quad g_2(t,u) = \frac{1}{(tu)^{1/2}}\,\widetilde{G}_2\Big(\frac{x}{t}, \frac{x^3t^{3/2}}{u^{1/2}}, \frac{x^3u^{3/2}}{t^{1/2}}, \frac{x}{u}\Big),$$

$$h_k(u) = \frac{1}{u^{k/3}}\widetilde{H}_k\Big(\frac{x^3}{u^{1/3}}, x^4u^{4/3}, \frac{x}{u}\Big)\ \ (k = 1{,}2), \qquad h_3(s) = \widetilde{H}_3\Big(\frac{x^2}{s^{1/2}}, x^4s, \frac{x^2}{s^{1/2}}\Big),$$

$$j_1(x) = \widetilde{J}_1(x^4, x^4), \qquad p(s,u) = 1 - \frac{x^2u^2}{s} - \frac{x}{u}, \qquad r(t,u) = \Big(1 - \frac{x}{t}\Big)\Big(1 - \frac{x}{u}\Big).$$

The combinatorial meaning of f, g_k, h_k, j_1 is the following. Let S be a shape contributing to the coefficient of a monomial $\mathbf{m}$ of one of these series. Then the exponent of x in $\mathbf{m}$ is equal to $2\int_0^4 \varphi(x)dx$ where $y=\varphi(x)$ is the equation of the upper boundary of S. The variables t,s,u correspond to the vertical lines $x=1$, $x=2$, $x=3$ respectively. If (x_0,y_0), (x_1,y_1), (x_2,y_2) are three consecutive vertices on the upper boundary of S, then the exponent of the variable corresponding to the vertical line $x=x_1$ is equal to the integer part of the difference of the slopes of the adjacent segments, that is

$$\left\lfloor \frac{y_1-y_0}{x_1-x_0} - \frac{y_2-y_1}{x_2-x_1} \right\rfloor,$$

in particular, $\mathbf{m}$ and the congruences in Figure 1 determine the upper boundary of S up to automorphisms of $\mathbb{Z}^2$ of the form $(x,y)\mapsto(x,y+ax+b)$. For example, the shapes for f_{32312} and $g^{(1)}_{3232}$ depicted in Figure 1 contribute to the coefficients of the monomials $x^{17}t^{-1}s^3u^{-3}$ (in f) and $x^{20}s^{-2}u^2$ (in g_1) respectively.

It follows that f,g_k,h_k,j_1 are elements of the ring

$$\mathbb{Z}[t^{\pm1},s^{\pm1},u^{\pm1}]((x))$$

of formal power series in x whose coefficients are Laurent polynomials in t,s,u. Division of such power series will be understood as the division in this ring.

Applying the relation from §2.2 we obtain

$$\begin{aligned} f(t,s,u)\,q(t,s,u) &= \frac{xt}{s^{1/2}}\,\widetilde{G}_1\Big(\frac{x}{t}\,\frac{xt}{s^{1/2}},\ \frac{xt}{s^{1/2}}\,\frac{x^2s^2}{tu},\ \frac{x^2u^2}{s},\ \frac{x}{u}\Big)\,p(s,u)\\ &\quad+\frac{xs}{(tu)^{1/2}}\,\widetilde{G}_2\Big(\frac{x}{t},\ \frac{x^2t^2}{s}\,\frac{xs}{(tu)^{1/2}},\ \frac{xs}{(tu)^{1/2}}\,\frac{x^2u^2}{s},\ \frac{x}{u}\Big)\,r(t,u)\\ &\quad+\frac{xu}{s^{1/2}}\,\widetilde{G}_1\Big(\frac{x}{u}\,\frac{xu}{s^{1/2}},\ \frac{xu}{s^{1/2}}\,\frac{x^2s^2}{tu},\ \frac{x^2t^2}{s},\ \frac{x}{t}\Big)\,p(s,t)\\ &\quad-\frac{xt}{s^{1/2}}\,\frac{xu}{s^{1/2}}\,\widetilde{H}_3\Big(\frac{x}{t}\,\frac{xt}{s^{1/2}},\ \frac{xt}{s^{1/2}}\,\frac{x^2s^2}{tu}\,\frac{xu}{s^{1/2}},\ \frac{xu}{s^{1/2}}\,\frac{x}{u}\Big). \end{aligned}$$

Given any Laurent series $\mathcal{F}(\xi,x_1,x_2,\dots)$, a monomial $\mathbf{m}=\mathbf{m}(x_1,x_2,\dots)$, and a new variable t, we have (cf. [7, Eq. (9)])

$$\mathfrak{cf}_{\xi^{-1}}\,\mathcal{F}(\xi,x_1,x_2,\dots)=\mathbf{m}\,\mathfrak{cf}_{t^{-1}}\,t^{\alpha-1}\mathcal{F}(t^\alpha\mathbf{m},x_1,x_2,\dots). \tag{2.3}$$

Using (2.3) with the substitutions $\xi=xt/s^{1/2}$ (in g_1), $\xi=xs/(tu)^{1/2}$ (in g_2), $\xi=xt^{1/2}/u^{1/6}$ (in h_1), $\xi=xs^{1/2}/u^{1/3}$ (in h_2), $\xi_1=xt/s^{1/2}$, $\xi=\xi_2=xu/s^{1/2}$ (in h_3), and $\xi=xu^{1/3}$ (in j_1), we obtain

$$\begin{aligned} g_1(s,u)p(s,u) &= \frac{1}{s^{1/2}}\Big\{p(s,u)\,\frac{x}{s^{1/2}}\,\mathfrak{cf}_{t^{-1}}\,\widetilde{F}\Big(\frac{x^2}{s^{1/2}}\,\frac{s^{1/2}}{xt},\ \frac{x^2t^2}{s},\ \frac{s^{1/2}}{xt}\,\frac{x^3s^{3/2}}{u},\ \frac{x^2u^2}{s},\frac{x}{u}\Big)\\ &\quad-\frac{xu}{s^{1/2}}\,\frac{x}{s^{1/2}}\,\mathfrak{cf}_{t^{-1}}\,\widetilde{G}_1\Big(\frac{x}{u}\,\frac{xu}{s^{1/2}},\ \frac{xu}{s^{1/2}}\,\frac{x^3s^{3/2}}{u}\,\frac{s^{1/2}}{xt},\ \frac{x^2t^2}{s},\ \frac{s^{1/2}}{xt}\,\frac{x^2}{s^{1/2}}\Big)\\ &\quad+\frac{xs^{1/2}}{u^{1/3}}\Big(1-\frac{x}{u}\Big)\widetilde{H}_1\Big(\frac{x^2}{s^{1/2}}\,\frac{xs^{1/2}}{u^{1/3}},\ \frac{x^2s}{u^{2/3}}\,\frac{x^2u^2}{s},\ \frac{x}{u}\Big)\\ &\quad+\frac{xu}{s^{1/2}}\widetilde{H}_3\Big(\frac{x^2}{s^{1/2}},\ \frac{x^3s^{3/2}}{u}\,\frac{xu}{s^{1/2}},\ \frac{xu}{s^{1/2}}\,\frac{x}{u}\Big)\Big\}, \end{aligned}$$

$$g_2(t,u)r(t,u) = \frac{1}{(tu)^{1/2}}\Big\{ r(t,u)\frac{x}{(tu)^{1/2}}\mathfrak{cf}_{s^{-1}}\widetilde{F}\Big(\frac{x}{t}, \frac{x^3t^{3/2}}{u^{1/2}}\,\frac{(tu)^{1/2}}{xs}, \frac{x^2s^2}{tu}, \frac{(tu)^{1/2}}{xs}\,\frac{x^3u^{3/2}}{t^{1/2}}, \frac{x}{u}\Big)$$
$$+ \frac{xt^{1/2}}{u^{1/6}}\Big(1-\frac{x}{u}\Big)\widetilde{H}_2\Big(\frac{x}{t}\,\frac{x^2t}{u^{1/3}}, \frac{xt^{1/2}}{u^{1/6}}\,\frac{x^3u^{3/2}}{t^{1/2}}, \frac{x}{u}\Big)$$
$$+ \frac{xu^{1/2}}{t^{1/6}}\Big(1-\frac{x}{t}\Big)\widetilde{H}_2\Big(\frac{x}{u}\,\frac{x^2u}{t^{1/3}}, \frac{xu^{1/2}}{t^{1/6}}\,\frac{x^3t^{3/2}}{u^{1/2}}, \frac{x}{t}\Big)\Big\},$$

$$h_1(u) = \frac{1}{u^{1/3}}\left\{\frac{x}{u^{1/6}}\,\mathfrak{cf}_{t^{-1}}\,\frac{1}{t^{1/2}}\widetilde{G}_2\Big(\frac{x^3}{u^{1/3}}\,\frac{u^{1/3}}{x^2t}, \frac{x^3t^{1/3}}{u^{1/2}}, \frac{u^{1/6}}{xt^{1/2}}\,x^4u^{4/3}, \frac{x}{u}\Big) + \frac{u^{1/3}}{x^3(1-x/u)}\right\},$$

$$h_2(u) = \frac{1}{u^{2/3}}\left\{\frac{x}{u^{1/3}}\,\mathfrak{cf}_{s^{-1}}\,\frac{1}{s^{1/2}}\widetilde{G}_1\Big(\frac{x^3}{u^{1/3}}\,\frac{u^{1/3}}{xs^{1/2}}, \frac{x^3s^{3/2}}{u}, \frac{u^{2/3}}{x^2s}\,x^4u^{4/3}, \frac{x}{u}\Big)\right\},$$

$$h_3(s) = \frac{2x}{s^{1/2}}\,\mathfrak{cf}_{u^{-1}}\,\widetilde{G}_1\Big(\frac{x^2}{s^{1/2}}, x^4s\,\frac{s^{1/2}}{xu}, \frac{x^2u^2}{s}, \frac{s^{1/2}}{xu}\,\frac{x^2}{s^{1/2}}\Big)$$
$$- \frac{x^2}{s}\,\mathfrak{cf}_{t^{-1}u^{-1}}\,\widetilde{F}\Big(\frac{x^2}{s^{1/2}}\,\frac{s^{1/2}}{xt}, \frac{x^2t^2}{s}, \frac{s^{1/2}}{xt}\,x^4s\,\frac{s^{1/2}}{xu}, \frac{x^2u^2}{s}, \frac{s^{1/2}}{xu}\,\frac{x^2}{s^{1/2}}\Big),$$

$$j_1(x) = x\,\mathfrak{cf}_{u^{-1}}\,\frac{1}{u^{2/3}}\widetilde{H}_2\Big(x^4\,\frac{1}{xu^{1/3}}, x^4u^{4/3}, \frac{1}{x^3u}\,x^4\Big).$$

Thus,

$$f(t,s,u)\,q(t,s,u) = xt\,g_1(s,u)p(s,u) + xs\,g_2(t,u)r(t,u) + xu\,g_1(s,t)p(s,t) - \frac{x^2tu}{s}h_3(s), \tag{2.4}$$

$$g_1(s,u)\,p(s,u) = \frac{x}{s}\,p(s,u)\,\mathfrak{cf}_{t^{-1}}\,f(t,s,u) - \frac{x^2u}{s}\,\mathfrak{cf}_{t^{-1}}\,g_1(s,t) + \frac{x(u-x)}{u}h_1(u) + \frac{xu}{s}h_3(s), \tag{2.5}$$

$$g_2(t,u) = \frac{x}{tu}\,\mathfrak{cf}_{s^{-1}}\,f(t,s,u) + \frac{xt}{t-x}h_2(u) + \frac{xu}{u-x}h_2(t), \tag{2.6}$$

$$h_1(u) = x\,\mathfrak{cf}_{t^{-1}}\,g_2(t,u) + \frac{u}{x^3(u-x)}, \tag{2.7}$$

$$h_2(u) = \frac{x}{u}\,\mathfrak{cf}_{s^{-1}}\,g_1(s,u), \tag{2.8}$$

$$h_3(s) = 2x\,\mathfrak{cf}_{u^{-1}}\,g_1(s,u) - \frac{x^2}{s}\,\mathfrak{cf}_{t^{-1}u^{-1}}\,f(t,s,u), \tag{2.9}$$

$$j_1(x) = x\,\mathfrak{cf}_{u^{-1}}\,h_2(u). \tag{2.10}$$

2.4. **Elimination of f, h_1, and h_2.** Let

$$\Phi_1(s,u) = \mathfrak{cf}_{t^{-1}}\,\frac{t}{q(t,s,u)}, \qquad \Psi_1(s,u) = 1 - \frac{x^2}{s}p(s,u)\Phi_1(s,u),$$
$$\Phi_2(t,u) = \mathfrak{cf}_{s^{-1}}\,\frac{s}{q(t,s,u)}, \qquad \Psi_2(t,u) = 1 - \frac{x^2}{tu}r(t,u)\Phi_2(t,u),$$
$$\Phi_3(s) = \mathfrak{cf}_{t^{-1}u^{-1}}\,\frac{tu}{q(t,s,u)}, \qquad \Psi_3(s) = 1 - \frac{x^4}{s^2}\Phi_3(s).$$

Remark 1. A `Wolfram Mathematica` code that checks the identities below up to $O(x^n)$ is available at `https://www.math.univ-toulouse.fr/~orevkov/tr45.html`.

