MULTIVARIATE SIGNATURES OF ITERATED TORUS LINKS

S. Yu. Orevkov

Abstract. We compute the multivariate signatures of any Seifert link (that is a union of some fibers in a Seifert homology sphere), in particular, of the union of a torus link with one or both of its cores (cored torus link). The signatures of cored torus links are used in Degtyarev-Florens-Lecuona splicing formula for computation of multivariate signatures of cables over links. We use Neumann's computation of equivariant signatures of such links.

For signatures of torus links with the core(s) we also rewrite the Neumann's formula in terms of integral points in a certain parallelogram, similar to Hirzebruch's formula for signatures of torus links (without cores) via integral points in a rectangle.

1. Introduction

An *iterated torus link* in the 3-sphere $\mathbb{S}^3$ is a link obtained from the unknot by successive cabling operations either with the core removed or retained (see §3 for a precise definition). The most important examples are links of singularities of plane complex analytic curves, and links at infinity of plane affine algebraic curves.

If K is an iterated torus knot, the Levine-Tristram signatures $\sigma_\zeta(K)$ (defined as the signature of $(1-\zeta)V + (1-\bar\zeta)V^T$ for a Seifert matrix V) can be recursively computed using:

- Hirzebruch's formula (see [1; §6], [13; §4]) which expresses the signatures of a torus knot $T(p,q)$ in terms of integral points in the rectangle $p \times q$;
- Litherland's formula [12; Thm. 2] which expresses the signatures of a (p,q)-cable of any knot K via those of K and $T(p,q)$

To extend this approach to iterated torus links, it is natural to consider a multivariate generalization of the Levine-Tristram signatures discussed in [3, 6, 10, 17]. In [6, 7], an analog of the Litherland's formula is obtained for cables over links. It expresses the multivariate signatures of a cable over L via those of L and those of the union of a torus link with one or both of its cores (the link $T_{m_1,m_2}(p,q)$, $m_1, m_2 \in \{0,1\}$, in the notation of §9 below), but the latter signatures have been unknown.

The initial aim of this paper was to fill this gap and to compute the multivariate signatures of $T_{m_1,m_2}(p,q)$. This (and a little more) is done in Proposition 10.1 which reduces the multivariate signatures of any Seifert link (in particular of $T_{m_1,m_2}(p,q)$) to Tristram-Levine signatures of this link. The latter were already computed by Walter Neumann in [14], [15].

More precisely, in Proposition 10.1, for any positive Seifert n-component link L (that is a union of n positively oriented fibers in a Seifert homology sphere), we show that its multivariate signature σ_L considered as a function on the open cube

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

$]0,1[^n$, is constant on each member of a family of parallel hyperplanes transverse to the main diagonal, so, all values of σ_L are determined by the values on the main diagonal, which coincide with the Levine-Tristram signatures up to a certain additive constant. The proof of Proposition 10.1 is based on the splice formula for multivariate signatures (which is the main result of [6]) combined with the observation that a Seifert link can be spliced in many different ways.

In Proposition 9.1 we show that in the case of a torus link $T_{m_1,m_2}(p,q)$, the Neumann's formula can be rewritten in such a way that it becomes almost identical with the Hirzebruch's formula but the parallelogram spanned by $(p+m_1, m_2)$ and $(m_1, q+m_2)$ is used instead of the $p \times q$ rectangle.

In §§2–8 we give necessary definitions and results from [6–8, 9, 13–15]. In §9 we prove Proposition 9.1; in §10 we prove Proposition 10.1; in §11 we consider some examples and formulate some questions for further research.

Acknowledgement. I am grateful to Alexander Degtyarev and Vincent Florens for useful discussions.

2. Splicing and splice diagrams (after Eisenbud and Neumann)

Let $(\Sigma', K' \cup L')$ and $(\Sigma'', K'' \cup L'')$ be two links in homology spheres. Here K' and K'' are components of the respective links. Let $T(K')$ and $T(K'')$ be their tubular neighborhood disjoint from L' and L'' respectively.

Definition 2.1. The *splice* of $(\Sigma', K' \cup L')$ and $(\Sigma'', K'' \cup L'')$ along K' and K'' is introduced in [9] (see also [15], [6]) as $(\Sigma, L' \cup L'')$ where $\Sigma = (\Sigma' \setminus T(K')) \cup_\varphi (\Sigma'' \setminus T(K''))$ and $\varphi : T(K') \to T(K'')$ is a homeomorphism which identifies the meridian of K' with the longitude of K'' and vice versa.

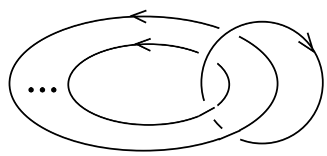


Figure 1. $\mathfrak{Seif}(0,1,\dots,1)$ (called in [6] a generalized Hopf link)

Let $a_1,\dots,a_n$ be positive pairwise coprime integers. Following [9, 15], we define *Seifert link* $\mathfrak{Seif}(a_1,\dots,a_n)$ to be the link $(\Sigma, S_1 \cup \dots \cup S_n)$ where $\Sigma = \Sigma(a_1,\dots,a_n)$ is the unique Seifert fibered homological 3-sphere which has fibers $S_1,\dots,S_n$ of degrees $a_1,\dots,a_n$ and no other fibers of degree >1 (see [9, §7] for a detailed construction). The orientation of Σ is chosen so that all the linking numbers $\mathrm{lk}(S_i,S_j)$, $i \neq j$, are positive. This definition extends to the case when $a_1,\dots,a_n$ are any pairwise coprime integers by setting

$$\Sigma(a_1,\dots,-a_i,\dots,a_n) = -\Sigma,$$
$$\mathfrak{Seif}(a_1,\dots,-a_i,\dots,a_n) = (-\Sigma, S_1 \cup \dots \cup (-S_i) \cup \dots \cup S_n),$$

and $\mathfrak{Seif}(0,1,\dots,1)$ to be the link in the 3-sphere shown in Figure 1. Finally, we define

$$\mathfrak{Seif}(a_1,\dots,a_k; a_{k+1},\dots,a_n) = (\Sigma, S_1 \cup \dots \cup S_k), \tag{1}$$

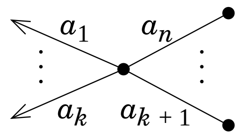


FIGURE 2. The splice diagram of $\mathfrak{Seif}(a_1, \dots, a_k; a_{k+1}, \dots, a_n)$.

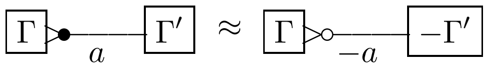
that is $\mathfrak{Seif}(a_1, \dots, a_n)$ with the last $n - k$ components removed.

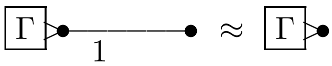
As shown in [9], any graph link in a *graph homology sphere* (i.e., a graph manifold which is a homology sphere) can be obtained as the result of splicing together Seifert links. The way of splicing is encoded in [9] by certain decorated trees (or forests) called *splice diagrams.* The splice diagram of $\varepsilon\mathfrak{Seif}(a_1, \dots, a_k; a_{k+1}, \dots, a_n)$, $\varepsilon = \pm 1$, is the graph in Figure 2 if $\varepsilon = +1$. If $\varepsilon = -1$, then we use the white color for the central vertex. If the orientation of some component is reversed, we put the minus sign near the corresponding arrowhead vertex (if the sign is not shown, we assume that it is plus). The splicing of $(\Sigma', L' \cup K')$ with $(\Sigma'', L'' \cup K'')$ along K' and K'' has the splice diagram $\boxed{\Gamma'}$——$\boxed{\Gamma''}$ where $\boxed{\Gamma'}\!\longrightarrow$ and $\longleftarrow\!\boxed{\Gamma''}$ are the splice diagrams of the links being spliced and the indicated arrowheads correspond to K' and K'' respectively. A disjoint union of diagrams corresponds to the split sum of the links, that is a connected sum of the ambient manifolds along balls disjoint from the links.

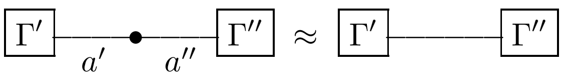
Let $-\Gamma$ denote the diagram Γ with the opposite signs of all the arrowhead vertices.

