

SEPARATING SEMIGROUP OF PLANE QUINTICS

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ABSTRACT. A real morphism f from a real algebraic curve X to $\mathbb{P}^1$ is called *separating* if $f^{-1}(\mathbb{RP}^1) = \mathbb{R}X$. Such a morphism defines a covering $\mathbb{R}X \rightarrow \mathbb{RP}^1$. Denote the components of $\mathbb{R}X$ by $X_1, \dots, X_r$ and the degree of the restriction of f to X_i by $d_i(f)$. Then, each separating morphism can be associated with the vector $d(f) = (d_1, \dots, d_r) \in \mathbb{N}^r$. Kummer and Shaw [KS20] defined the separating semigroup of X as the set of all vectors $d(f)$ where f is a separating morphism $X \rightarrow \mathbb{P}^1$.

In the present paper, we describe the separating semigroup of plane quintic curves.

1. INTRODUCTION

By a *real curve* we mean a complex algebraic curve X equipped with an anti-holomorphic involution $\text{conj}: X \rightarrow X$. Its real points set is $\mathbb{R}X := \{p \in X \mid \text{conj}(p) = p\}$. All curves considered here are smooth and irreducible, unless otherwise specified.

A real curve X is called *separating* or a curve of *type I* if $X \setminus \mathbb{R}X$ is disconnected. In this case $X \setminus \mathbb{R}X$ consist of two connected components which are interchanged by an anti-holomorphic involution.

Ahlfors [Ahl50] showed that a real curve is separating if and only if there exists a *separating morphism* $f: X \rightarrow \mathbb{P}^1$, that is, a morphism such that $f^{-1}(\mathbb{RP}^1) = \mathbb{R}X$.

Every separating morphism defines a covering map $f|_{\mathbb{R}X}: \mathbb{R}X \rightarrow \mathbb{RP}^1$. Let $X_1, \dots, X_r$ be the components of $\mathbb{R}X$. Denoting by $d_i(f)$ the covering degree of the restriction of f to X_i , we may associate every separating morphism with the vector $d(f) = (d_1(f), \dots, d_r(f)) \in \mathbb{N}^r$. Kummer and Shaw [KS20] defined the *separating semigroup* of X as

$$\text{Sep}(X) = \{d(f) \mid f: X \rightarrow \mathbb{P}^1 \text{ is a separating morphism}\}.$$

It is easy to prove that $\text{Sep}(X)$ is indeed a semigroup, see [KS20, Prop. 2.1]. Denote

$$\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 1\}, \quad \mathbb{N}_0 = \{n \in \mathbb{Z} \mid n \geq 0\}.$$

In [KS20] Kummer and Shaw computed the separating semigroups of M -curves of arbitrary genus and all real curves of genus $g \leq 2$. Namely, if X is an M -curve of genus g (i.e. $b_0(\mathbb{R}X) = g+1$), then $\text{Sep}(X) = \mathbb{N}^{g+1}$. If X is a separating curve of genus 1, then $\text{Sep}(X) = \mathbb{N}^2$, and if X is a separating curve of genus 2, then $\text{Sep}(X) = \mathbb{N}^3$ or $2 + \mathbb{N}_0$. Subsequently, in [Ore19, Ore24] separating semigroups of all hyperelliptic curves and all curves of genus ≤ 4 were computed. In [Man24] Manzarolli presented some results on separating semigroups of $(M-2)$ -curves (i.e. curves of genus g with $b_0(\mathbb{R}X) = g-1$).

In the present paper we describe the separating semigroups of plane curves of degree 5 (which have genus 6). It is well-known (see e.g. [Rok78, p. 95]) that only three isotopy types of plane quintics can be realized by separating curves: the M -quintic, the $(M-2)$ -quintic, and the hyperbolic quintic (a plane real curve is called *hyperbolic* if the linear projection from some point on it is a separating morphism). If X is an M -quintic, then, as we mentioned above, $\text{Sep}(X) = \mathbb{N}^7$, so we are interested in the latter two cases.

The hyperbolic quintic and the $(M - 2)$ -quintic are drawn in Fig. 1 (see Lemma 4.1). The doubled arrows in Fig. 1 represent the *complex orientation*, i.e. the boundary orientation induced to $\mathbb{R}X$ from one of the halves of $X \setminus \mathbb{R}X$.