We eliminate f, h_1, h_2 from (2.4)–(2.9) by plugging (2.4), (2.7), (2.8) into (2.5), (2.6), (2.9). After a simplification we obtain the following system of equations for g_1, g_2, and h_3:

$$\begin{aligned}\Psi_1(s,u)g_1(s,u) &= \frac{x^2u}{s}\,\mathfrak{cf}_{t^{-1}}\left(\frac{p(s,t)}{q(t,s,u)} - \frac{1}{p(s,u)}\right)g_1(s,t)\\ &\quad+\frac{x^2(u-x)}{u}\,\mathfrak{cf}_{t^{-1}}\left(\frac{t-x}{t\,q(t,s,u)} + \frac{1}{p(s,u)}\right)g_2(t,u)\\ &\quad+\frac{xu\Psi_1(s,u)}{s\,p(s,u)}h_3(s) \;+\; \frac{1}{x^2p(s,u)},\end{aligned} \tag{2.11}$$

$$\begin{aligned}\Psi_2(t,u)g_2(t,u) &= \frac{x^2}{u}\,\mathfrak{cf}_{s^{-1}}\left(\frac{p(s,u)}{q(t,s,u)} + \frac{t}{t-x}\right)g_1(s,u)\\ &\quad+\frac{x^2}{t}\,\mathfrak{cf}_{s^{-1}}\left(\frac{p(s,t)}{q(t,s,u)} + \frac{u}{u-x}\right)g_1(s,t) \;-\; \mathfrak{cf}_{s^{-1}}\,\frac{x^3h_3(s)}{s\,q(t,s,u)},\end{aligned} \tag{2.12}$$

$$\Psi_3(s)h_3(s) = 2x\,\mathfrak{cf}_{u^{-1}}\,\Psi_1(s,u)g_1(s,u) \;-\; x^3\,\mathfrak{cf}_{t^{-1}u^{-1}}\,\frac{r(t,u)}{q(t,s,u)}\,g_2(t,u). \tag{2.13}$$

Eliminating $h_2(u)$ from (2.8) and (2.10) we express j_1 via g_1:

$$j_1(x) = x^2\,\mathfrak{cf}_{s^{-1}u^{-1}}\,\frac{g_1(s,u)}{u}. \tag{2.14}$$

2.5. **Estimates for the radii of convergence.** Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$ centered at 0. For a series $\mathcal{F} \in \mathbb{Z}[t^{\pm1}, s^{\pm1}, u^{\pm1}]((x))$ we define its *x-radius of convergence* denoted by $R_x(\mathcal{F})$ as the supremum of positive x_0 such that $\mathcal{F}$ converges in a neighborhood of the set $[0,x_0]\times\mathbb{T}^3$ in $\mathbb{C}^4$. If all the coefficients of $\mathcal{F}$ are positive, then $R_x(\mathcal{F})$ coincides with the radius of convergence of $\mathcal{F}(x,1,1,1)$.

Lemma 2.1. *All coefficients of f, g_k, h_k, j_1, Φ_k, $1/q$, $1/p$, $1/r$, $1/(p\Psi_1)$, $1/(r\Psi_2)$, $1/\Psi_3$ are positive, and $R_x(1/q) = 1/2$, $R_x(1/p) = (-1+\sqrt{5})/2 \approx 0.618$, $R_x(1/r) = 1$.*

Proof. Let "Pos($\mathcal{F}$)" mean "all coefficients of $\mathcal{F}$ are positive". It is enough to prove $\mathrm{Pos}(1/q)$, $\mathrm{Pos}(1/(p\Psi_1))$, and $\mathrm{Pos}(1/(r\Psi_2))$ because $\mathrm{Pos}(1/q) \Rightarrow \mathrm{Pos}(\Phi_k)$, $\mathrm{Pos}(\Phi_3) \Rightarrow \mathrm{Pos}(1/\Psi_3)$, and the statement for the other series is evident.

$\mathrm{Pos}(1/q)$. Let $F_0(x,y,z,w,v)$ be defined as F but counting only the primitive triangulation such that the projection of any non-vertical edge to the horizontal axis has length 1, and the bottom of the shapes is the segment $[(0,0),(4,0)]$. Let f_0 be obtained from F_0 by the substitutions in §2.3. Then (see [7, Example 2.3]) we have $F_0 = 1/Q$. Hence $1/q = f_0$ has all positive coefficients. Since $q(1,1,1) = (1-2x)(1-2x^2)$, we have $R_x(1/q) = 1/2$.

$\mathrm{Pos}(1/(p\Psi_1))$. Let F_1 and G_{11} be defined as F and G_1 but counting only the primitive triangulations such that the projection of any non-vertical edge to the horizontal axis is either the segment $[0,2]$ or a segment of length 1, and the bottom of the shapes is the union of the segments $[(0,-1),(2,0)]\cup[(2,0),(4,0)]$. Then we have (cf. §2.2)

$$\begin{aligned}F_1(x,y,z,w,v)Q(x,y,z,w,v) &= y^{1/2}G_{11}(xy^{1/2},y^{1/2}z,w,v)(1-w-v),\\ G_{11}(x,z,w,v)(1-w-v) &= \mathfrak{cf}_{\xi^{-1}}\,F_1\Big(\frac{x}{\xi},\xi^2,\frac{z}{\xi},w,v\Big)(1-w-v)+\frac{1}{x}.\end{aligned}$$

Let f_1 and g_{11} be obtained from F_1 and G_{11} by the substitutions in §2.3. Then the analogue of (2.11) takes the form $p\,\Psi_1\,g_{11} = 1/x^2$, hence the coefficients of $1/(p\Psi_1)$ are positive.

$\mathrm{Pos}(1/(r\Psi_2))$. Let F_2 and G_{22} be defined as F and G_2 but counting only the primitive triangulations such that the projection of any non-vertical edge to the horizontal axis is either the segment $[1,3]$ or a segment of length 1, and the bottom of the shapes is the polygonal chain $[(0,-1),(1,0),(3,1),(4,1)]$. Let f_2 and g_{22} be obtained from F_2 and G_{22} by the substitutions in §2.3. Proceeding as in §§2.2–2.3, we obtain $r\,\Psi_2\,g_{22} = x^3$, whence the result. □

Remark 2. All the coefficients of $1-\Psi_k$ and $1/\Psi_k$ that we computed are also positive.

Lemma 2.2. *(a). For any $x \in [0, \frac{1}{2})$ the function $q(t,s,u)$ does not vanish on $\mathbb{T}^3$.*

(b). For any fixed $(x,s,u) \in (0,\frac{1}{2}) \times \mathbb{T}^2$, the equation $q(t,s,u) = 0$ has two roots $t_1(x,s,u), t_2(x,s,u)$ in the disk $|t| < 1$ and two roots in its complement.

For any fixed $(x,t,u) \in (0,\frac{1}{2}) \times \mathbb{T}^2$, the equation $q(t,s,u) = 0$ has two roots $s_1(x,t,u), s_2(x,t,u)$ in the disk $|s| < 1$ and two roots in its complement.

(c). One has

$$\Phi_1(s,u) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{t\,dt}{q(t,s,u)} = \frac{t_1}{q'_t(t_1,s,u)} + \frac{t_2}{q'_t(t_2,s,u)},$$

$$\Phi_2(t,u) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{s\,ds}{q(t,s,u)} = \frac{s_1}{q'_s(t,s_1,u)} + \frac{s_2}{q'_s(t,s_2,u)},$$

$$\Phi_3(s) = \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2} \frac{t\,u\,dt\,du}{q(t,s,u)} = \frac{1}{2\pi i}\int_{\mathbb{T}} \Phi_1(s,u)u\,du.$$

Proof. (a). Follows from Lemma 2.1.

(b). By (a), the number of roots in the unit disk is constant on $[0,\frac{1}{2}) \times \mathbb{T}^2$, and one easily checks that it is equal to 2 at $(x,1,1)$ for a small x.

(c). Follows from (b) by the residue formula for the Cauchy integrals. □

Remark 3. The same arguments give a computation-free proof of Lemma 4.1 in [7].

Recall that $R_x(J^*)$ is denoted in §2.1 by β_4.

Lemma 2.3. *(a). The x-radii of convergence of the series f, g_k, h_k, j_1 are greater than $\beta_4^{1/4}$, and we have $\beta_4 > 0$.*

(b). We have $R_x(\Phi_1) = R_x(\Phi_2) = R_x(\Phi_3) = 1$ and

$$R_x(\Psi_1) = 0.495375..., \qquad R_x(\Psi_2) = 0.495455..., \qquad R_x(\Psi_3) = 0.499999... \tag{2.15}$$

Proof. (a). The fact that $\beta_4 > 0$ follows from the exponential upper bound $f(m,n) < 8^{mn}$. By the arguments as in [7, Lemma 3.2], it is easy to show that the radius of convergence of each of $f(1,1,1)$, $g_k(1,1)$, $h_k(1)$, j_1 coincides with that of $J(x^4)$, hence it is greater than $\beta_4^{1/4}$. Since these Laurent series have positive coefficients, the same is true for the x-radii of convergence of f, g_k, h_k, j_1.

(b). We have $R_x(\Phi_k) \ge R_x(1/q) \ge 1/2$ by Lemma 2.1. Analyzing the behavior of the integrals in Lemma 2.2(c) when $x \to 1/2$ one can show that $\Phi_k(1,1) \sim C_k(1-2x)^{-1/2}$, $k = 1,2$, and $\Phi_3(1) \sim C_3\log(1-2x)$ for some constants C_k. We omit the details since in fact we need only the lower bounds for the radii of convergence.

The functions $p\Psi_1$, $r\Psi_2$, Ψ_3 with $t = s = u = 1$ decrease on the interval $[0,1/2)$ by Lemma 2.1 and they can be computed with any precision using Lemma 2.2(c). Then it is easy to find numerically the zero of each of them on this interval; see also Figure 2. □

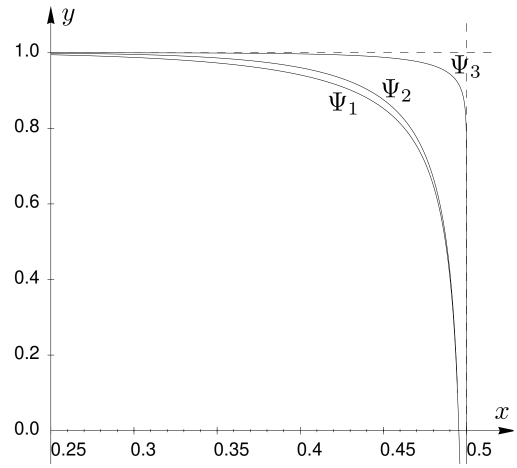

FIGURE 2. The graphs of $\Psi_1(x;1,1)$, $\Psi_2(x;1,1)$, $\Psi_3(x;1)$ on $\left[\frac{1}{4},\frac{1}{2}\right]$.