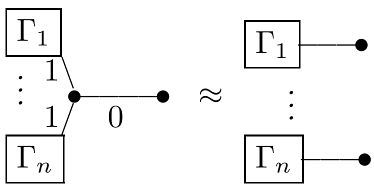
Theorem 2.2. (See [9, Thm. 8.1].) *Two splice diagrams determine the same link if and only if they are obtained from each other by a sequence of the following equivalences:*

(1) $\Gamma \approx -\Gamma$,

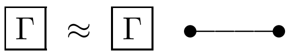
(2) 

(3) 

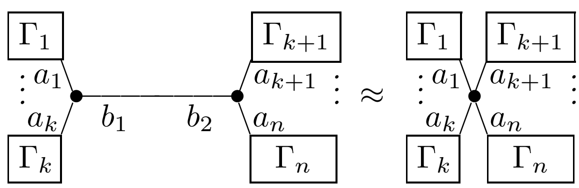


(4) 

(disjoint union)

(5) 

(disjoint union);

(6) 

if $\prod_{i \le k} a_i = b_2$ *and* $\prod_{i > k} a_i = b_1$.

Proposition 2.3. (See [9, §10].) *Let* $(\Sigma, S_1 \cup \cdots \cup S_k)$ *be as in (1). Then the linking numbers are* $\mathrm{lk}(S_i, S_j) = a_1 \dots \hat{a}_i \dots \hat{a}_j \dots a_n$, $1 \le i < j \le k$ *(as usual, the hat means the omission of the corresponding factor).*

In fact, in [9, §10], the linking numbers are expressed in terms of a splice diagram for any graph link in a graph homology 3-sphere.

3. Cabling and iterated torus links

Given an oriented link $L \cup K$ in a homology sphere Σ where K is a knot and a pair of integers $(p, q) \ne (0, 0)$, we set $d = \gcd(p, q)$, $p_1 = p/d$, $q_1 = q/d$, and we define the (p, q)*-cabling of* $L \cup K$ *along* K *with the core removed* (resp. *with the core retained*) as $L \cup L_{p,q}$ (resp. $L \cup K \cup L_{p,q}$) where $L_{p.q}$ is the union of d disjoint knots $K_1 \cup \cdots \cup K_d$ such that for some tubular neighborhood T of K disjoint from L, one has $K_j \subset \partial T$, $[K_j] = p_1[K] \in H_1(T)$, and $\mathrm{lk}(K_j, K) = q_1$ for each $j = 1, \dots, d$.

An iterated torus link in $\mathbb{S}^3$ is defined as a link obtained from the unknot by successive cabling. Note that reversion of orientation of a component is equivalent to the $(-1, 0)$-cabling with the core removed.

The (p, q)-cabling of $(\Sigma, L \cup K)$ along K with the core retained (resp. removed) is equivalent to the splicing of $(\Sigma, L \cup K)$ with

$$\mathfrak{Seif}(q_1, \underbrace{1, \dots, 1}_{d}, p_1) \quad \text{(resp. with } \mathfrak{Seif}(q_1, \underbrace{1, \dots, 1}_{d}; p_1))$$

along (K, S_1) where S_1 is the component corresponding to the weight q_1 (see [9, Prop. 9.1] and the paragraph after it).

4. Multivariate link signatures

Let L be an oriented link in a homology sphere Σ. For $\zeta \in \mathbb{C} \setminus \{1\}$, we denote the Levine-Tristram signature and nullity at ζ by $\sigma_\zeta(L)$ and nullity $\mathrm{n}_\zeta(L)$ respectively (see the introduction). There are different conventions for the nullity in the litterature. We define $\mathrm{n}_\zeta(L)$ as the nullity of $(1 - \zeta)V + (1 - \bar{\zeta})V^T$ where V is the Seifert form on a **connected** Seifert surface.

Let $\mathbb{S}^1$ be the unit circle in $\mathbb{C}$ and let $\mathcal{T} = \{e^{2\pi i\theta} \mid \theta \in \mathbb{Q} \text{ and } 0 < \theta < 1\} \subset \mathbb{S}^1 \setminus \{1\}$. Let $L_1, \dots, L_\mu$ be the components of L. A *multivariate signature and nullity* of L defined in [3, 6, 10] are functions $\sigma_L, \mathrm{n}_L : \mathcal{T}^\mu \to \mathbb{Z}$. As shown in [17] (see also [5]), these functions can be defined on $(\mathbb{S}^1 \setminus \{1\})^\mu$. It is natural to extend them to $(\mathbb{S}^1)^\mu$ by interpreting the value 1 of the i-th argument as the removal of L_i (see [6, 7]).

Remark 4.1. In fact, the multivatiate signatures are defined in [3, 10, 17] for any *colored link*, i.e., a link L with a fixed decomposition into a disjoint union of sublinks (not necessarily connected) $L = L_1 \cup \cdots \cup L_\mu$. In this case the i-th argument of σ_L and n_L corresponds to L_i. In this paper we consider only the case when each L_i is connected. However, everything can be easily extended to the case of arbitrary colored links due to [3, Prop. 2.5].

We refer to [17], [5], or [8] for a definition of σ_L and n_L. Here we just mention some properties of them.

Proposition 4.2. (See [3, Prop. 2.7].)

$$\begin{aligned} \sigma_L(u_1,\ldots,u_\mu) &= \sigma_L(u_1^{-1},\ldots,u_\mu^{-1}),\\ \mathrm{n}_L(u_1,\ldots,u_\mu) &= \mathrm{n}_L(u_1^{-1},\ldots,u_\mu^{-1}). \end{aligned} \tag{2}$$

Proposition 4.3. (See [3, Prop. 2.8].) *Let L' be obtained from $L = L_! \cup\cdots\cup L_\mu$ by reversing the orientation of L_i. Then $\sigma_L(u) = \sigma_{L'}(u')$ and $\mathrm{n}_L(u) = \mathrm{n}_{L'}(u')$ where $u = (u_1,\ldots,u_\mu)$ and $u' = (u_1,\ldots,u_i^{-1},\ldots,u_\mu)$.*

Proposition 4.4. (See [3, Prop. 2.5].) *Let $\lambda \in \mathbb{S}^1 \setminus \{1\}$. Then*

$$\sigma_\lambda(L) = \sigma_L(\lambda,\ldots,\lambda) - \sum_{1\le i<j\le\mu} \mathrm{lk}(L_i,L_j), \qquad \mathrm{n}_\lambda(L) = \mathrm{n}_L(\lambda,\ldots,\lambda).$$

The following fact is proven in [3] for links in the 3-sphere only but the proof extends (with certain efforts) to links in any homology sphere.

Proposition 4.5. (See [3].) *There exists a matrix $A = A(t_1,\ldots,t_\mu)$ whose entries are Laurent polynomials and such that:*

- *the multivariate Alexander polynomial $\Delta_L(t_1,\ldots,t_n)$ is equal to* $\det A$ *up to some factors of the form $\pm t_i^{\pm1}$ or $\pm(t_i-1)$;*
- *for any $u \in (\mathbb{S}^1\setminus\{1\})^\mu$, the signature and nullity of $A(u)$ are $\sigma_L(u)$ and $\mathrm{n}_L(u)$ respectively.*

Corollary 4.6. *If the multivariate Alexander polynomial $\Delta_L(t_1,\ldots,t_\mu)$ is not identically zero, then σ_L is constant on each connected component of the complement of the zero set of Δ_L in $(\mathbb{S}^1\setminus\{1\})^\mu$.*

The following fact should be known and it is easy to prove.

Lemma 4.7. *Let $A(t) = (a_{ij}(t))$, $t\in\mathbb{R}$, be a Hermitian matrix such that* $\operatorname{Re} a_{ij}$ *and* $\operatorname{Im} a_{ij}$ *are real analytic functions of t. Suppose that* $\det A(t)$ *has a simple root at $t = t_0$, and there are no other roots in an interval $(t_0-2\varepsilon, t_0+2\varepsilon)$. Let s_0 and $s_\pm$ be the signature of $A(t)$ at $t=t_0$ and $t = t_0\pm\varepsilon$ respectively. Then $|s_+-s_-| = 2$, $s_0 = (s_++s_-)/2$, and the nullity of $A(t_0)$ is* 1. □

By combining Proposition 4.5 with Lemma 4.7, we obtain:

Corollary 4.8. *Let $u = (u_1,\ldots,u_\mu) \in (\mathbb{S}^1\setminus\{1\})^\mu$ be such that the gradient of $\Delta_L(t_1,\ldots,t_\mu)$ at u is non-zero. Then, for a small neighborhood $U \subset (\mathbb{S}^1\setminus\{1\})^\mu$ of u, the restrictions $\sigma_L|_U$ and $\mathrm{n}_L|_U$ depend only on the sign of Δ_L, and one has $|s_1 - s_{-1}| = 2$, $s_0 = (s_1+s_{-1})/2$, $n_{\pm1} = 0$, and $n_0 = 1$ where $s_t = \sigma_L|_{U_t}$ and $n_t = \mathrm{n}_L|_{U_t}$ for $U_t = U\cap\{\operatorname{sign}\Delta_L = t\}$, $t = -1,0,1$.*