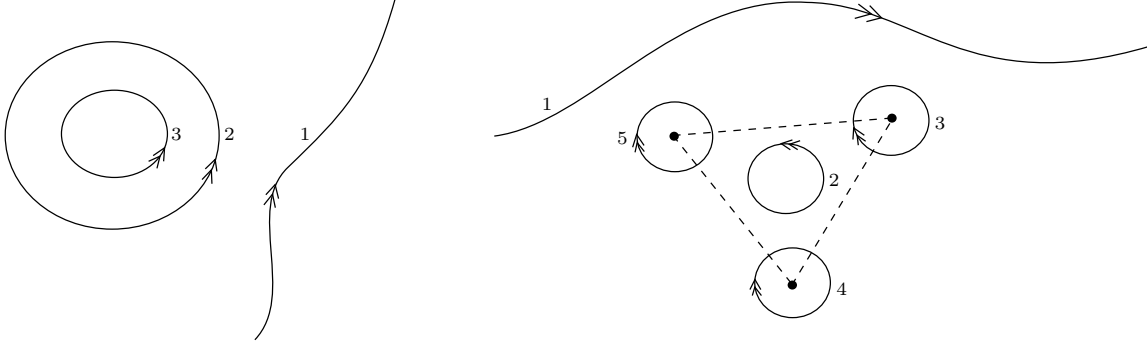


Figure 1. Isotopy types of separating non- M -quintics.

Let us formulate the main result of the present paper.

Theorem 1.1. Let X be a plane separating quintic. Then, if $b_0(\mathbb{R}X) = 3$, we have

$$\text{Sep}(X) = \{(1, 2, 1), (1, 2, 2)\} \cup ((2, 3, 1) + \mathbb{N}_0^3) \cup ((1, 3, 2) + \mathbb{N}_0^3).$$

and if $b_0(\mathbb{R}X) = 5$, we have

$$\text{Sep}(X) = (2, 1, 1, 1, 1) + \mathbb{N}_0^5.$$

Our proofs are mainly based on the techniques from [Ore21, Ore24] and some recent results from [MO25].

The present paper is organized as follows. In §2.1 we recall the necessary preliminary material from [Ore21]. Next, in §2.2 we present the general method for constructing separating morphisms. This method first appeared in [Ore24] and is based on the combination of the Abel-Jacobi theorem with the adjunction formula in terms of Poincaré residues. Sections 3 and 4 are devoted to the computation of separating semigroups of hyperbolic quintics and $(M - 2)$ -quintics, respectively.

2. PRELIMINARIES

2.1. Chess-board orientations. For the purposes of the present paper here we consider only plane curves. However, one can see that the results from this subsection and from §2.2 hold in the general setting, see [Ore21].

In addition to the complex orientation on the set of real points of a separating real curve, we will need the *D-orientation* associated with a divisor $D \in |X + K_{\mathbb{P}^2}|$.

Let X be a real curve in $\mathbb{P}^2$ and D be a real divisor from the adjoint linear system $|X + K_{\mathbb{P}^2}|$. Assume that D does not have X as a component and write $D = D_1 + 2D_0$, where D_0 and D_1 are effective and D_1 is a reduced curve.

As $D - X \sim K_{\mathbb{P}^2}$, it is a divisor of some meromorphic 2-form ω on $\mathbb{CP}^2$. Since D and X are real, ω can also be chosen real. Then ω defines a «chess-board» orientation on $\mathbb{RP}^2 \setminus (\mathbb{R}X \cup \mathbb{R}D_1)$: this orientation changes each time we cross $\mathbb{R}X \cup \mathbb{R}D_1$ at its smooth point (see Fig. 2). This orientation induces a boundary orientation on $\mathbb{R}X \setminus \mathbb{R}D$. The orientation on $\mathbb{R}X \setminus \mathbb{R}D$ described above is called *D-orientation*.

The D -orientation, similarly to the complex orientations, is defined up to simultaneous reversal on each connected component of $\mathbb{R}X$. In fact, we can choose the orientation which corresponds to ω or the orientation which corresponds to $-\omega$. Hereafter, by choosing a D -orientation, we will mean precisely the choice of one of these two orientations.

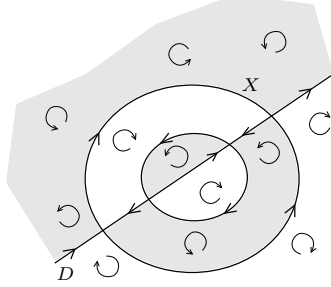


Figure 2. D -orientation.

Remark 2.1. Let ω_D be the Poincaré residue of the form ω . Then the D -orientation coincides with the orientation defined by ω_D in the following sense: for $v \in T(\mathbb{R}X)$ we have $\omega_D(v) > 0$ if the D -orientation is positive on v . Also, for $v \in T_p(\mathbb{R}X)$ we have $\omega_D(v) = 0 \iff p \in \mathbb{R}X \cap \mathbb{R}D$.

Often it is more convenient to speak not of separating morphisms themselves, but of divisors corresponding to their fibers. Namely, let us call a totally real divisor $P = p_1 + \dots + p_n$ on a real curve X *separating* if there exists a separating morphism $f: X \rightarrow \mathbb{P}^1$ and a point $p_0 \in \mathbb{R}\mathbb{P}^1$ such that $P = f^{-1}(p_0)$. In this case, we say that P has *degree partition* $d(P) := d(f) \in \text{Sep}(X)$.

The following theorem expresses the connections between complex orientations, D -orientations and separating morphisms. The notation in the theorem is as given above.