2.6. **The system of integral equations and its discretization.** Recall that β_4 is the first real root of the equation $J(x)=1$ (see §2.1). The value of c_4 announced in the introduction corresponds to $\beta_4 = 0.054114\ldots$ but our aim is to justify this computation and we do not assume yet that β_4 is close to this value. Let us set $\beta_4^+ = 0.05414$ (see Figure 3), $x_0 = \beta_4^{1/4}$, and $x_0^+ = (\beta_4^+)^{1/4} \approx 0.482369$.

In the previous subsection we have shown that the series involved in the equations (2.11)–(2.13) converge in a neighborhood of $\{x\}\times\mathbb{T}^3$ whenever $x<\min(x_0,x_0^+)$. Hence, for such x, we may replace $\mathfrak{cf}_{\xi^{-1}}(\ldots)$ by $\frac{1}{2\pi i}\int_{\mathbb{T}}(\ldots)d\xi$ (here ξ stands for t, s, or u). Then for each x we obtain a system of three integral equations for the restrictions of g_1, g_2, and h_3 to $\mathbb{T}^2$ and $\mathbb{T}$.

We are going to prove that this system has a unique solution for any fixed $x\in[0,x_0^+]$ and the solution analytically depends on x. Then, by the identity theorem for analytic functions, the series g_1, g_2, h_3 converge on this interval and, replacing the integrals by Riemann sums, we can compute g_1 at any point by solving numerically the resulting system of linear equations. This would allow us to find $J(x^4)$ (from (2.14) and (2.2)) for any $x\in[0,x_0^+]$, and (since J is monotone) to solve the equation $J(x)=1$.

In the space of operators acting on analytic functions in a neighborhood of $\mathbb{T}^2$, let us define the following subspaces $\mathcal{F}$, $\mathcal{F}_1$, $\mathcal{F}_2$:

$$\begin{aligned} T\in\mathcal{F} \quad &\Leftrightarrow\quad (Tg)(t,s)=\int_{\mathbb{T}^2}K(t,s;u,v)\,g(u,v)\,du\,dv,\\ \widehat{T}\in\mathcal{F}_1 \quad &\Leftrightarrow\quad (\widehat{T}g)(t,s)=\int_{\mathbb{T}}\widehat{K}(t,s;u)\,g(u,s)\,du,\\ \widehat{T}\in\mathcal{F}_2 \quad &\Leftrightarrow\quad (\widehat{T}g)(t,s)=\int_{\mathbb{T}}\widehat{K}(t,s;v)\,g(t,v)\,dv \end{aligned}$$

for some analytic functions K and $\widehat{K}$. We also set

$$\mathcal{F}_j'=\{\sigma\circ\widehat{T}\mid\widehat{T}\in\mathcal{F}_j\}\qquad(j=1,2),\qquad\text{where }(\sigma g)(t,s)=g(s,t).$$

By Fredholm's theory, each operator $T\in\mathcal{F}$ is compact and, if $I-T$ is invertible, then $(I-T)^{-1}=I+S$ with $S\in\mathcal{F}$.

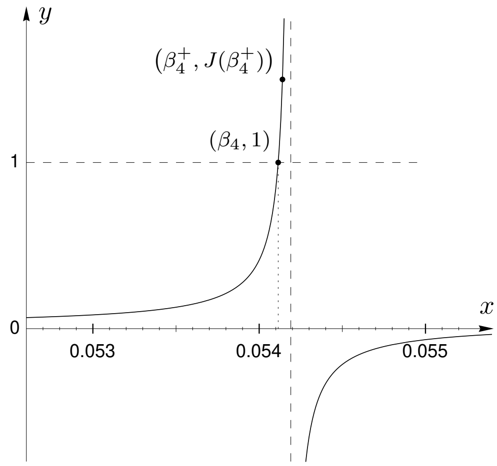

FIGURE 3. The graph of $J(x)$.

If $\widehat{T} \in \mathcal{F}_1$, then $\widehat{T}$ is no longer compact. However, $\widehat{T}$ can be considered as an analytic family of classical Fredholm operators $\{\widehat{T}_s\}_{s\in\mathbb{T}}$ acting on functions of t. Thus, if $I - \widehat{T}_s$ is invertible for each $s \in \mathbb{T}$, then $(I - \widehat{T}_s)^{-1}$ is an operator of the same form which analytically depends on s (see, e.g., [9, Ch. VI], [7, §5.4]) and hence $I - \widehat{T}$ is invertible and

$$(I - \widehat{T})^{-1} = I + \widehat{S}, \qquad \widehat{S} \in \mathcal{F}_1. \tag{2.16}$$

It is clear that $\mathcal{F}$, $\mathcal{F}_j$ are closed by composition and, for $T \in \mathcal{F}$, $\widehat{T}_j \in \mathcal{F}_j$, we have

$$T\widehat{T}_j,\ \widehat{T}_jT,\ \widehat{T}_j\widehat{T}_{3-j} \in \mathcal{F}, \qquad \sigma T_j\sigma \in \mathcal{F}_{3-j}. \tag{2.17}$$

Eliminating h_3 from (2.12) and (2.13), we obtain a system of two equations for indeterminate functions g_1, g_2 on $\mathbb{T}^2$. By (2.17) it is of the form

$$g_1 = \widehat{T}_{11}g_1 + (\widehat{T}_{12} + T_{12})g_2 + \phi, \tag{2.18}$$

$$g_2 = (\widehat{T}_{21} + \widehat{T}'_{21} + T_{21})g_1 + T_{22}\,g_2, \tag{2.19}$$

where the $\phi = 1/(x^2p\Psi_1)$, $T_{ij} \in \mathcal{F}$, and $\widehat{T}_{11}, \widehat{T}_{12}, \widehat{T}_{21}, \widehat{T}'_{21}$ belong to $\mathcal{F}_2, \mathcal{F}_1, \mathcal{F}_1, \mathcal{F}'_1$ respectively. For the integration kernels we shall use the same notation as for the operators but with T replaced by K.

One can check numerically that $\|K_{22}\|_2 < 0.06$ for $x < x_0^+$, hence the operator $I - T_{22}$ is invertible and the inverse is $I + \widetilde{T}_{22}$ with $\widetilde{T}_{22} \in \mathcal{F}$. The integration kernel $\widetilde{K}_{22}$ can be computed at any point with any precision. Resolving the equation (2.19) with respect to g_2 and plugging the result into (2.18) we obtain an equation of the form

$$(I - \widehat{T}_1 - \widehat{T}_2 - T)g_1 = \phi, \tag{2.20}$$

where $\widehat{T}_1 = \widehat{T}_{12}\widehat{T}_{21} \in \mathcal{F}_1$, $\widehat{T}_2 = \widehat{T}_{11} \in \mathcal{F}_2$, and $T \in \mathcal{F}$ by (2.17). A computation shows that $\|\widehat{K}_1|_{\mathbb{T}\times\{t\}}\|_2 \le 0.83$ and $\|\widehat{K}_2|_{\{t\}\times\mathbb{T}}\|_2 \le 0.53$ for any $x \in [0, x_0^+]$ and any fixed $t \in \mathbb{T}$. Therefore (see (2.16)) the operators $\widehat{T}_j$ are invertible and by (2.17) we have

$$(I - \widehat{T}_1)^{-1}(I - \widehat{T}_2)^{-1}(I - \widehat{T}_1 - \widehat{T}_2 - T) = I + \widetilde{T}, \qquad \widetilde{T} \in \mathcal{F}.$$

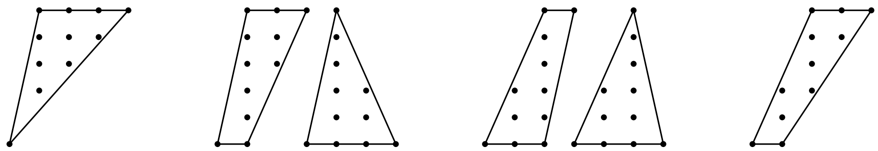

FIGURE 4. Trapezoids that contribute to $\ell^*_{\lambda\mu}(3)$ and $\ell_{\lambda\mu}(3)$.

A computation shows that $\|\widetilde{T}^{64}\|_2 \le 0.7$ (we successively computed $\widetilde{T}^2$, $\widetilde{T}^4$, $\widetilde{T}^8, \dots$), hence the operator in (2.20) is invertible and the solution of (2.20) (and hence of the initial system) analytically depends on x. Thus it coincides with g_1 for each $x \in [0, x_0^+]$.

If we replace everywhere the integrals by the n-th Riemann sums, then the integral equation at each step transforms to a system of usual linear equations for the values of the involved functions at points all whose coordinates are n-th roots of unity (see [7, §5.2] for estimates of the approximation error). The discretizations of (2.20) is equivalent to the discretization of the system (2.11)–(2.13). In our computation of c_4 we used the system (2.11)–(2.13) because the matrix of its discretization is sparse (it has only $O(n^3)$ non-zero entries) but the sparseness is lost after the elimination of h_3 (see §4.2 for the programming details).

3. STRIPS OF WIDTH 5

In this section we express c_5 via solution of a certain system of integral equations on the generating functions, which allows us to compute c_5 with 15 decimal digits. The computation and proofs are similar to those in §2 and we expose them with less details.

3.1. **Reduction to trapezoids.** Let $\lambda, \mu \in \{1, 2\}$ and let a_0, a_5 be positive integers such that $a_0 - \lambda \equiv a_5 \pm \mu \mod 5$. We set $\ell^*_{\lambda\mu}(a_0, a_5)$ to be the number of primitive triangulations of the trapezoid $T_{5,\lambda}(a_0, a_5)$ spanned by $(0,0), (\lambda, 5), (\lambda + a_5, 5), (a_0, 0)$. Let $\ell_{\lambda\mu}(a_0, a_5)$ be the number of primitive triangulations of $T_{5,\lambda}$ without interior side-to-side edges (i.e. edges of the form $[(k_0, 0), (k_5, 5)]$). We set $\ell^*_{\lambda\mu}(0,0) = 1$, $\ell_{\lambda\mu}(0,0) = 0$, and $\ell^*_{\lambda\mu}(a_0, a_5) = \ell_{\lambda\mu}(a_0, a_5) = 0$ when $a_0 - \lambda \not\equiv a_5 \pm \mu \mod 5$. Consider the generating functions (see Figure 4)

$$L^*_{\lambda\mu}(x) = \sum_{n \ge 0} \ell^*_{\lambda\mu}(n) x^n = \sum_{a_0, a_5 \ge 0} \ell^*_{\lambda\mu}(a_0, a_5) x^{a_0 + a_5},$$

$$L_{\lambda\mu}(x) = \sum_{n \ge 0} \ell_{\lambda\mu}(n) x^n = \sum_{a_0, a_5 \ge 0} \ell_{\lambda\mu}(a_0, a_5) x^{a_0 + a_5}.$$

Then the matrices

$$\boldsymbol{L}(x) = \begin{pmatrix} L_{11}(x) & L_{12}(x) \\ L_{21}(x) & L_{22}(x) \end{pmatrix}, \qquad \boldsymbol{L}^*(x) = \begin{pmatrix} L^*_{11}(x) & L^*_{12}(x) \\ L^*_{21}(x) & L^*_{22}(x) \end{pmatrix}$$

are symmetric and satisfy the relation (here $\boldsymbol{I}$ is the identity matrix)

$$\boldsymbol{L}^* = \boldsymbol{I} + \boldsymbol{L} + \boldsymbol{L}^2 + \boldsymbol{L}^3 + \dots = (\boldsymbol{I} - \boldsymbol{L})^{-1}.$$