5. Multivariate signatures of a splice (after Degtyarev, Florens and Lecuona)

For $\ell = (\ell_1,\ldots,\ell_\mu) \in \mathbb{Z}^\mu$, we define the *defect function* $\delta_\ell : (\mathbb{S}^1)^\mu \to \mathbb{Z}$ by setting

$$\delta_\ell(u_1,\ldots,u_\mu) = \operatorname{ind}\Big(\sum \ell_i \operatorname{Log} u_i\Big) - \sum \ell_i \operatorname{ind}(\operatorname{Log} u_i)$$

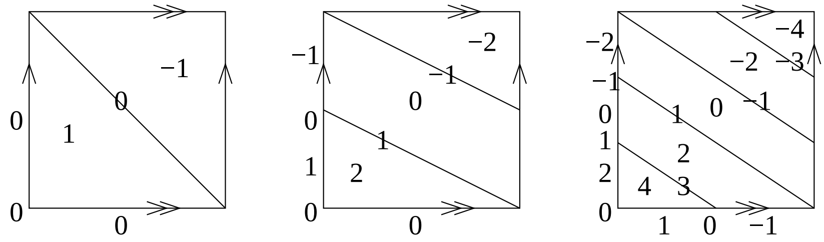


FIGURE 3. The values of δ_ℓ on $(\mathbb{S}^1)^2$ for $\ell = (1,1)$, $(1,2)$, and $(2,3)$.

(see Figure 3) where Log : $\mathbb{S}^1 \to [0,1[$ and ind : $\mathbb{R} \to \mathbb{Z}$ are defined by

$$\operatorname{Log}(e^{2\pi i t}) = t, \qquad \operatorname{ind}(x) = \lfloor x \rfloor - \lfloor -x \rfloor.$$

Let $(\Sigma, L' \cup L'')$ be the splice of $(\Sigma', K' \cup L')$ and $(\Sigma'', K'' \cup L'')$ along K' and K'' (see Definition 2.1). Let $L'_1, \dots, L'_{\mu'}$ and $L''_1, \dots, L''_{\mu''}$ be the components of L' and L'' respectively. Let ℓ' and ℓ'' be the vectors of linking numbers

$$\ell' = \big(\operatorname{lk}(K', L'_1), \dots, \operatorname{lk}(K', L'_{\mu'})\big), \qquad \ell'' = \big(\operatorname{lk}(K'', L''_1), \dots, \operatorname{lk}(K'', L''_{\mu''})\big).$$

Theorem 5.1. *(a).* (See [6, Thm. 2.2], [7, Thm. 5.2].) *Assume that L' and L'' are non-empty. Let $u' \in (\mathbb{S}^1)^{\mu'}$ and $u'' \in (\mathbb{S}^1)^{\mu''}$ be such that $(v', v'') \neq (1,1)$ where $v' = (u')^{\ell'}$ and $v'' = (u'')^{\ell''}$. Then*

$$\sigma_{L'\cup L''}(u', u'') = \sigma_{K'\cup L'}(v'', u') + \sigma_{K''\cup L''}(v', u'') + \delta_{\ell'}(u')\delta_{\ell''}(u''),$$
$$\mathrm{n}_{L'\cup L''}(u', u'') = \mathrm{n}_{K'\cup L'}(v'', u') + \mathrm{n}_{K''\cup L''}(v', u'').$$

(b). (See [6, Addendum 2.7].) *Assume that $L' = \varnothing$ and $L'' \neq \varnothing$. Then, for any $u \in (\mathbb{S}^1)^{\mu''}$ one has*

$$\sigma_{(\Sigma, L'')}(u) = \sigma_{K'}(u^{\lambda''}) + \sigma_{K''\cup L''}(u),$$
$$\mathrm{n}_{(\Sigma, L'')}(u) = \mathrm{n}_{K'}(u^{\lambda''}) + \mathrm{n}_{K''\cup L''}(u).$$

In the case when $(v', v'') = (1,1)$, Theorem 5.1(a) does not apply but the signature and nullity of the splice can be, however, often computed by the same formulas with a correction term which ranges in $[-2, 2]$. The correction term is a function of the values of a certain invariant called *slope* evaluated on each of the splice components, see details in [7, 8] (see also Question 11.1 below).

6. Equivariant link signatures

Another kind of link signatures considered in this paper are *equivariant signatures* $\sigma^\pm_\lambda$. We define them for fibered links only though the definition can be extended to the general case (see surveys [4], [11]). So, let L be a fibered link, F be the fiber (in particular, $\partial F = L$), and $h : H_1(F) \to H_1(F)$ be the monodromy operator. Then the one-variable Alexander polynomial $\Delta_L(t)$ is the characteristic polynomial of h. Let $H = H_1(F) \otimes \mathbb{C}$ and let $H = \bigoplus_\lambda H_\lambda$ be the splitting of H according to the eigenvalues of h. Let l be the Seifert form on $H_1(F)$ extended to a sesquilinear form on H. Then σ^+_λ (resp. σ^-_λ) is defined as the signature of the hermitian form $l + l^*$ (resp. $i(l - l^*)$) restricted to H_λ.

Proposition 6.1. (See [13], [14].) *For any fibered link L, we have:*

(a). $\sigma^+_{\bar\lambda} = \sigma^+_\lambda$ and $\sigma^-_\lambda = (\operatorname{sign}\operatorname{Im}\lambda)\sigma^+_\lambda$, in particular $\sigma^-_\lambda = -\sigma^-_{\bar\lambda}$ and $\sigma^-_{\pm1} = 0$;

(b). if $|\lambda| \ne 1$ or λ is not a root of the Alexander polynomial, then $\sigma^+_\lambda = \sigma^-_\lambda = 0$;

(c). Let $\omega = e^{i\varphi}$, $0 < \varphi \le \pi$. If h is semisimple or ω is not a root of $\Delta_L(t)$, then the Levine–Tristram signature and nullity are

$$\sigma_\omega(L) = \sigma^+_\omega + 2 \sum_{\substack{0\le\theta<\varphi\\ \lambda=e^{i\theta}}} \sigma^+_\lambda \qquad \text{and} \qquad \mathrm{n}_\omega(L) = \dim H_\omega,$$

i.e., $\mathrm{n}_\omega(L)$ is the multiplicity of ω as a root of $\Delta_L(t)$.

Remark 6.2. The orientation conventions in [13] and those in [14, 15] are different. In [13] they are chosen so that $l^*h = l$ whereas in [14, 15] so that $lh = l^*$. By this reason, the signs of σ^-_λ in these sources also differ. We use the convention from [13], which seems to be more common nowadays.

7. The sawtooth function and an identity for it

Let

$$((x)) = \begin{cases} \frac12 - x + \lfloor x\rfloor, & x \notin \mathbb{Z} \\ 0, & x \in \mathbb{Z} \end{cases}$$

where $\lfloor x\rfloor = \max\{n \in \mathbb{Z} \mid n \le x\}$ is the integer part of x. Let, for a set Ω,

$$\mathbf{1}_\Omega(x) = \begin{cases} 1, & x \in \Omega, \\ 0, & x \notin \Omega. \end{cases}$$

Lemma 7.1. *Let $x, y, z_1, \dots, z_n \in \mathbb{R}$. Suppose that $x + y + z_1 + \dots + z_n = s \in \mathbb{Z}$ and $0 < z_i < 1$ for all $i = 1, \dots, n$. Then*

$$((x)) + ((y)) + ((z_1)) + \dots + ((z_n)) = n/2 - \#\big(\mathbb{Z} \cap]x, s-y[\big) - \tfrac12 \mathbf{1}_\mathbb{Z}(x) - \tfrac12 \mathbf{1}_\mathbb{Z}(y).$$

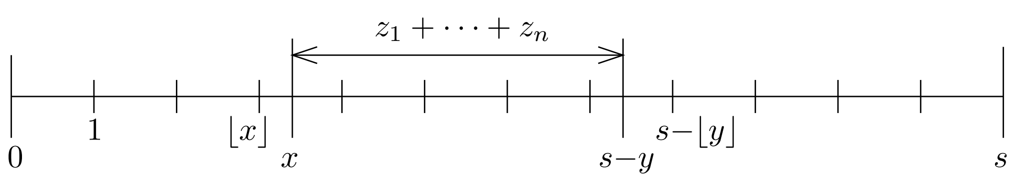


Figure 4

Proof. We consider the case $x, y \notin \mathbb{Z}$ and we leave the other cases to the reader. Since $\lfloor z_1\rfloor = \dots = \lfloor z_n\rfloor = 0$, the left hand side is equal to

$$\frac{n+2}{2} - (x - \lfloor x\rfloor) - (y - \lfloor y\rfloor) - \sum z_i = \frac{n+2}{2} - s + \lfloor x\rfloor + \lfloor y\rfloor$$

and we have $1 + \#\big(\mathbb{Z} \cap]x, s-y[\big) = (s - \lfloor y\rfloor) - \lfloor x\rfloor$ (see Figure 4). $\square$

8. Neumann's formula for equivariant signatures of Seifert links

Let $L = \mathfrak{Seif}(a_1,\dots,a_k;a_{k+1},\dots,a_n)$ with positive $a_1,\dots,a_n$. We set:

$$m_j = \begin{cases} 1, & j\le k,\\ 0, & j>k, \end{cases} \qquad a'_j = (a_1\dots a_n)/a_j, \qquad m = \sum_{j=1}^n m_j a'_j. \tag{3}$$

Let us choose $b_1,\dots,b_n$ so that $b_j a'_j \equiv 1 \mod a_j$ for each $j=1,\dots,n$, and let

$$s_j = (m_j - b_j m)/a_j, \qquad j=1,\dots,n.$$

Theorem 8.1. (Neumann [14, 15].) *The link L is fibered, its monodromy operator is semisimple, and one has: $\sigma^+_1(L) = 1-k$, $\sigma^+_{-1}(L)=0$, and, for $\lambda\ne 1$,*

$$\sigma^-_\lambda(L) = \begin{cases} -2\sum_{j=1}^n((s_jk/m)) & \text{if } \lambda=\exp(2\pi ik/m) \text{ with } k\in\mathbb{Z},\\ 0 & \text{otherwise} \end{cases}$$

where $((\dots))$ is the sawtooth function defined in §7.

Recall that the sign of σ^-_λ in [14, 15] is opposite, see Remark 6.2.

Corollary 8.2. *For $\omega = e^{i\varphi}$, $0<\varphi\le\pi$, one has*

$$\sigma_\omega(L) = 1-k+\sigma^-_\omega(L) + 2\sum_{\substack{0<\theta<\varphi\\ \lambda=e^{i\theta}}}\sigma^-_\lambda(L)$$

and $\mathrm{n}_\omega(L)$ is equal to the multiplicity of ω as a root of $\Delta_L(t)$.

By [9, Thm. 12.1], if $k\ge 2$, the multivariate Alexander polynomial of L is

$$\Delta_L(t_1,\dots,t_k) = \big(t_1^{a'_1}\dots t_k^{a'_k}-1\big)^{n-2}\prod_{j=k+1}^n\big(t_1^{a'_1/a_j}\dots t_k^{a'_k/a_j}-1\big)^{-1} \tag{4}$$

and hence the one-variable Alexander polynomial is $\Delta_L(t) = (t-1)\Delta_L(t,\dots,t)$. (If $k=1$, then the right hand side of (4) should be multiplied by t_1-1.) The formula (4) is a specialization of a general formula given in [9] for the Alexander polynomial of any graph multilink in a graph homology sphere.

Remark 8.3. In fact, the results of [15, Thms. 5.1–5.3] are much stronger than Theorem 8.1: for any algebraic link (not only for positive Seifert links) a decomposition of the Hermitian isometric structure into irreducible factors is computed there, which includes a description of the Seifert form up to congruence over $\mathbb{C}$ (there is a misprint in [15, Thm. 5.3]: the factor i should be omitted in the Seifert part of Λ^2_λ).

9. Hirzebruch-type formula for signatures of a torus link with the core(s)

Given four integers a,b,c,d with $ad-bc>0$, we denote the **open** parallelogram in $\mathbb{R}^2$ spanned by the vectors (a,b) and (c,d) by $\Pi=\Pi(a,b,c,d)$. Thus

$$\Pi(a,b,c,d) = \{s(a,b)+t(c,d)\mid 0<s<1 \text{ and } 0<t<1\}.$$

Let $u:\mathbb{R}^2\to\mathbb{R}$ be the linear function such that $u(a,b)=u(c,d)=1$. For $0\le\theta<1$, we set (see Figure 5):

$$\begin{aligned}
N_\theta^- &= N_\theta^-(a,b,c,d) = \#\{(x,y)\in\mathbb{Z}^2\cap\Pi \mid \theta<u(x,y)<\theta+1\},\\
N_\theta^+ &= N_\theta^+(a,b,c,d) = \#\{(x,y)\in\mathbb{Z}^2\cap\Pi \mid u(x,y)<\theta \text{ or } \theta+1<u(x,y)\},\\
M_\theta^- &= M_\theta^-(a,b,c,d) = \#\{(x,y)\in\mathbb{Z}^2\cap\Pi \mid u(x,y)=\theta+1\},\\
M_\theta^+ &= M_\theta^+(a,b,c,d) = \#\{(x,y)\in\mathbb{Z}^2\cap\Pi \mid u(x,y)=\theta\}.
\end{aligned}$$

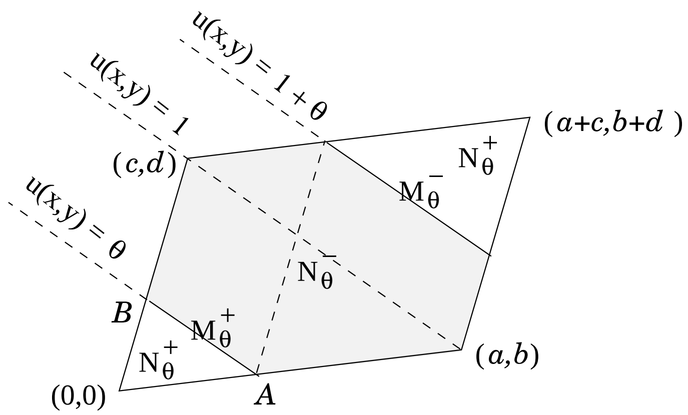


FIGURE 5. Definition of $N_\theta^\pm$ and $M_\theta^\pm$

Let p and q be positive integers (not necessarily coprime) and let $m_1,m_2\in\{0,1\}$. We define the torus link $T_{m_1,m_2}(p,q)$ as the intersection of the unit sphere in $\mathbb{C}^2$ with the algebraic curve $\{(z_1,z_2)\mid z_1^{m_1}z_2^{m_2}(z_1^p-z_2^q)=0\}$ endowed with the boundary orientation induced from the intersection of the curve with the unit ball, thus the linking number of any two components is positive. If $m_1=m_2=0$, this is the torus link $T(p,q)$.

Proposition 9.1. *Let $(a,b)=(p,0)+(m_1,m_2)$ and $(c,d)=(0,q)+(m_1,m_2)$. Then the equivariant signature and Levine-Tristram signature and nullity of $T_{m_1,m_2}(p,q)$ at $\lambda=e^{2\pi i\theta}$ for $0<\theta<1$ are:*

$$\sigma_\lambda^- = M_\theta^+ - M_\theta^-,\qquad \sigma_\lambda = N_\theta^+ - N_\theta^- - m_1 - m_2,\qquad \mathrm{n}_\lambda = M_\theta^+ + M_\theta^-$$

where $N_\theta^\pm=N_\theta^\pm(a,b,c,d)$, $M_\theta^\pm=M_\theta^\pm(a,b,c,d)$.

Remark 9.2. In the case $m_1=m_2=0$ this is Hirzebruch's formula (see [1; §6] for $\theta=1/2$ and [13; §4] for any θ).

Remark 9.3. In [16; Prop. 8.1], I gave the following formulas for the signature and nullity of the braid closure L of the braid Δ^n with $2k+1$ strings:

$$\sigma(L)=\sigma_{-1}(L)=\begin{cases} -nk(k+1)+(-1)^{(n-1)/2} & \text{if } k\equiv n\equiv 1 \mod 2,\\ -nk(k+1) & \text{otherwise,}\end{cases}$$

$$\mathrm{n}(L)=\mathrm{n}_{-1}(L)=\begin{cases} 2k & \text{if } n\equiv 0 \mod 4,\\ 0 & \text{otherwise.}\end{cases}$$

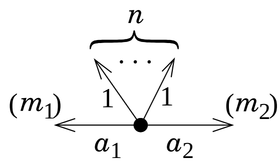


FIGURE 6. Splice diagram of $T_{m_1,m_2}(na_1, na_2)$

For a proof I gave just a reference to [15]. Now Proposition 9.1 provides missing details of the computation because the link in question is $T_{0,1}(nk, 2k)$.