We have

$$\boldsymbol{L} = \begin{pmatrix} 2x^2 + 79x^3 + 1075x^4 + \dots & x + 5x^2 + 84x^3 + 2104x^4 + \dots \\ x + 5x^2 + 84x^3 + 2104x^4 + \dots & x + 8x^2 + 111x^3 + 3419x^4 + \dots \end{pmatrix},$$

$$\boldsymbol{L}^* = \boldsymbol{I} + \begin{pmatrix} 3x^2 + 90x^3 + 1296x^4 + \dots & x + 6x^2 + 101x^3 + 2469x^4 + \dots \\ x + 6x^2 + 101x^3 + 2469x^4 + \dots & x + 10x^2 + 140x^3 + 3965x^4 + \dots \end{pmatrix}.$$

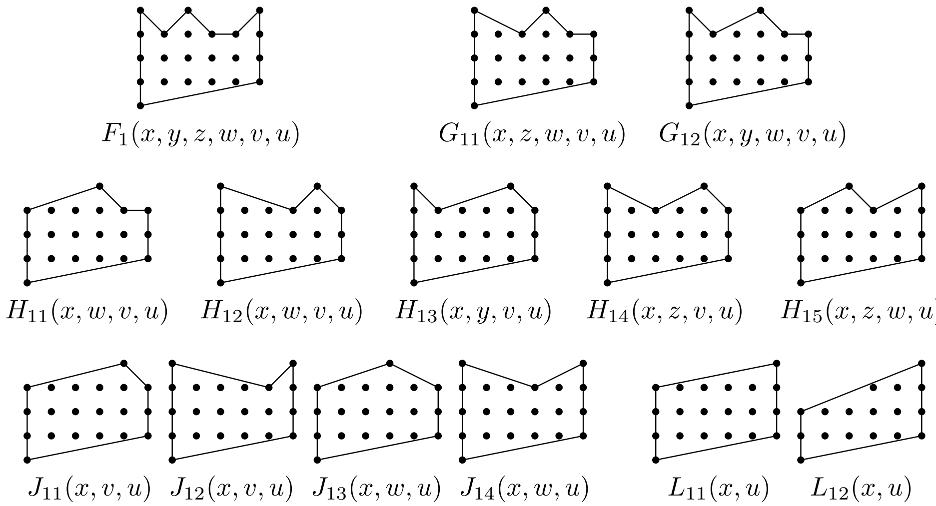

FIGURE 5. The shapes of $F_1, G_{1\nu}, H_{1\nu}, J_{1\nu}, L_{1\nu}$.

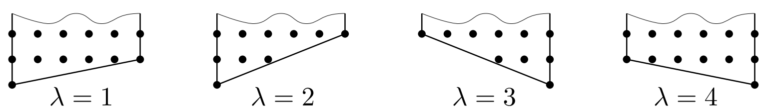

FIGURE 6. The bottom of the shapes of $F_\lambda, G_{\lambda\nu}, H_{\lambda\nu}, J_{\lambda\nu}, L_{\lambda\nu}$.

As in [7] and in §2, for each λ, μ we have $\lim_n f(5,n)^{1/n} = \lim_n \ell^*_{\lambda\mu}(2n)^{1/n}$, and we deduce

$$c_5 = -\frac{2}{5}\log_2 \beta_5, \tag{3.1}$$

where β_5 is the first real root of the equation $\det(\boldsymbol{I} - \boldsymbol{L}(x)) = 1$.

Remark 4. The analog of (2.1) and (3.1) for rectangles of an arbitrary width $m \ge 2$ is $c_m = -\frac{2}{m}\log_2 \beta_m$ where β_m is the first real root of the equation $\det(\boldsymbol{I} - \boldsymbol{L}(x)) = 1$ and $\boldsymbol{L}(x)$ is a square matrix of size $\varphi(m)/2$ defined in a similar way (φ is the Euler function).

3.2. **Recurrence relations.** We define the generating functions $F_1, G_{11}, G_{12}, \dots$ as in §2.2 but according to Figure 5. By replacing the bottom parts of the corresponding shapes as in Figure 6 we define $F_\lambda, G_{\lambda 1}, G_{\lambda 2}, \dots$ (with the same arguments) for $\lambda = 1, 2, 3, 4$. The symmetrizations $\widetilde{F}_\lambda, \widetilde{G}_{\lambda 1}, \widetilde{G}_{\lambda 2}, \dots$ $(\lambda = 1, 2)$ can be equivalently defined by setting

$$\widetilde{F}_\lambda = F_\lambda + F_{5-\lambda}, \quad \widetilde{X}_{\lambda,\nu} = X_{\lambda,\nu} + X_{5-\lambda,\nu}, \qquad X = G, H, J, L, \quad \lambda = 1, 2,$$

with the same arguments. Notice that

$$L_{\lambda\mu}(x) = x^{2-\lambda}\widetilde{L}_{\lambda\mu}(x,x) \tag{3.2}$$

(see §3.1 for the definition of $L_{\lambda\mu}(x)$).

Since most of the computations do not depend on λ, we omit the first subscript. In a few cases when they do depend on λ, we use the Kronecker symbol $\delta_{\lambda\mu}$. In this notation, the recurrence relations from [7, §2.1] (cf. §2.2) take the following form.

$$\begin{aligned}
\widetilde{F}(x,\dots,u)Q(x,\dots,u) &= y^{1/2}\widetilde{G}_1\big(xy^{1/2}, y^{1/2}z, w, v, u\big)P_1(w,v,u)\\
&+ z^{1/2}\widetilde{G}_2\big(x,\ yz^{1/2}, z^{1/2}w,\ v, u\big)P_2(x,v,u)\\
&+ w^{1/2}\widetilde{G}_2\big(u, vw^{1/2}, w^{1/2}z, y, x\big)P_2(u,y,x)\\
&+ v^{1/2}\widetilde{G}_1\big(uv^{1/2}, v^{1/2}w,\ z,\ y,\ x\big)P_1(z,y,x)\\
&- y^{1/2}w^{1/2}\widetilde{H}_4\big(xy^{1/2}, y^{1/2}zw^{1/2}, w^{1/2}v, u\big)(1-u)\\
&- y^{1/2}v^{1/2}\widetilde{H}_5\big(xy^{1/2}, y^{1/2}z, wv^{1/2}, v^{1/2}u\big)\\
&- z^{1/2}v^{1/2}\widetilde{H}_4\big(uv^{1/2}, v^{1/2}wz^{1/2}, z^{1/2}y, x\big)(1-x),
\end{aligned}$$

where

$$\begin{aligned}
Q(x,y,z,w,v,u) &= 1 - x - y - z - w - v - u\\
&+ xz + xw + xv + xu + yw + yv + yu + zv + zu + wu\\
&- xzv - xzu - xwu - ywu,
\end{aligned}$$

$$P_1(w,v,u) = 1 - w - v - u + wu, \qquad P_2(x,v,u) = (1-x)(1-v-u),$$

$$\begin{aligned}
\widetilde{G}_1(x,z,w,v,u)P_1(w,v,u) &= \mathfrak{cf}_{\xi^{-1}}\,\widetilde{F}\Big(\frac{x}{\xi}, \xi^2, \frac{z}{\xi}, w, v, u\Big)P_1(w,v,u)\\
&+ z^{1/3}\widetilde{H}_1\big(xz^{1/3}, z^{2/3}w, v, u\big)(1-u-v)\\
&+ w^{1/2}\widetilde{H}_4\big(x, zw^{1/2}, w^{1/2}v, u\big)(1-u)\\
&+ v^{1/2}\widetilde{H}_5\big(x, z, wv^{1/2}, v^{1/2}u\big)\\
&- w^{1/2}\,\mathfrak{cf}_{\xi^{-1}}\,\widetilde{G}_2\Big(u, vw^{1/2}, w^{1/2}\frac{z}{\xi}, \xi^2, \frac{x}{\xi}\Big)(1-u)\\
&- v^{1/2}\,\mathfrak{cf}_{\xi^{-1}}\,\widetilde{G}_1\Big(uv^{1/2}, v^{1/2}w, \frac{z}{\xi}, \xi^2, \frac{x}{\xi}\Big)\\
&- z^{1/3}v^{1/2}\widetilde{J}_3\big(xz^{1/3}, z^{2/3}wv^{1/2}, v^{1/2}u\big),
\end{aligned}$$

$$\begin{aligned}
\widetilde{G}_2(x,y,w,v,u)P_2(x,v,u) &= \mathfrak{cf}_{\xi^{-1}}\,\widetilde{F}\Big(x, \frac{y}{\xi}, \xi^2, \frac{w}{\xi}, v, u\Big)P_2(x,v,u)\\
&+ y^{1/3}\widetilde{H}_2\big(xy^{2/3}, y^{1/3}w, v, u\big)(1-v-u)\\
&+ w^{1/3}\widetilde{H}_3\big(x, yw^{1/3}, w^{2/3}v, u\big)(1-x)(1-u)\\
&+ v^{1/2}\widetilde{H}_4\big(uv^{1/2}, v^{1/2}w, y, x\big)(1-x)\\
&- y^{1/3}v^{1/2}\widetilde{J}_4\big(xy^{2/3}, y^{1/3}wv^{1/2}, v^{1/2}u\big)\\
&- v^{1/2}\,\mathfrak{cf}_{\xi^{-1}}\,\widetilde{G}_1\Big(uv^{1/2}, v^{1/2}\frac{w}{\xi}, \xi^2, \frac{y}{\xi}, x\Big)(1-x),
\end{aligned}$$

$$\begin{aligned}
\widetilde{H}_1(x,w,v,u)(1-v-u) &= \mathfrak{cf}_{\xi^{-1}}\,\widetilde{G}_2\Big(\frac{x}{\xi^2}, \xi^3, \frac{w}{\xi}, v, u\Big)(1-v-u)\\
&+ w^{1/4}\widetilde{J}_1\big(xw^{1/4}, w^{3/4}v, u\big)(1-u)\\
&+ v^{1/2}\widetilde{J}_3\big(x, wv^{1/2}, v^{1/2}u\big)\\
&- v^{1/2}\,\mathfrak{cf}_{\xi^{-1}}\,\widetilde{H}_4\Big(uv^{1/2}, v^{1/2}\frac{w}{\xi}, \xi^3, \frac{x}{\xi^2}\Big),
\end{aligned}$$

$$\begin{aligned}\widetilde{H}_2(x,w,v,u)(1-v-u) = \mathfrak{cf}_{\xi^{-1}}\,\widetilde{G}_1\Big(\frac{x}{\xi},\xi^3,\frac{w}{\xi^2},v,u\Big)(1-v-u)\\ + v^{1/2}\widetilde{J}_4\big(x,wv^{1/2},v^{1/2}u\big)\\ - v^{1/2}\,\mathfrak{cf}_{\xi^{-1}}\,\widetilde{H}_5\Big(\frac{x}{\xi},\xi^3,\frac{w}{\xi^2}v^{1/2},v^{1/2}u\Big),\end{aligned}$$

$$\widetilde{H}_3(x,y,v,u) = \mathfrak{cf}_{\xi^{-1}}\,\widetilde{G}_2\Big(u,\frac{v}{\xi},\xi^3,\frac{y}{\xi^2},x\Big) + \frac{v^{1/4}}{1-u}\widetilde{J}_2\big(uv^{3/4},v^{1/4}y,x\big),$$

$$\begin{aligned}\widetilde{H}_4(x,z,v,u) = \mathfrak{cf}_{\xi^{-1}}\,\widetilde{G}_1\Big(x,\frac{z}{\xi},\xi^2,\frac{v}{\xi},u\Big) \;+\; \mathfrak{cf}_{\xi^{-1}}\,\widetilde{G}_2\Big(u,v,\frac{z}{\xi},\xi^2,\frac{x}{\xi}\Big)\\ + \frac{v^{1/3}}{1-u}\Big\{\widetilde{J}_4\big(uv^{2/3},v^{1/3}z,x\big) \;-\; \mathfrak{cf}_{\xi^{-1}}\,\widetilde{H}_2\Big(uv^{2/3},v^{1/3}\frac{z}{\xi},\xi^2,\frac{x}{\xi}\Big)\Big\}\\ - \mathfrak{cf}_{\xi_1^{-1}\xi_2^{-1}}\,\widetilde{F}\Big(\frac{x}{\xi_1},\xi_1^2,\frac{z}{\xi_1\xi_2},\xi_2^2,\frac{v}{\xi_2},u\Big),\end{aligned}$$

$$\begin{aligned}\widetilde{H}_5(x,z,w,u) = \mathfrak{cf}_{\xi^{-1}}\,\widetilde{G}_1\Big(x,z,\frac{w}{\xi},\xi^2,\frac{u}{\xi}\Big) \;+\; \mathfrak{cf}_{\xi^{-1}}\,\widetilde{G}_1\Big(u,w,\frac{z}{\xi},\xi^2,\frac{x}{\xi}\Big)\\ + z^{1/3}\widetilde{J}_3\big(xz^{1/3},z^{2/3}w,u\big) \;+\; w^{1/3}\widetilde{J}_3\big(uw^{1/3},w^{2/3}z,x\big)\\ - z^{1/3}\,\mathfrak{cf}_{\xi^{-1}}\,\widetilde{H}_1\Big(xz^{1/3},z^{2/3}\frac{w}{\xi},\xi^2,\frac{u}{\xi}\Big) \;-\; w^{1/3}\,\mathfrak{cf}_{\xi^{-1}}\,\widetilde{H}_1\Big(uw^{1/3},w^{2/3}\frac{z}{\xi},\xi^2,\frac{x}{\xi}\Big)\\ - \mathfrak{cf}_{\xi_1^{-1}\xi_2^{-1}}\,\widetilde{F}\Big(\frac{x}{\xi_1},\xi_1^2,\frac{z}{\xi_1},\frac{w}{\xi_2},\xi_2^2,\frac{u}{\xi_2}\Big),\end{aligned}$$

$$\begin{aligned}\widetilde{J}_1(x,v,u) &= \mathfrak{cf}_{\xi^{-1}}\,\widetilde{H}_3\Big(\frac{x}{\xi^3},\xi^4,\frac{v}{\xi},u\Big) + \frac{\delta_{\lambda,1}}{x(1-u)},\\ \widetilde{J}_2(x,v,u) &= \mathfrak{cf}_{\xi^{-1}}\,\widetilde{H}_2\Big(\frac{x}{\xi},\xi^4,\frac{v}{\xi^3},u\Big),\end{aligned}$$

$$\begin{aligned}\widetilde{J}_3(x,w,u) &= \mathfrak{cf}_{\xi^{-1}}\,\widetilde{H}_1\Big(x,\frac{w}{\xi},\xi^2,\frac{u}{\xi}\Big) + \mathfrak{cf}_{\xi^{-1}}\,\widetilde{H}_4\Big(u,\frac{w}{\xi},\xi^3,\frac{x}{\xi^2}\Big)\\ &\quad - \mathfrak{cf}_{\xi_1^{-1}\xi_2^{-1}}\,\widetilde{G}_2\Big(\frac{x}{\xi_1^2},\xi_1^3,\frac{w}{\xi_1\xi_2},\xi_2^2,\frac{u}{\xi_2}\Big),\\ \widetilde{J}_4(x,w,u) &= \mathfrak{cf}_{\xi^{-1}}\,\widetilde{H}_2\Big(x,\frac{w}{\xi},\xi^2,\frac{u}{\xi}\Big) + \mathfrak{cf}_{\xi^{-1}}\,\widetilde{H}_5\Big(u,\frac{w}{\xi^2},\xi^3,\frac{x}{\xi}\Big)\\ &\quad - \mathfrak{cf}_{\xi_1^{-1}\xi_2^{-1}}\,\widetilde{G}_1\Big(\frac{x}{\xi_1},\xi_1^3,\frac{w}{\xi_1^2\xi_2},\xi_2^2,\frac{u}{\xi_2}\Big) + \frac{x\delta_{\lambda,2}}{u},\end{aligned}$$

$$\widetilde{L}_1(x,u) = \mathfrak{cf}_{\xi^{-1}}\,\widetilde{J}_2\Big(\frac{u}{\xi},\xi^5,\frac{x}{\xi^4}\Big), \qquad \widetilde{L}_2(x,u) = \mathfrak{cf}_{\xi^{-1}}\,\widetilde{J}_3\Big(\frac{x}{\xi^2},\xi^5,\frac{u}{\xi^3}\Big).$$

3.3. **A change of variables.** We set

$$
\begin{aligned}
f(t,s,u,v) &= \widetilde{F}\Big(\frac{x}{t}, \frac{x^2t^2}{s}, \frac{x^2s^2}{tu}, \frac{x^2u^2}{sv}, \frac{x^2v^2}{u}, \frac{x}{v}\Big),\\
g_1(s,u,v) &= \frac{1}{s^{1/2}}\,\widetilde{G}_1\Big(\frac{x^2}{s^{1/2}}, \frac{x^3s^{3/2}}{u}, \frac{x^2u^2}{sv}, \frac{x^2v^2}{u}, \frac{x}{v}\Big),\\
g_2(t,u,v) &= \frac{1}{(tu)^{1/2}}\,\widetilde{G}_2\Big(\frac{x}{t}, \frac{x^3t^{3/2}}{u^{1/2}}, \frac{x^3u^{3/2}}{t^{1/2}v}, \frac{x^2v^2}{u}, \frac{x}{v}\Big),\\
h_k(u,v) &= \frac{1}{u^{k/3}}\widetilde{H}_k\Big(\frac{x^3}{u^{1/3}}, \frac{x^4u^{4/3}}{v}, \frac{x^2v^2}{u}, \frac{x}{v}\Big), \qquad k=1,2,\\
h_3(t,v) &= \frac{1}{t^{2/3}v^{1/3}}\widetilde{H}_3\Big(\frac{x}{t}, \frac{x^4t^{4/3}}{v^{1/3}}, \frac{x^4v^{4/3}}{t^{1/3}}, \frac{x}{v}\Big),\\
h_4(s,v) &= \frac{1}{sv^{1/2}}\widetilde{H}_4\Big(\frac{x^2}{s^{1/2}}, \frac{x^4s}{v^{1/2}}, \frac{x^3v^{3/2}}{s^{1/2}}, \frac{x}{v}\Big),\\
h_5(s,u) &= \frac{1}{(su)^{1/2}}\widetilde{H}_5\Big(\frac{x^2}{s^{1/2}}, \frac{x^3s^{3/2}}{u}, \frac{x^3u^{3/2}}{s}, \frac{x^2}{u^{1/2}}\Big),\\
j_1(v) &= \frac{1}{v^{1/4}}\widetilde{J}_1\Big(\frac{x^4}{v^{1/4}}, x^5v^{5/4}, \frac{x}{v}\Big), \qquad j_3(u) = \frac{1}{v^{5/6}}\widetilde{J}_3\Big(\frac{x^3}{v^{1/3}}, x^5u^{5/6}, \frac{x^2}{u^{1/2}}\Big),\\
j_2(v) &= \frac{1}{v^{3/4}}\widetilde{J}_2\Big(\frac{x^4}{v^{1/4}}, x^5v^{5/4}, \frac{x}{v}\Big), \qquad j_4(u) = \frac{1}{v^{1/6}}\widetilde{J}_4\Big(\frac{x^3}{v^{1/3}}, x^5u^{5/6}, \frac{x^2}{u^{1/2}}\Big),\\
l_1(x) &= \widetilde{L}_1\big(x^5, x^5\big), \qquad l_2(x) = \widetilde{L}_2\big(x^5, x^5\big).
\end{aligned}
$$

We also set

$$
\begin{aligned}
q(t,s,u,v) &= Q\Big(\frac{x}{t}, \frac{x^2t^2}{s}, \frac{x^2s^2}{tu}, \frac{x^2u^2}{sv}, \frac{x^2v^2}{u}, \frac{x}{v}\Big),\\
p_1(s,u,v) &= P_1\Big(\frac{x^2u^2}{sv}, \frac{x^2v^2}{u}, \frac{x}{v}\Big), \qquad p_2(t,u,v) = P_2\Big(\frac{x}{t}, \frac{x^2v^2}{u}, \frac{x}{v}\Big),\\
r_1(u,v) &= 1 - \frac{x^2v^2}{u} - \frac{x}{v}, \qquad r_2(t,v) = \Big(1-\frac{x}{t}\Big)\Big(1-\frac{x}{v}\Big).
\end{aligned}
$$

Then the recurrence relations from §3.2 take the following form (cf. §2.3):

$$
\begin{aligned}
f(t,s,u,v)q(t,s,u,v) = {} & xt\,p_1(s,u,v)g_1(s,u,v) + xs\,p_2(t,u,v)g_2(t,u,v)\\
& + xv\,p_1(u,s,t)\,g_1(u,s,t) + xu\,p_2(v,s,t)\,g_2(v,s,t)\\
& - x^2tu\Big(1-\frac{x}{v}\Big)h_4(s,v) - x^2sv\Big(1-\frac{x}{t}\Big)h_4(u,t) - x^2tv\,h_5(s,u),
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
g_1(s,u,v)p_1(s,u,v) = {} & \frac{x}{s}\,p_1(s,u,v)\,\mathfrak{cf}_{t^{-1}}\,f(t,s,u,v)\\
& + x\,r_1(u,v)h_1(u,v) + xu\Big(1-\frac{x}{v}\Big)h_4(s,v) + xv\,h_5(s,u)\\
& - \frac{x^2u}{s}\Big(1-\frac{x}{v}\Big)\,\mathfrak{cf}_{t^{-1}}\,g_2(v,s,t) - \frac{x^2v}{s}\,\mathfrak{cf}_{t^{-1}}\,g_1(u,s,t) - x^2v\,j_3(u),
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
g_2(t,u,v)p_2(t,u,v) = {} & \frac{x}{tu}\,p_2(t,u,v)\,\mathfrak{cf}_{s^{-1}}\,f(t,s,u,v)\\
& + x\,r_1(u,v)h_2(u,v) + x\,r_2(t,v)h_3(t,v) + xv\Big(1-\frac{x}{t}\Big)h_4(u,t)\\
& - \frac{x^2v}{u}\,j_4(u) - \frac{x^2v}{ut}\Big(1-\frac{x}{t}\Big)\,\mathfrak{cf}_{s^{-1}}\,g_1(u,s,t),
\end{aligned} \tag{3.5}
$$

$$h_1(u,v)r_1(u,v) = x\,r_1(u,v)\,\mathfrak{cf}_{t^{-1}}\,g_2(t,u,v) + x\Big(1-\frac{x}{v}\Big)j_1(v) + xv\,j_3(u) - x^2v\,\mathfrak{cf}_{t^{-1}}\,h_4(u,t), \tag{3.6}$$

$$h_2(u,v)r_1(u,v) = \frac{x}{u}\,r_1(u,v)\,\mathfrak{cf}_{s^{-1}}\,g_1(s,u,v) + \frac{xv}{u}j_4(u) - \frac{x^2v}{u}\,\mathfrak{cf}_{s^{-1}}\,h_5(s,u), \tag{3.7}$$

$$h_3(t,v) = \frac{x}{t}\,\mathfrak{cf}_{s^{-1}}\,g_2(v,s,t) + \frac{vx}{v-x}\,j_2(t), \tag{3.8}$$