Remark 9.4. If m_1 and m_2 are any non-negative integers, we define $T_{m_1,m_2}(p, q)$ by the same formula as above but we interpret it as a multilink in the sense of [9, 15]. In this case the equality $\sigma^-_\lambda = M^+_\theta - M^-_\theta$ and its proof hold true.

Proof of Proposition 9.1. Let $p = a_1 n$ and $q = a_2 n$ where $n = \gcd(p, q)$. Then the splice diagram of the (multi)link $T_{m_1,m_2}(p, q)$ is as shown in Figure 6. We have $M^+_0 = n - 1$, $M^-_0 = 0$, and the number of link components is $n + m_1 + m_2 = (M^+_0 - M^-_0) + 1 + m_1 + m_2$. It is also easy to check that

$$N^+_\varphi - N^-_\varphi = (M^+_\varphi - M^-_\varphi) + 2 \sum_{0\le\theta<\varphi} (M^+_\theta - M^-_\theta).$$

Thus, by Corollary 8.2, the computation of σ_λ reduces to that of σ^-_λ. To compute σ^-_λ, we apply Theorem 8.1 with $a_j = m_j = 1$ for $j = 3, \dots, n + 2$. Let the a'_j, b_j, and s_j be defined as in §8 with the range of the indices appropriately changed. We have $a'_1 = a_2$, $a'_2 = a_1$, and $a'_j = a_1 a_2$ for $j \ge 3$ whence

$$m = m_1 a_2 + m_2 a_1 + n a_1 a_2.$$

Hence the linear function u involved in the definition of $M^\pm_\theta$, $N^\pm_\theta$ takes the form $u(x, y) = (a_2 x + a_1 y)/m$. So, when $m\theta \notin \mathbb{Z}$, we have $M^+_\theta - M^-_\theta = 0 - 0 = \sigma^-_\theta$. Let us fix $\theta = k/m$ with $k \in \mathbb{Z}$, $0 < k < m$. Then, by Theorem 8.1, we have $\sigma^-_\theta = -2 \sum ((x_j))$ where $x_j = s_j k/m$, $j = 1, \dots, n + 2$.

We choose b_1 and b_2 so that $a_1 b_2 + a_2 b_1 = 1$ and we set $b_j = 0$ for $j \ge 3$. Then $\sum b_j a'_j = 1$, hence

$$a_1 \dots a_{n+2} \sum s_j = \sum (m_j - b_j m) a'_j = m - m \sum b_j a'_j = 0$$

and we obtain $\sum x_j = \sum s_j k/m = 0$. We have also $0 < x_j = k/m < 1$ for $j \ge 3$. Thus we may apply Lemma 7.1 which yields

$$\sigma^-_\lambda = -n + 2\#\{l \in \mathbb{Z} \mid x_1 < l < -x_2\} + \mathbf{1}_\mathbb{Z}(x_1) + \mathbf{1}_\mathbb{Z}(x_2). \tag{5}$$

The inequalities $x_1 < l < -x_2$ in (5) can be transformed as follows:

$$\begin{array}{lclcl} x_1 < l & \Leftrightarrow & (m_1 - b_1 m)k/m < a_1 l & \Leftrightarrow & b_1 k + a_1 l > m_1 k/m, \\ l < -x_2 & \Leftrightarrow & (m_1 - b_2 m)k/m < -a_2 l & \Leftrightarrow & b_2 k - a_2 l > m_2 k/m. \end{array}$$

Since $\varphi : t \mapsto (b_1 k + a_1 t, b_2 k - a_2 t)$ is a parametrization of the line $\{u = \theta\}$ such that $\varphi(\mathbb{Z}) = \mathbb{Z}^2 \cap \{u = \theta\}$ and since $\varphi(m_1 k/m) = A$ and $\varphi(m_2 k/m) = B$ are the intersection points of $\{u = \theta\}$ with $\partial\Pi$ (see Figure 5), we can rewrite (5) in the form

$$\sigma_\lambda^- = -n + 2M_\theta^+ + \partial M_\theta^+$$

where ∂M_θ^+ stands for $\mathbf{1}_{\mathbb{Z}^2}(A) + \mathbf{1}_{\mathbb{Z}^2}(B)$. By combining this equation with

$$n = M_0^- + \tfrac{1}{2}\partial M_0^- = M_\theta^- + M_\theta^+ + \tfrac{1}{2}(\partial M_\theta^- + \partial M_\theta^+) \quad \text{and} \quad \partial M_\theta^+ = \partial M_\theta^-, \qquad (6)$$

we obtain the desired expression for σ_λ^-.

Now we apply Corollary 8.2 to compute n_λ. By (4) we have

$$\Delta_L(t) = \frac{(t-1)(t^m-1)^{n+m_1+m_2-2}}{(t^{m/a_1}-1)^{1-m_1}(t^{m/a_2}-1)^{1-m_2}}$$

and by (6) we have $M_\theta^+ + M_\theta^- = n - \partial M_\theta^+$. Then in the case $(m_1, m_2) = (1, 1)$ we immediately conclude that $M_\theta^+ + M_\theta^-$ is the multiplicity of $e^{2\pi i\theta}$ as a root of $\Delta_L(t)$. We leave the cases $(m_1, m_2) = (1, 0)$ and $(m_1, m_2) = (0, 0)$ to the reader. □

10. Multivariate signatures of Seifert links

In this section we compute the multivariate signatures and nullities of any Seifert link with any orientations of its components. Proposition 10.1 combined with Corollary 8.2 allows us to compute σ_L and n_L for any positive Seifert link $L = \mathfrak{Seif}(a_1, \dots, a_k; a_{k+1}, \dots, a_n)$, $a_i \ge 0$. Theorem 2.2(2) combined with Proposition 4.3 allows us to reduce the general case to the case of positive Seifert links. Note that a reversing of orientation of $\Sigma(a_1, \dots, a_n)$ (switching the color of the central node of a splice diagram) changes the sign of σ_L.

In fact, in most cases the computation of σ_L and n_L for a positive Seifert link immediately follows from the facts formulated in previous sections. Indeed, let us consider the coordinates $(\theta_1, \dots, \theta_n)$ on $(\mathbb{S}^1 \setminus \{1\})^n$, $u_j = e^{2\pi i\theta_j}$. Then the restriction of σ_L and n_L to the diagonal $\theta_1 = \dots = \theta_n$ is determined by Theorem 8.1 and Proposition 4.4. By (4), the zero set of the multivariate Alexander polynomial is a union of parallel hyperplanes transverse to the diagonal. Thus the extension of σ_L and n_L from the diagonal to $(\mathbb{S}^1 \setminus \{1\})^n$ is determined by Corollary 4.8 everywhere except the hyperplanes corresponding to the multiple roots of Δ_L. Thus the only thing remaining to do is to extend σ_L and n_L to the multiple components of $\{\Delta_L = 0\}$. This is done using the splicing formula in Theorem 5.1 combined with the observation that a Seifert link can be spliced in many different ways. Let us give the exact statement and a formal proof.

Proposition 10.1. *Let $L = L_1 \cup \dots \cup L_k = \mathfrak{Seif}(a_1, \dots, a_k;\, a_{k+1}, \dots, a_n)$ with non-negative $a_1, \dots, a_n$. Let $u = (u_1, \dots, u_k)$ where $u_j = \exp(2\pi\theta_j)$, $0 < \theta_j < 1$, Then $\sigma_L(u)$ and $\mathrm{n}_L(u)$ depend only on the sum $\sum_{j=1}^k a'_j\theta_j$ (the a'_j are as in (3)), thus*

$$\sigma_L(u) = \sigma_L(\lambda, \dots, \lambda) = \sigma_\lambda(L) + \sum_{1 \le i < j \le k} \mathrm{lk}(L_i, L_j)$$

and $\mathrm{n}_L(u) = \mathrm{n}_L(\lambda, \dots, \lambda) = \mathrm{n}_\lambda(L)$ for

$$\lambda = \exp(2\pi i\theta), \qquad \theta = \frac{a'_1\theta_1 + \dots + a'_k\theta_k}{a'_1 + \dots + a'_k}.$$

Proof. If $a_j = 0$ for some j, then $(a_1, \dots, a_n)$ is a permutation of $(0, 1, \dots, 1)$, hence either L is a trivial k-component link (if $j > k$; Theorem 2.2(4)), or L is the link in Figure 1, and then the result follows from [6; Theorem 2.10]. So, we assume that all a_j are positive.