$$h_4(s,v) = \frac{x}{sv}\,\mathfrak{cf}_{u^{-1}}\,g_1(s,u,v) + \frac{x}{s}\,\mathfrak{cf}_{t^{-1}}\,g_2(v,s,t) + \frac{xv}{s(v-x)}j_4(s) - \frac{x^2}{s^2v}\,\mathfrak{cf}_{t^{-1}u^{-1}}\,f(t,s,u,v) - \frac{x^2v}{s(v-x)}\,\mathfrak{cf}_{t^{-1}}\,h_2(s,t), \tag{3.9}$$

$$h_5(s,u) = \frac{x}{u}\,\mathfrak{cf}_{v^{-1}}\,g_1(s,u,v) + \frac{x}{s}\,\mathfrak{cf}_{t^{-1}}\,g_1(u,s,t) + x\,j_3(u) + x\,j_3(s) - \frac{x^2}{u}\,\mathfrak{cf}_{v^{-1}}\,h_1(u,v) - \frac{x^2}{s}\,\mathfrak{cf}_{t^{-1}}\,h_1(s,t) - \frac{x^2}{su}\,\mathfrak{cf}_{t^{-1}v^{-1}}\,f(t,s,u,v), \tag{3.10}$$

$$j_1(v) = x\,\mathfrak{cf}_{t^{-1}}\,h_3(t,v) + \frac{v\delta_{\lambda,1}}{x^4(v-x)}, \tag{3.11}$$

$$j_2(v) = \frac{x}{v}\,\mathfrak{cf}_{u^{-1}}\,h_2(u,v), \tag{3.12}$$

$$j_3(u) = \frac{x}{u}\,\mathfrak{cf}_{v^{-1}}\,h_1(u,v) + x\,\mathfrak{cf}_{t^{-1}}\,h_4(u,t) - \frac{x^2}{u}\,\mathfrak{cf}_{t^{-1}v^{-1}}\,g_2(t,u,v), \tag{3.13}$$

$$j_4(u) = x\,\mathfrak{cf}_{v^{-1}}\,h_2(u,v) + x\,\mathfrak{cf}_{s^{-1}}\,h_5(u,s) - \frac{x^2}{u}\,\mathfrak{cf}_{s^{-1}v^{-1}}\,g_1(s,u,v) + x\delta_{\lambda,2}, \tag{3.14}$$

$$l_1(x) = x\,\mathfrak{cf}_{v^{-1}}\,j_2(v), \qquad l_2(x) = x\,\mathfrak{cf}_{u^{-1}}\,j_3(u). \tag{3.15}$$

3.4. **Elimination of f, h_1, h_2, j_1, j_2, j_4.** Let

$$\Phi_1(s,u,v) = \mathfrak{cf}_{t^{-1}}\,\frac{t}{q(t,s,u,v)}, \qquad \Psi_1(s,u,v) = 1 - \frac{x^2}{s}\,p_1(s,u,v)\Phi_1(s,u,v),$$

$$\Phi_2(t,u,v) = \mathfrak{cf}_{s^{-1}}\,\frac{s}{q(t,s,u,v)}, \qquad \Psi_2(t,u,v) = 1 - \frac{x^2}{tu}\,p_2(t,u,v)\Phi_2(t,u,v),$$

$$\Phi_{13}(s,v) = \mathfrak{cf}_{t^{-1}u^{-1}}\,\frac{tu}{q(t,s,u,v)}, \qquad \Psi_{13}(s,v) = 1 - \frac{x^4(v-x)}{s^2v^2}\Phi_{13}(s,v),$$

$$\Phi_{14}(s,u) = \mathfrak{cf}_{t^{-1}v^{-1}}\,\frac{tv}{q(t,s,u,v)}, \qquad \Psi_{14}(s,u) = 1 - \frac{x^4}{su}\Phi_{14}(s,u),$$

$$\Phi_0(u) = \mathfrak{cf}_{v^{-1}}\,\frac{v}{r_1(u,v)}, \qquad \Psi_0(u) = 1 - \frac{x^2}{u}\Phi_0(u).$$

Remark 5. We have (see OEIS [10], A025174)

$$\frac{x^2}{u}\Phi_0(u) = \sum_{n=1}^{\infty}\binom{3n-1}{n-1}\frac{x^{4n}}{u^n}.$$

Lemma 3.1. *One has*

$$h_2(u,v) = \frac{x}{u}\,\mathfrak{cf}_{s^{-1}}\,g_1(s,u,v) + \frac{x^2v\delta_{\lambda,2}}{u\,r_1(u,v)\Psi_0(u)}. \tag{3.16}$$

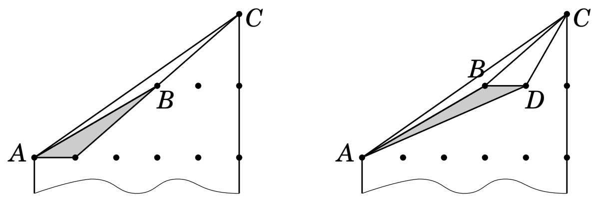

FIGURE 7. To the proof of Lemma 3.2.

Proof. Eliminating $j_4(u)$ from (3.7) and (3.14), we obtain the equation

$$\chi(u,v) = \frac{x^2 v}{u\, r_1(u,v)} \big(\delta_{\lambda,2} + \mathfrak{cf}_{w^{-1}}\, \chi(u,w)\big), \tag{3.17}$$

where

$$\chi(u,v) = h_2(u,v) - \frac{x}{u}\, \mathfrak{cf}_{s^{-1}}\, g_1(s,u,v).$$

By equating the coefficients of v^{-1} in both sides of (3.17), we obtain

$$\mathfrak{cf}_{v^{-1}}\, \chi(u,v) = \frac{x^2}{u} \Phi_0(u) \big(\delta_{\lambda,2} + \mathfrak{cf}_{w^{-1}}\, \chi(u,w)\big). \tag{3.18}$$

Noting that $\mathfrak{cf}_{v^{-1}}\, \chi(u,v) = \mathfrak{cf}_{w^{-1}}\, \chi(u,w)$, we find $\mathfrak{cf}_{w^{-1}}\, \chi(u,w)$ from (3.18) and plug it into (3.17), which yields (3.16) after simplifications. The lemma is proven. ☐

Lemma 3.2. *One has*

$$l_2(x) = x^2\, \mathfrak{cf}_{t^{-1}u^{-1}}\, h_4(u,t) + x^3\, \mathfrak{cf}_{v^{-1}}\, j_1(v) \tag{3.19}$$

Proof. Any primitive triangulation contributing to $l_2(x)$ has the triangle ABC (see Figure 7). There are exactly two possibilities for the triangle adjacent to the edge AB from below: the gray triangles in Figure 7. In the second case we necessarily have also the edge CD. In terms of the generating functions these observations mean (3.19). The lemma is proven. ☐

Remark 6. A `Wolfram Mathematica` code that checks the identities below up to $O(x^n)$ is available at `https://www.math.univ-toulouse.fr/~orevkov/tr45.html`.

By plugging (3.3), (3.6), and then (3.11), into (3.4) we obtain

$$\begin{aligned} \Psi_1(s,u,v) g_1(s,u,v) &= \frac{\delta_{\lambda,1}}{x^2 p_1(s,u,v)} \;+\; \frac{x\Psi_1(s,u,v)}{v\, p_1(s,u,v)} \big\{u(v-x)h_4(s,v) \;+\; v^2 h_5(s,u)\big\} \\ &+ \mathfrak{cf}_{t^{-1}} \bigg\{ \frac{x^2 v}{s} \left(\frac{p_1(u,s,t)}{q(t,s,u,v)} - \frac{1}{p_1(s,u,v)} \right) g_1(u,s,t) \;+\; \frac{x^3(v-x)}{v\, p_1(s,u,v)} h_3(t,v) \bigg\} \\ &+ \frac{x^2 u(v-x)}{sv}\, \mathfrak{cf}_{t^{-1}} \left(\frac{r_1(s,t)}{q(t,s,u,v)} - \frac{1}{p_1(s,u,v)} \right) g_2(v,s,t) \\ &+ \mathfrak{cf}_{t^{-1}} \left(\frac{x^2(t-x)}{t\, q(t,s,u,v)} + \frac{x^2}{p_1(s,u,v)} \right) \big\{ r_1(u,v) g_2(t,u,v) - xv\, h_4(u,t) \big\}. \end{aligned} \tag{3.20}$$

By plugging (3.3) and (3.7) into (3.5) we obtain

$$\begin{aligned}\Psi_2(t,u,v)g_2(t,u,v) &= \frac{x(v-x)}{v\,r_1(u,v)}h_3(t,v) \;+\; \frac{xv\Psi_2(t,u,v)}{r_1(u,v)}h_4(u,t)\\ &+ \mathfrak{cf}_{s^{-1}}\left\{\frac{x^2v}{tu}\left(\frac{p_1(u,s,t)}{q(t,s,u,v)} - \frac{1}{r_1(u,v)}\right)g_1(u,s,t) \;+\; \frac{x^2p_2(v,s,t)}{t\,q(t,s,u,v)}g_2(v,s,t)\right\}\\ &+ \frac{x^2}{u}\,\mathfrak{cf}_{s^{-1}}\left(\frac{p_1(s,u,v)}{q(t,s,u,v)} + \frac{t}{t-x}\right)g_1(s,u,v) \;-\; \frac{x^3(v-x)}{v}\,\mathfrak{cf}_{s^{-1}}\,\frac{h_4(s,v)}{q(t,s,u,v)}\\ &- \frac{x^3v}{u}\,\mathfrak{cf}_{s^{-1}}\left(\frac{1}{q(t,s,u,v)} + \frac{1}{p_2(t,u,v)}\right)h_5(s,u).\end{aligned} \tag{3.21}$$

By plugging (3.3) and (3.14) (with t,s,u,v replaced by v,u,s,t respectively) into (3.9) we obtain

$$\begin{aligned}\Psi_{13}(s,v)h_4(s,v) &= \frac{x^2v\,\delta_{\lambda,2}}{s(v-x)} \;+\; \frac{x}{sv}\,\mathfrak{cf}_{u^{-1}}\Psi_1(s,u,v)g_1(s,u,v) \;+\; \frac{x}{s}\,\mathfrak{cf}_{t^{-1}}\Psi_2(v,s,t)g_2(v,s,t)\\ &- \frac{x^3}{s}\,\mathfrak{cf}_{t^{-1}u^{-1}}\left\{\left(\frac{v}{s(v-x)} + \frac{p_1(u,s,t)}{s\,q(t,s,u,v)}\right)g_1(u,s,t) \;+\; \frac{p_2(t,u,v)}{v\,q(t,s,u,v)}g_2(t,u,v)\right\}\\ &+ \mathfrak{cf}_{u^{-1}}\left(\frac{x^2v}{s(v-x)} + \frac{x^4}{s^2}\Phi_1(s,u,v)\right)h_5(s,u) \;+\; \mathfrak{cf}_{t^{-1}u^{-1}}\frac{x^4(t-x)}{ts\,q(t,s,u,v)}h_4(u,t).\end{aligned} \tag{3.22}$$

By plugging (3.3), (3.13), and (3.13) with t,s,u,v replaced by v,u,s,t into (3.10) we obtain

$$\begin{aligned}\Psi_{14}(s,u)h_5(s,u) &= \frac{x}{u}\,\mathfrak{cf}_{v^{-1}}\,\Psi_1(s,u,v)g_1(s,u,v) \;+\; \frac{x}{s}\,\mathfrak{cf}_{t^{-1}}\,\Psi_1(u,s,t)g_1(u,s,t)\\ &+ \mathfrak{cf}_{v^{-1}}\Big(x^2 + \frac{x^4(v-x)}{sv}\Phi_1(s,u,v)\Big)h_4(s,v) \;+\; \mathfrak{cf}_{t^{-1}}\Big(x^2 + \frac{x^4(t-x)}{tu}\Phi_1(u,s,t)\Big)h_4(u,t)\\ &- \mathfrak{cf}_{t^{-1}v^{-1}}\left\{\frac{x^3}{u}\Big(1 + \frac{p_2(t,u,v)}{q(t,s,u,v)}\Big)g_2(t,u,v) \;+\; \frac{x^3}{s}\Big(1 + \frac{p_2(v,s,t)}{q(t,s,u,v)}\Big)g_2(v,s,t)\right\}.\end{aligned} \tag{3.23}$$

Eliminating $h_2(u,v)$ from (3.12) and (3.16) we obtain

$$j_2(v) = \mathfrak{cf}_{s^{-1}u^{-1}}\,\frac{x^2}{uv}g_1(s,u,v) \;+\; \mathfrak{cf}_{u^{-1}}\,\frac{x^3\delta_{\lambda,2}}{u\,r_1(u,v)\Psi_0(u)}. \tag{3.24}$$

Then (3.8) and (3.20)–(3.24) is a system of six equations for g_1, g_2, h_3, h_4, h_5, j_2. Our final goals are l_1 and l_2. The series l_1 is already expressed via j_2 in (3.15). Eliminating j_1 from (3.19) and (3.11) we express l_2 via h_3 and h_4. Thus

$$l_1 = x\,\mathfrak{cf}_{v^{-1}}\,j_2(v), \qquad l_2 = x^4\,\mathfrak{cf}_{t^{-1}v^{-1}}h_3(t,v) \;+\; x^2\,\mathfrak{cf}_{s^{-1}v^{-1}}h_4(s,v) \;+\; \delta_{\lambda,1}. \tag{3.25}$$

Notice also that (3.24) combined with (3.25) yields

$$l_1 = \mathfrak{cf}_{(suv)^{-1}}\,\frac{x^3}{uv}g_1(s,u,v) + x^5\delta_{\lambda,2}.$$

3.5. **Computation of c_5.** The computation of c_5 and its justification are as in Sections 2.5 and 2.6 and we omit the details. In brief, for a fixed $x \in [0, \beta_5^{1/5} + \varepsilon]$, we replace the "$\mathfrak{cf}$" by Cauchy integrals in (3.8) and (3.20)–(3.24), discretize the obtained system of integral equations, solve them, plug the solution to (3.19) and (3.11), and compute $\boldsymbol{L}$ by (3.2). Then we solve numerically the equation $\det(\boldsymbol{I} - \boldsymbol{L}(x)) = 1$ and find c_5 by (3.1).

4. Computational issues

4.1. **Improving the rate of convergence of Riemann sums.** Given an analytic function f on $\mathbb{T}$, the rate of convergence of the Riemann sums to the Cauchy integral

$$\frac{1}{n}\sum_{k=1}^{n} f(e^{2\pi ik/n}) \;\longrightarrow\; \int_0^1 f(e^{2\pi it})\,dt = \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f(z)}{z}dz$$

depends on the maximal width of an annulus of the form $1/r < [z| < r$ to which f extends analytically; see [7, Lemma 5.1]. Thus, if μ is a linear fractional transformation such that $\mu(\mathbb{T}) = \mathbb{T}$ and the function $f\circ\mu$ extends to a wider annulus of this form, then the Riemann sums converge faster after the change of variable $z = \mu(\zeta)$. It turns out that the accuracy of computation of the functions $H(x)$ (in [7, §4.4]), $J(x)$ (in §2), and $L_{\lambda\mu}(x)$ (in §3) becomes almost twice better after the change of variable

$$z = (\zeta + b)/(b\zeta + 1) \tag{4.1}$$

with $b = 1/3$ applied at all steps of the computation; see the middle column of Table 1.

Remark 7. Surprisingly, the upper bounds of the L^2-norms of the integration kernels in §2.6 become worse after this variable change with $b > 0$ but better with $b < 0$.

Another simple trick (we call it the ¼*-shift*), which allowed us to improve the rate of convergence is based on the fact that all the series we consider have real coefficients. Namely, let f be as above and let $\sum_{m\in\mathbb{Z}} c_m z^m$ be its Laurent series. Since f is analytic in a neighborhood of $\mathbb{T}$, we have $|c_m| = o(r^{|m|})$, $0 < r < 1$. Let $\omega = e^{2\pi i/n}$. Then the approximation error for the Riemann sum over the nodes ω^k, $k = 1,\dots,n$ is

$$\begin{aligned}\int_0^1 f(e^{2\pi it})\,dt - \frac{1}{n}\sum_{k=1}^{n} f(\omega^k) &= c_0 - \frac{1}{n}\sum_{k=1}^{n} f(\omega^k) = c_0 - \frac{1}{n}\sum_{k=1}^{n}\sum_{m\in\mathbb{Z}} c_m\omega^{km}\\ &= c_0 - \frac{1}{n}\sum_{m\in\mathbb{Z}} c_m\sum_{k=1}^{n}\omega^{km} = c_0 - \sum_{m\equiv 0(n)} c_m = o(r^n),\end{aligned}$$

whereas, if all the c_m are real, then the approximation error for the real part of the Riemann sum over the shifted nodes $\omega_0\omega^k$, where $\omega_0 = e^{2\pi i/4n}$ (and hence $\omega_0^n = i$), is

$$\begin{aligned}\int_0^1 f(e^{2\pi it})\,dt - \frac{1}{n}\operatorname{Re}\sum_{k=1}^{n} f(\omega_0\omega^k) &= c_0 - \frac{1}{n}\operatorname{Re}\sum_{k=1}^{n} f(\omega_0\omega^k)\\ &= c_0 - \frac{1}{n}\operatorname{Re}\sum_{k=1}^{n}\sum_{m\in\mathbb{Z}} c_m\omega_0^m\omega^{km} = c_0 - \frac{1}{n}\operatorname{Re}\sum_{m\in\mathbb{Z}} c_m\omega_0^m\sum_{k=1}^{n}\omega^{km}\\ &= -\operatorname{Re}(\,\dots ic_{-3n} - c_{-2n} - ic_{-n} + ic_n - c_{2n} - ic_{3n} + \dots\,) = o(r^{2n}).\end{aligned}$$

We illustrate the efficiency of these improvements in the following example. In [7, Figure 7] is presented a `Wolfram Mathematica` function `H` that computes the n-th approximation of $H(x)$ for a given x, where H is the function such that the first real solution x_0 of $H(x) = 1$ determines c_3 by the formula $c_3 = -\frac{2}{3}\log_2 x_0$. In [7, Table 7], the results of computation of $H(x_0) - 1$ by this program are given for $n = 100, 200, \dots, 1200$. They are reproduced in the left column of Table 1 here.

In Figure 8 we present a modification of the program from [7] implementing the above improvements. We see in Table 1 that the impact of both improvements is significant. The program used for the middle column is as in Figure 8 but without the ¼-shift, i.e., with "`omega0=`..." replaced by "`omega0=1`" (here we used x_0 computed to 500 digits).

```
P = u^2*t^2-(u+t)u*t*x+(1-t^3-u^3)u*t*x^2+(t^4+u^4)x^3;
Psi = Function[{x0,t0,prec},Module[{P0,u0,i},
   P0=P/.{x->x0,t->t0}; u0 = NRoots[0==P0,u,prec];
   u0=Sort[Table[{Abs[u0[[i,2]]],u0[[i,2]]},{i,4}]];
   1-x0^2(t0-x0)Sum[(u^3/D[P0,u])/.u->u0[[i,2]],{i,2}] ]];

H2 = Function[{x3,n,prec}, Module[
   {x0,z,P0,Id,K,F,G,j,k,Tj,Uk,PsiTj,Pjk,dTj,dUk,b=1/3,zeta,omega0},
   x0=N[x3^(1/3),prec]; b=1/3;
   zeta=N[Exp[2Pi*I/n],prec]; omega0=N[Exp[Pi*I/2/n],prec];
   K=Id=IdentityMatrix[n]; F=K[[1]]; P0=P/.x->x0;
   Do[ z=omega0*zeta^j; Tj=(z+b)/(z*b+1); PsiTj=Psi[x0,Tj,prec];
      F[[j]] = Tj^2/(Tj-x0)/PsiTj;
      Do[ z=omega0*zeta^k; Uk = (z+b)/(z*b+1);
         dUk = z(1-b^2)/(b*z+1)^2; Pjk=P0/.{t->Tj,u->Uk};
         K[[j,k]] = x0^2*Tj^3(Uk-x0)dUk/Pjk/PsiTj/n,
      {k,n}],
   {j,n}];
   G = Inverse[Id-K].F;
   Re[x0^2*Sum[z=omega0*zeta^k;G[[k]]z(1-b^2)/(b*z+1)/(z+b),{k,n}]/n]
]];
```

FIGURE 8. The program from [7] improved using (4.1) and the $1/4$-shift.

Table 1. Approximations of $H(x_0) - 1$.
b is from (4.1); $t_{\mathtt{H}}$ and $t_{\mathtt{H2}}$ are the CPU time for `H` and `H2`.

n	`H` from [7]	$b = \frac{1}{3}$, no $1/4$-shift	`H2` in Fig.8	$t_{\mathtt{H2}}/t_{\mathtt{H}}$
100	1.44×10^{-10}	-4.05×10^{-21}	7.96×10^{-30}	1.19
200	5.01×10^{-22}	-6.95×10^{-42}	3.60×10^{-60}	1.22
300	1.73×10^{-33}	-8.63×10^{-63}	1.79×10^{-90}	1.31
400	6.02×10^{-45}	-9.56×10^{-84}	9.35×10^{-121}	1.33
500	2.09×10^{-56}	-9.92×10^{-105}	4.99×10^{-151}	1.37
600	7.26×10^{-68}	-9.88×10^{-126}	2.71×10^{-181}	1.41
700	2.52×10^{-79}	-9.58×10^{-147}	1.49×10^{-211}	1.47
800	8.78×10^{-91}	-9.09×10^{-168}	8.25×10^{-242}	1.57
900	3.06×10^{-102}	-8.49×10^{-189}	4.61×10^{-272}	1.66
1000	1.06×10^{-113}	-7.84×10^{-210}	2.59×10^{-302}	2.01
1100	3.72×10^{-125}	-7.16×10^{-231}	1.45×10^{-332}	2.01
1200	1.29×10^{-136}	-6.49×10^{-252}	8.27×10^{-363}	2.03

4.2. **The software used for solving big linear systems with high precision.** The systems of linear equations in [7] were not too big, and we used `Wolfram Mathematica` to solve them (see [7, Figure 7] and Figure 8). The size of the linear systems in the present paper exceeds the capacity of `Mathematica`, and we used `MATLAB` (the code is available at `https://www.math.univ-toulouse.fr/~orevkov/tr45.html`). For computations in §3, the standard precision is enough because anyway it is impossible to find c_5 with a higher precision since the number of equations grows too rapidly.

As for computations in §2, it is possible to do them with high precision but since we do not have access to a high precision version of MATLAB, we combined MATLAB and Mathematica. Namely, in order to solve a matrix equation $AX = B$ we computed successive approximations of the solution $X_0, X_1, X_2, \dots$ where $X_0 = 0$, $X_{k+1} = X_k + x_k$, and x_k is a solution of $AX = B_k$ computed by MATLAB where B_k is $B - AX_k$ computed by Mathematica with a suitable higher precision.

The computation of B_k can be performed without storing the whole high precision matrix A to the memory, because each entry of A is used only once. Since the matrix of the system (2.11)–(2.13) is sparse (its dimension is of order n^2 but the number of non-zero entries is of order n^3), the computation of B_k with Mathematica takes a reasonable time.