If $n = 2$, the statement is evident because L is either an unknot or a 2-component Hopf link. If $n = 3$, the statement follows from Corollary 4.8 because the multivariate Alexander polynomial is of the form (see (4))

$$\Delta_L(t_1, \dots, t_k) = f(t_1^{a'_1} \dots t_k^{a'_k} - 1),$$

where $f(t)$ is a polynomial without multiple roots. Hence the zero set of Δ_L considered as an analytic function of $\theta_1, \dots, \theta_k$ is a union of parallel hyperplanes transverse to the main diagonal, and the gradient of Δ_L does not vanish on it.

If $k = 2$, we consider the splice in Figure 7. The obtained link coincides with L by the edge contraction property (Theorem 2.2(6)). Hence the result follows from Theorem 5.1(b) combined with the case $n = 3$ of the proposition that we are proving.

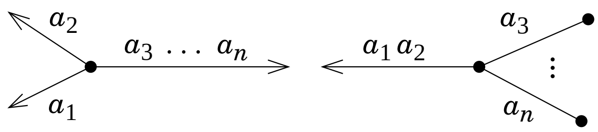


FIGURE 7. Splicing when $k = 2$ in the proof of Prop. 10.1.

So, we assume that $k \geq 3$. We need to prove that the signature and nullity (considered as functions of $\theta_1, \dots, \theta_k$) are locally constant on each open $(k-1)$-dimensional polytope $P_c = \{a'_1\theta_1 + \dots + a'_k\theta_k = c\} \cap]0, 1[^k$. To this end it is enough to show that they are constant on each open interval in P_c defined by the condition that only two of the θ_k's vary and the others are fixed. Moreover, it is enough to establish the constancy along the subintervals of some fixed length. Thus we reduce the problem to the following assertion.

Let $k \geq 3$ and let $\tilde\theta_2$ and $\tilde\theta_3$ satisfy the conditions:

$$0 < \frac{\tilde\theta_2 - \theta_2}{a_2} = \frac{\theta_3 - \tilde\theta_3}{a_3} < \frac{1}{a_1}, \qquad 0 < \tilde\theta_2 < 1, \quad 0 < \tilde\theta_3 < 1. \tag{7}$$

Then $\sigma_L(u) = \sigma_L(u_1, \tilde u_2, \tilde u_3, u_4, \dots, u_k)$ where $\tilde u_j = \exp(2\pi i\tilde\theta_j)$, $j = 2, 3$.

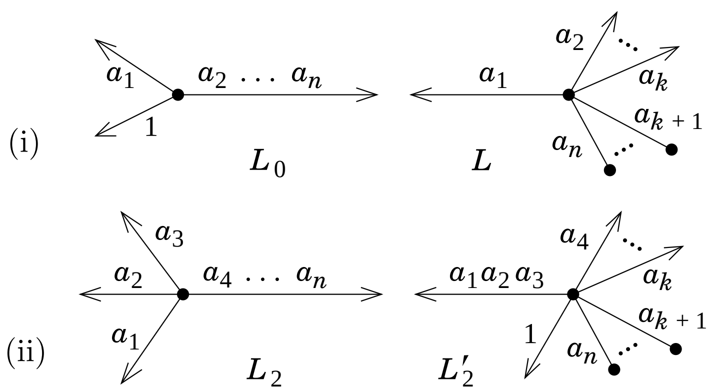


FIGURE 8. Two splicings of L_1 in the proof of Prop. 10.1.

Let us prove it. Indeed, let L_1 be the link obtained by the splicing according to Figure 8(i). We have $L_1 = \mathfrak{Seif}(1, a_1, \dots, a_k;\, a_{k+1}, \dots, a_n)$ by the edge contraction property (Theorem 2.2(6)). By the same reason L_1 can be spliced also as in Figure 8(ii).

We set $A_j = a_j a_{j+1} \dots a_n$ and $u_{\vec{j}} = (u_j, \dots, u_k)$, in particular $u_{\vec{1}} = u$. Due to the splicing in Figure 8(i), for any u_0 we have by Theorem 5.1(a):

$$\sigma_L(u) = \sigma_{L_1}(u_0, u_1^*, u_{\vec{2}}) - \sigma_{L_0}(v, u_0, u_1^*) - \delta_L(u)\delta_{L_0}(v, u_0, u_1^*) \tag{8}$$

where $u_1^* = u_1 u_0^{-a_1}$ and $v = \prod_{j=2}^k u_j^{A_2/a_j}$ (the notation δ_L and δ_{L_0} must be clear). This formula applies when $(u_1, v) \neq (1,1)$ which is the case because $u_1 \neq 1$.

Similarly, if $(v_2, v_2') \neq (1,1)$, then the splicing in Figure 8(ii) yields

$$\sigma_{L_1}(u_0, u_1^*, u_{\vec{2}}) = \sigma_{L_2}(v_2', u_1^*, u_2, u_3) + \sigma_{L_2'}(v_2, u_0, u_{\vec{4}}) + \delta_{L_2}(u_1^*, u_2, u_3)\delta_{L_2'}(u_0, u_{\vec{4}}) \tag{9}$$

where $v_2' = u_0^{A_4} \prod_{j=4}^k u_j^{A_4/a_j}$ and $v_2 = (u_1^*)^{a_2 a_3} u_2^{a_1 a_3} u_3^{a_1 a_2}$.

Further, for the link L_2 (the left hand side of Figure 8(ii)), we consider the two splicings shown in Figure 9. Then, if $(v_3, v_3') \neq (1,1) \neq (v_4, v_4')$, we have

$$\sigma_{L_2}(v_2', u_1^*, u_2, u_3) = \sigma_{L_3}(v_3', u_1^*, u_2) + \sigma_{L_3'}(v_3, u_3, v_2') + \delta_{L_3}(u_1^*, u_2)\delta_{L_3'}(u_3, v_2'), \tag{10}$$

$$\sigma_{L_2}(v_2', u_1^*, \tilde{u}_2, \tilde{u}_3) = \sigma_{L_4}(v_4', u_1^*, \tilde{u}_3) + \sigma_{L_4'}(v_4, \tilde{u}_2, v_2') + \delta_{L_4}(u_1^*, \tilde{u}_3)\delta_{L_4'}(\tilde{u}_2, v_2') \tag{11}$$

where $v_3 = (u_1^*)^{a_2} u_2^{a_1}$, $v_3' = (v_2')^{a_3} u_3^{A_4}$, $v_4 = (u_1^*)^{a_3} \tilde{u}_3^{a_1}$, $v_4' = (v_2')^{a_2} \tilde{u}_2^{A_4}$.

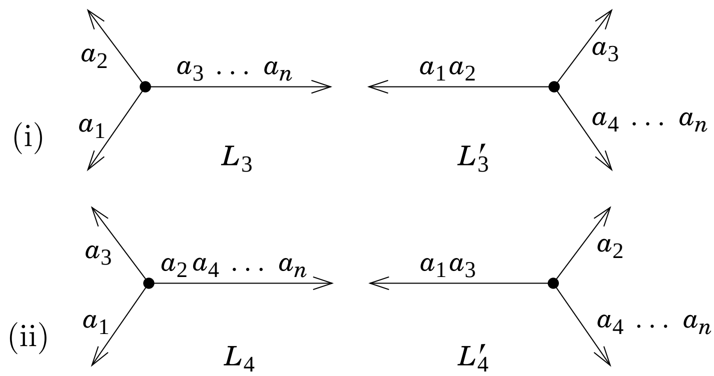


FIGURE 9. Two splicings of L_2 in the proof of Prop. 10.1.