5. Exact values and empirical estimates

5.1. **Exact values.** In [7, §2.2] we reported about some computed exact values of $f(m,n)$. Since then we have modified the C-program that was used for these computations: optimized the memory allocation (storing in most cases 32-bit offsets instead of 64-bit pointers) and parallelized the computations using the `pthread` library. This allowed us to compute $f(6,n)$ for $n = 51, \dots, 57$, $f(7,n)$ for $n = 21, \dots, 34$, $f(8,n)$ for $n = 14, \dots, 20$, $f(9,n)$ for $n = 10, 11, 12$, and $f(10,10)$. In particular,

$$f(10,10) = 14961827914933606361147029768437248214296633789595109069939881 2$$

(this was the most extensive computation) and

$$\begin{aligned} f(6,57) = {} & 557421896540695911900399221054546689165982929838646018 3 \\ & 474962258786874190268329847808221439278716261308942938 9 \\ & 359865279444834094942128569360082968955999678802006849 9 \\ & 19175179038598729639782780684957850920957806320909 88, \end{aligned}$$

which gives $c_{6,57} \approx 2.10531$. This is the largest computed capacity of a rectangle but it is less than $c_5 \approx 2.11801$. All the computed exact values of $f(m,n)$ are available at

`https://www.math.univ-toulouse.fr/~orevkov/tr.html`.

5.2. **Empirical estimates.** In this subsection we present empirical estimates of c_m via the known exact values of $f(m,n)$. We use the method proposed in [4, §6]. For a fixed m with known values of $f(m,1), \dots, f(m, 2k+2)$, we find A, $a_1, \dots, a_k$, and $b_1, \dots, b_k$ such that

$$\frac{f(m,n+1)}{f(m,n)} = A \frac{n^k + a_1 n^{k-1} + \dots + a_k}{n^k + b_1 n^{k-1} + \dots + b_k} \tag{5.1}$$

for $n = 1, \dots, 2k+1$, and hope that the equation (5.1) is a good approximation of the quotient $f(m,n+1)/f(m,n)$ for all n. If $a_1 \neq b_1$, this assumption implies that (see [4, §6])

$$f(m,n) \sim \mathrm{const} \cdot A^n \, n^\alpha, \qquad \alpha = a_1 - b_1.$$

Let $A_m^{(k)}$ and $\alpha_m^{(k)}$ be the constants A and α found in this way for given m and k, and let $c_m^{(k)} = \frac{1}{m} \log_2 A_m^{(k)}$. We have

$$\begin{aligned} &|c_3^{(399)} - c_3| \approx 10^{-40}, \qquad |c_4^{(99)} - c_4| \approx 10^{-12}, \qquad |c_5^{(56)} - c_5| \approx 10^{-8}, \\ &|\alpha_3^{(399)} + 0.5| \approx 10^{-35}, \qquad |\alpha_4^{(99)} + 0.5| \approx 10^{-8}, \qquad |\alpha_5^{(56)} + 0.5| \approx 10^{-5}. \end{aligned} \tag{5.2}$$

In Figure 9 we show the difference $c_m^{(k)} - c_m$ for $m = 3,4,5$. We see that, for the most of the values of k, this difference is less than $10^{-2-0.11\,k}$, though there are some rare exceptions, for example so are $c_4^{(12)} = 2.0927$, $c_5^{(13)} = 2.1725$, and $c_6^{(20)} = 2.0714$; see the dashed lines in

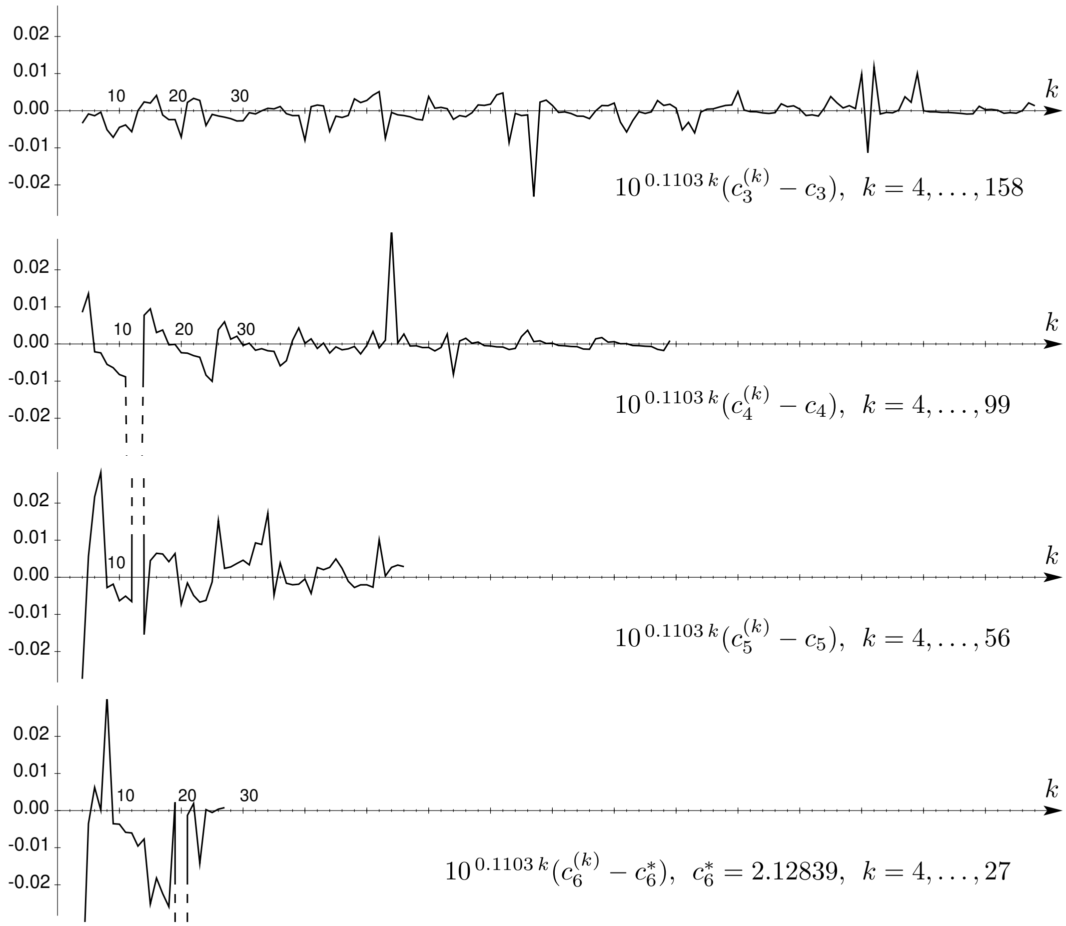

FIGURE 9. Convergence of the empirical estimates to c_3, c_4, c_5, c_6^*.

Figure 9. Based on these data, one can expect that $c_6^* = 2.12839$ and $c_7^* = 2.136$ are good approximations of c_6 and c_7. (see Figure 9 and Tables 2 and 3). The extrapolation using (5.1) with the computed values and empirical estimates of $\lim_n f(m,n)^{1/n}$ for $m = 0, \dots, 7$ (as well as for $m = 1, \dots, 6$ or $m = 2, \dots, 7$) allows us to expect that $2.2 < c < 2.3$.

Table 2. Empirical estimates of c_6.

k	$c_6^{(k)}$	k	$c_6^{(k)}$	k	$c_6^{(k)}$	k	$c_6^{(k)}$
4	2.106546	10	2.127929	16	2.127728	22	2.128409
5	2.127148	11	2.127791	17	2.127728	23	2.128267
6	2.130178	12	2.127888	18	2.127766	24	2.128391
7	2.128419	13	2.127743	19	2.128433	25	2.128386
8	2.134234	14	2.127968	20	2.071426	26	2.128391
9	2.127842	15	2.127263	21	2.128372	27	2.128392

Table 3. Empirical estimates of c_7.

k	$c_7^{(k)}$	k	$c_7^{(k)}$	k	$c_7^{(k)}$	k	$c_7^{(k)}$
5	2.14004	8	2.15126	11	2.13562	14	2.13599
6	2.14037	9	2.12815	12	2.13597	15	2.13657
7	2.13348	10	2.12209	13	2.13657	16	2.13628

The second line in (5.2) leads to a conjecture that $f(m,n) \sim \mathrm{const}\cdot 2^{c_m n}/\sqrt{n}$ for any m. A computation shows that the convergence of $\alpha_m^{(k)}$ to $-1/2$ looks as good as for the numbers $c_m^{(k)}$. For $m=1$ this asymptotics follows from Stirling's formula: $f(1,n) = \binom{2n}{n} \sim 4^n/\sqrt{\pi n}$.

Notice that usually such asymptotics mean that the first real singularity $x = x_0$ of the generating function is a brunching point and the leading term of the Laurent-Puiseux expansion at it is $C(x-x_0)^{-1/2}$ (see, e.g., [2, Figure VII.24], [8, p. 596]).

The factor $n^{-1/2}$ might also mean that there is a sort of Central Limit Theorem for the numbers $j^*_{a,n-a}$ (defined §2.1) and their analogues for $m > 4$.

5.3. **Convexity conjecture.** In [7, §2.3] we formulated the following convexity conjecture: $f(m,n-1)f(m,n+1) \ge f(m,n)^2$ for all $m,n \ge 1$, which implies the bound

$$c_m \ge (n+1)c_{m,n+1} - nc_{m,n} \tag{5.3}$$

for any m,n, in particular, $c \ge c_{115} \ge 5c_{115,5} - 4c_{115,4} \approx 2.16848$. The newly computed exact values of $f(m,n)$ still confirm this conjecture.

Passing to the limit in (5.3) we obtain $c \ge (n+1)c_{n+1} - nc_n$. Using the computed values of $c_2,\dots,c_5$ the conjecture implies lower bounds

$$c \ge 3c_3 - 2c_2 \approx 2.14641, \quad c \ge 4c_4 - 3c_3 \approx 2.16413, \quad c \ge 5c_5 - 4c_4 \approx 2.17436,$$

and $c \ge 6c_6^* - 5c_5 \approx 2.1803$ (see §5.2), which agrees with the expected bounds $2.2 < c < 2.3$ from §5.2 (recall that the best proven bounds are $2.118 < c < 2.777$).

6. Non-primitive lattice triangulations

Denote the number of all (not necessarily primitive) lattice triangulations of the $m \times n$ rectangle by $f^{\mathrm{np}}(m,n)$. In this section we prove the following asymptotic lower bound for these numbers (in [7, §6] we gave a rather coarse upper bound for them):

$$\lim_{n\to\infty} f^{\mathrm{np}}(n,n)^{1/n^2} \ge 5. \tag{6.1}$$

Indeed, given integers n and $k \le n$, consider triangulations of the rectangle $[0,n]^2$ such that each vertical segment $\{m\} \times [0,n]$, $m \in \mathbb{Z}$, is the union of k edges of the triangulation. There are $\binom{n}{k-1}$ ways to choose vertices on each vertical line and, for fixed vertices, $\binom{2k}{k}$ triangulations of each vertical strip (see [3, Eq. (2.1)]). Thus, the total number of such triangulations is $\binom{n}{k-1}^{n+1}\binom{2k}{k}^n$. Let $f_x(n,n)$, $x \in [0,1]$, be this number for $k = \lfloor xn \rfloor$. By Stirling formula we obtain

$$\lim_{n\to\infty} \ln f_x(n,n)^{1/n^2} = x\ln 4 - x\ln x - (1-x)\ln(1-x).$$

The first two derivatives of the right hand side are $\ln(4-4x) - \ln x$ and $1/(x-x^2)$, hence the maximum is attained at $x = 4/5$ and it is equal to $\ln 5$, whence the bound (6.1).

It is easy to check that the limit will be the same if we consider all lattice triangulations of $[0,n]^2$ such that each vertical strip of width 1 is a union of triangles.

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