Let $\theta_1^* \in \,]0,1[$ be such that $u_1^* = e^{2\pi i \theta_1^*}$ (recall that $u_1^* = u_1 u_0^{-a_1}$). Let us choose u_0 so that $v_j \neq 1$ for $j = 2,3,4$ (thus (8)–(11) hold) and θ_1^* satisfies the condition

$$0 < \tilde{\theta}_1 < \theta_1^* < 1 \quad \text{where } \tilde{\theta}_1 \text{ is defined by} \quad \frac{\theta_1^* - \tilde{\theta}_1}{a_1} = \frac{\tilde{\theta}_2 - \theta_2}{a_2} = \frac{\theta_3 - \tilde{\theta}_3}{a_3}. \tag{12}$$

Such u_0 exists due to (7). Similarly to (10) and (11), for $\tilde{u}_1 = e^{2\pi i \tilde{\theta}_1}$ we have

$$\sigma_{L_2}(v_2', \tilde{u}_1, \tilde{u}_2, u_3) = \sigma_{L_3}(v_3', \tilde{u}_1, \tilde{u}_2) + \sigma_{L_3'}(v_3, u_3, v_2') + \delta_{L_3}(\tilde{u}_1, \tilde{u}_2)\delta_{L_3'}(u_3, v_2'), \tag{13}$$

$$\sigma_{L_2}(v_2', \tilde{u}_1, \tilde{u}_2, u_3) = \sigma_{L_4}(v_4', \tilde{u}_1, u_3) + \sigma_{L_4'}(v_4, \tilde{u}_2, v_2') + \delta_{L_4}(\tilde{u}_1, u_3)\delta_{L_4'}(\tilde{u}_2, v_2'). \tag{14}$$

Since the case $n = 3$ is already done, the condition (12) implies

$$\sigma_{L_3}(v_3', u_1^*, u_2) = \sigma_{L_3}(v_3', \tilde u_1, \tilde u_2), \qquad \delta_{L_3}(u_1^*, u_2) = \delta_{L_3}(\tilde u_1, \tilde u_2),$$
$$\sigma_{L_4}(v_4', u_1^*, \tilde u_2) = \sigma_{L_4}(v_4', \tilde u_1, u_2), \qquad \delta_{L_4}(u_1^*, \tilde u_2) = \delta_{L_4}(\tilde u_1, u_2).$$

Hence all terms in the right hand sides of (10) and (11) are equal to the corresponding terms in (13) and (14). Therefore the left hand sides are equal as well and we obtain

$$\sigma_{L_2}(v_2', u_1^*, u_2, u_3) \overset{(10),(13)}{=} \sigma_{L_2}(v_2', \tilde u_1, \tilde u_2, u_3) \overset{(11),(14)}{=} \sigma_{L_2}(v_2', u_1^*, \tilde u_2, \tilde u_3).$$

Thus all terms in the right hand side of (9) do not change when we replace u_2 and u_3 with $\tilde u_2$ and $\tilde u_3$, whence the same is true for (8). This completes the proof for the signature. The proof for the nullity is the same. □

11. Some examples

11.1. Singularity link of two transverse cusps. We start with the example considered in [15, §7]. Let L be the link which is cut out by the curve

$$(x^3 - y^2)(x^2 - y^3) = 0 \tag{15}$$

on a sufficiently small sphere in $\mathbb{C}^2$ centered at the origin. Then L is obtained from the Hopf link by the $(2, 3)$-cabling along each of its component. This corresponds to the splicing in Figure 10 where we write the variable names near the corresponding arrowhead vertices. We denote the splice components by $L_j \cup K_j$, $j = 1, 2$, where L_j is a trefoil knot and K_j is its core.

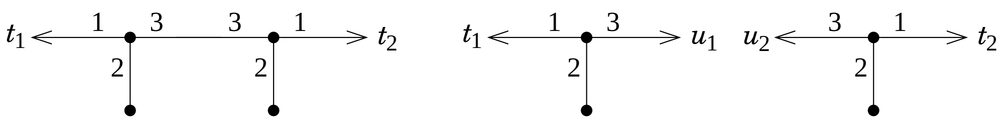


Figure 10. Splice diagram and splice components for the link in §11.1

The Alexander polynomial of L is

$$\Delta_L(t_1, t_2) = \frac{(t_1^6 t_2^4 - 1)(t_1^4 t_2^6 - 1)}{(t_1^3 t_2^2 - 1)(t_1^2 t_2^3 - 1)} = (t_1^3 t_2^2 + 1)(t_1^2 t_2^3 + 1)$$

and those of the splice components are $t_1^3 u_1 + 1$ and $t_2^3 u_2 + 1$.

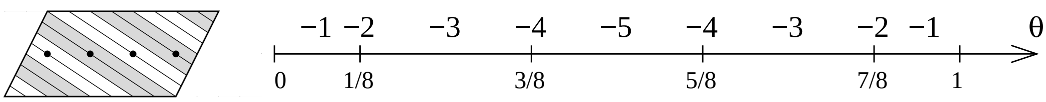


Figure 11. $\Pi(3 + 1, 0, 1, 2)$ and the values of $\sigma_\lambda(K_j \cup L_j)$, $\lambda = e^{2\pi i\theta}$

By Proposition 9.1, the Tristram–Levine signatures of each splice component are as in Figure 11. Therefore, by Proposition 10.1, the multivariate signatures $\sigma_{K_j\cup L_j}(t_j,u_j)$ for $t_j=e^{2\pi i\theta_j}$, $u_j=e^{2\pi i\varphi_j}$ are as in Figure 12 on the left. The nullity $\mathrm{n}_{K_j\cup L_j}$ is $1-|\operatorname{sign}(\Delta_{L_j})|$ with only two exceptions: $\mathrm{n}_{K_j\cup L_j}(-1,1)=\mathrm{n}_{K_j\cup L_j}(1,-1)=0$ (the small white circles in Figure 12).

By Theorem 5.1, for $(t_1,t_2)\in(\mathbb{S}^1)^2$ we have

$$\sigma_L(t_1,t_2)=\sigma_{K_1\cup L_1}(t_2^2,t_1)+\sigma_{K_2\cup L_2}(t_1^2,t_2)+\delta_{(2)}(t_1)\delta_{(2)}(t_2) \tag{16}$$

unless $(t_1^2,t_2^2)=(1,1)$. In Figure 13 we show the values for two of the three terms of the right hand side of (16) (the pictures for $\sigma_{K_1\cup L_1}(t_2^2,t_1)$ and for $\sigma_{K_2\cup L_2}(t_1^2,t_2)$ are symmetric with respect to the diagonal $\theta_1=\theta_2$). By summing the three terms we obtain the values of σ_L shown in Figure 13. We do not write the values of σ_L on the line segments composing the set $\Delta_L=0$, since by Lemma 4.7 its value on each such segment is the half-sum of the values on the two sides of the segments.

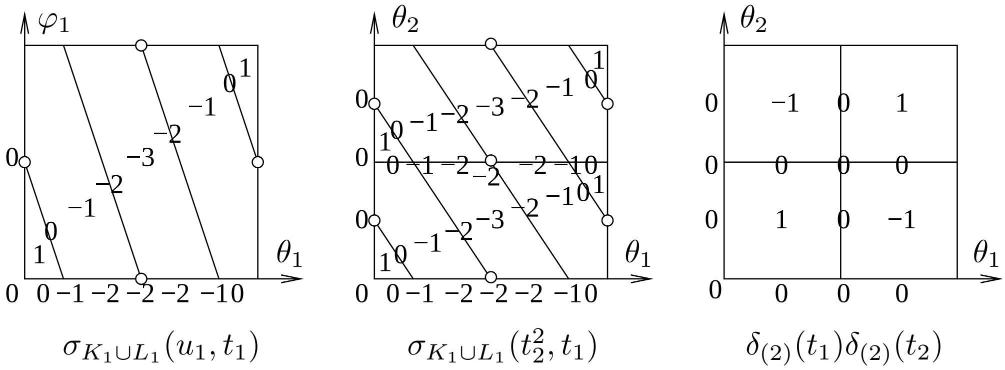


FIGURE 12. The values of summands in the right hand side of (16)

In fact Theorem 5.1 states that σ_L is as shown in Figure 13 only for $(t_1,t_2)\neq(\pm1,\pm1)$. However, a correction term $\Delta\sigma$ can be computed using [7, Thm. 5.3] and it turns out to be zero. In the following computation of $\Delta\sigma$ we use the terminology and notation from [7]. We perform the computation at $(t_1,t_2)=(-1,-1)$ only. The Conway potential function of $K_j\cup L_j$ and that of the trefoil knot K_j are

$$\nabla_{K_j\cup L_j}(u_j,t_j)=t_j^3u_j+t_j^{-3}u_j^{-1},\qquad \nabla_{K_j}(t_j)=(t_j^2-1+t_j^{-2})/(t-t^{-1})$$

(see [2]). Hence the slopes are $\kappa_j=(L_j/K_j)(-1)=-(\nabla_{K_j\cup L_j})'_{u_j}(1,i)/2\nabla_{K_j}(i)=2/3$, $j=1,2$ (see [7, Thm. 3.21]). Hence $\Delta\sigma=0$ (see [7, Thm. 5.3 and Rem. 5.4]).

Similarly to the signatures, Theorem 5.1 and [6, Thm. 5.3] also allow us to compute $\mathrm{n}_L(t_1,t_2)$ for any $(t_1,t_2)\in(\mathbb{S}^1)^2$. It is equal to the number of components of $\Delta_L=0$ passing through the point (t_1,t_2) unless (t_1,t_2) is one of $(-1,-1)$, $(1,i^k)$, or $(i^k,1)$, $k=1,2,3$ (the small white circles in Figure 13) where it is less by 1.

By Proposition 4.4, we deduce that the Tristram-Levine signatures of L are as in Figure 14 and the non-zero nullities are: $\mathrm{n}_{-1}(L)=1$ and $\mathrm{n}_\lambda(L)=2$ for $\lambda=e^{2\pi ik/10}$ with $k\in\{1,3,7,9\}$. This fact well agrees with the Seifert form of L computed in [15, §7] (see, however, Remarks 6.2 and 8.3).

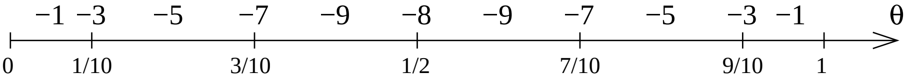


FIGURE 14. The values of $\sigma_\lambda(L)$, $\lambda=e^{2\pi i\theta}$

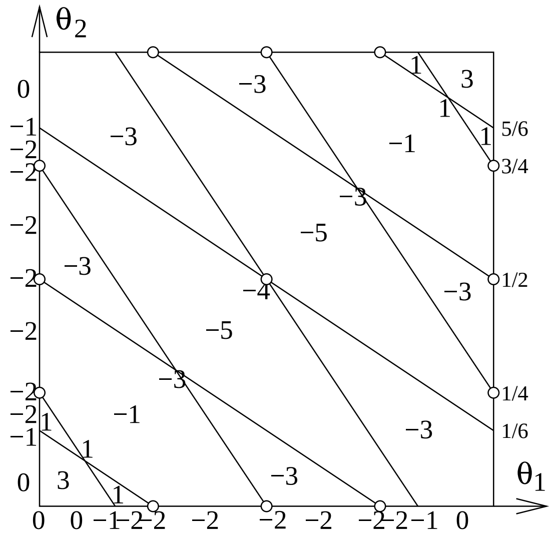


FIGURE 13. The values of $\sigma_L(t_1,t_2)$, $t_j = e^{2\pi i\theta_j}$

Question 11.1. Is it true that the slopes of all iterated torus links (all graph links in graph 3-spheres?) are defined and can be computed by [7, Thm. 3.21]?

11.2. The link at infinity of two half-cubic parabolas.

Let L^∞ be the link which is cut out by the same curve (15) but on a sufficiently large sphere in $\mathbb{C}^2$ centered at the origin. The splice diagram is obtained from Figure 10 by exchanging the labels "2" and "3". The Alexander polynomial is

$$(t_1^6t_2^4 + t_1^3t_2^2 + 1)(t_1^4t_2^6 + t_1^2t_2^3 + 1).$$

Performing the same computations as in the previous example, we obtain the values of σ_{L^∞} and n_{L^∞} as in Figure 15 (as in §11.1, the small white circles mark the points where n_{L^∞} is one less than the number of components of Δ_{L^∞}). The Tristram–Levine signatures of L^∞ are shown in Figure 16. The nullity is 1 (resp. 2) at primitive 3rd (resp. 15th) roots of unity.

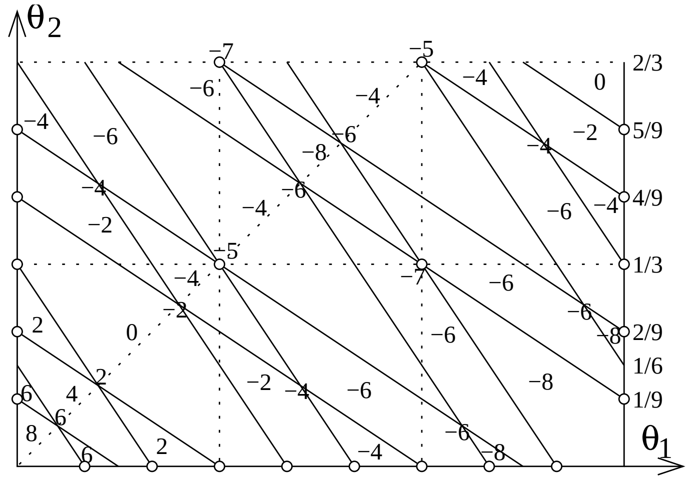


FIGURE 15. The values of $\sigma_{L^\infty}(t_1,t_2)$, $t_j = e^{2\pi i\theta_j}$

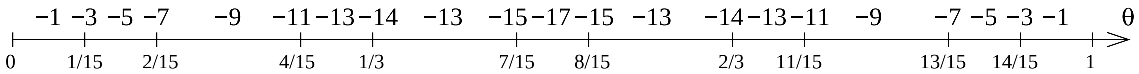


Figure 16. The values of $\sigma_\lambda(L^\infty)$, $\lambda = e^{2\pi i\theta}$

Question 11.2. One observes that the decomposition of the Hermitian isometric structure of L^∞ is given by the same formulas as in [14], [15] with the only exception that the factors Λ^2_λ appear with the negative sign. Is this true for all links at infinity of affine algebraic curves in $\mathbb{C}^2$ or for some natural subclass of such links?

References

1. E. Brieskorn, *Beispiele zur Diferentialtopologie von Singularitäten*, Invent. Math. **2** (1966), 1–14.
2. D. Cimasoni, *The Conway potential function of a graph link*, Math. Proc. Cambridge Phil. Soc. **136** (2004), 557–563.
3. D. Cimasoni, V. Florens, *Generalized Seifert surfaces and signatures of colored links*, Trans. Amer. Math. Soc. **360** (2008), no. 3, 1223–1264.
4. A. Conway, *The Levine-Tristram signature: a survey*, arXiv:1903.04477.
5. A. Conway, M. Nagel, E. Toffoli, *Multivariable signatures, genus bounds and 1-solvable cobordisms*, arXiv:1703.07540.
6. A. Degtyarev, V. Florens, A. Lecuona, *The signature of a splice*, Int. Math. Res. Notices (IMRN) 2017, No. 8, 2249–2283.
7. A. Degtyarev, V. Florens, A. Lecuona, *Slopes and signatures of links*, arXiv:1802.01836.
8. A. Degtyarev, V. Florens, A. Lecuona, *Slopes of links and signature formulas*, arXiv:2002.02790.
9. D. Eisenbud, W. D. Neumann, *Three-dimensional link theory and invariants of plane curve singularities*, Ann. of Math. Studies 110, Princeton University Press, 1985.
10. V. Florens, *Signatures of colored links with application to real algebraic curves*, J. Knot Theory Ramifications **14** (2005), no. 7, 883–918.
11. C. M. Gordon, *Some aspects of classical knot theory*, in: Knot Theory Proceedings. Plans-sur-Bex, Switzerland, 1977, Lect. Notes in Math. vol. 685, Springer, Berlin, 1978, pp. 1–60.
12. R. A. Litherland, *Signatures of iterated torus knots*, in: Topology of low-dimensional manifolds (Proc. Second Sussex Conf., Chelwood Gate, 1977), Lecture Notes in Math., vol. 722, Springer, Berlin, 1979, pp. 71–84.
13. T. Matumoto, *On the signature invariants of a non-singular complex sesqui-linear form*, J. Math. Soc. Japan **29** (1977), no. 1, 67–71.
14. W. D. Neumann, *Invariants of plane curve singularities*, in: Noeuds, tresses et singularités, C. R. Sémin., Plans-sur-Bex 1982, Monogr. Enseign. Math. 31, 1983, pp. 223–232.
15. W. D. Neumann, *Splicing algebraic links*, in: Complex analytic singularities, Adv. Studies in Pure Math., 8, 1987, pp. 349–361.
16. S. Yu. Orevkov, *Plane real algebraic curves of odd degree with a deep nest*, J. Knot Theory Ramifications **14** (2005), no. 4, 497–522.
17. O. Viro, *Twisted acyclicity of a circle and signatures of a link*, J. Knot Theory Ramifications **18** (2009), no. 6, 729–755.

Steklov Mathematical Institute, Gubkina 8, Moscow, Russia

IMT, l'université Paul Sabatier, 118 route de Narbonne, Toulouse, France

AGHA Laboratory, Moscow Institute of Physics and Technology, Russia