Theorem 2.2 (see [Ore21, Theorem 3.2]). Let P be a separating divisor on a real curve X . If $P \not\subset D$, then the D -orientation cannot coincide with the complex orientation at all points of $P \setminus D$.

2.2. Construction of separating morphisms. In this section we discuss a general method for constructing separating morphisms. This method was used in the calculation of separating semigroups of hyperelliptic curves and all curves of genus ≤ 4 , see [Ore24]. Since this method works for curves of arbitrary genus, we will provide its general formulation and, for the convenience of the reader, give a proof here (though it follows almost verbatim the arguments given in [Ore24, latter part of the proof of Lemma 2.3]). To simplify further formulations we introduce the following notation.

Let X be a separating real curve. Fix the complex orientation and some D -orientation on $\mathbb{R}X$. For $p \in \mathbb{R}X \setminus \mathbb{R}D$ let us write that

$$\text{sign}_D(p) := \begin{cases} +1, & \text{if the complex orientation coincides with the } D\text{-orientation at the point } p, \\ -1, & \text{otherwise.} \end{cases}$$

Setup 2.3. Let X be a real plane separating curve. Fix the complex orientation on $\mathbb{R}X$. Consider a divisor $P = p_1 + p_2 + \dots + p_n$, where $p_i \in \mathbb{R}X$ are pairwise distinct points. Suppose that

- $h^0(P) \geq 2$;

- the complete linear system $|P|$ is base-point-free.

Under these conditions there exists a smooth deformation $P_t = p_1(t) + \dots + p_n(t) \in |P|$, where $p_j: [0, t_0] \rightarrow X$ are smooth paths such that $p_j(0) = p_j$ and $v_j := p'_j(0) \neq 0$, see [Ore24, Lemma 2.2].

Lemma 2.4. Assume Setup 2.3. Suppose that there exists a real divisor $D \in |X + K_{\mathbb{P}^2}|$ such that two following conditions hold:

- $P \setminus D = \{p_i, p_j\}$;
- $\text{sign}_D(p_i) = -\text{sign}_D(p_j)$.

Then the complex orientation is of the same sign on the vectors v_i and v_j .

Proof. Let ω be a meromorphic 2-form with the divisor $D - X$ and ω_D be its Poincaré residue (see §2.1). Since for all $t \in [0, t_0]$ $P_t \in |P|$, by the Abel-Jacobi theorem we have

$$\forall t \in [0, t_0] \quad \sum_{j=1}^n \int_{p_j([0, t])} \omega_D = 0.$$

Differentiating this identity at $t = 0$ we get

$$(*) \quad \omega_D(v_1) + \omega_D(v_2) + \dots + \omega_D(v_n) = 0.$$

Since $\omega_D(v_k) = 0 \iff p_k \in \mathbb{R}X \cap \mathbb{R}D$ and $P \setminus D = \{p_i, p_j\}$, the equality $(*)$ simplifies to

$$\omega_D(v_i) = -\omega_D(v_j).$$

By the definition, $\text{sign}_D(p_i) = -\text{sign}_D(p_j)$ means that the D -orientation coincides with the complex orientation at one of the points p_i and p_j and they are opposite at the other. Hence the complex orientation has the same sign on v_i and v_j . $\square$

The next Lemma follows immediately from [KS20, Prop. 2.11].

Lemma 2.5. Assume Setup 2.3. Let $p_{i_1}, \dots, p_{i_k}$ denote the points of P lying on the components of $\mathbb{R}X$, which contain more than one point of P . Suppose that the complex orientation has the same sign on v_{i_j} , $j = 1, \dots, k$. Then P is separating.

Also we will use the following recent result:

Theorem 2.6 ([MO25]). Let X be a real curve of genus g with $b_0(\mathbb{R}X) = r$ and $P = p_1 + \dots + p_g$ be a separating divisor on X with $h^1(P) = 1$. Then $d(P) + \mathbb{N}_0^r \subset \text{Sep}(X)$.

3. SEPARATING SEMIGROUP OF HYPERBOLIC QUINTICS

Let X be a hyperbolic quintic, fix the numeration of the components of $\mathbb{R}X$ as in Fig. 1: let X_1 be the pseudoline, X_2 the outer oval and X_3 the inner oval.¹

It is well-known (see e.g. [KS20, Example 2.8]) that the linear projection from a point on the inner oval of X defines a separating morphism with degree partition $(1, 2, 1)$. The linear projection centered at a point inside the inner oval defines a separating morphism with degree partition $(1, 2, 2)$.

Lemma 3.1. The following hold:

- (1) For all $n, m \in \mathbb{N}$ we have $(n, 1, m) \notin \text{Sep}(X)$.
- (2) For $n, m \in \mathbb{N}$ with $n + m \geq 4$ we have $(n, 2, m) \notin \text{Sep}(X)$. Moreover, $(2, 2, 1) \notin \text{Sep}(X)$.
- (3) For all $n \geq 3$ we have $(1, n, 1) \notin \text{Sep}(X)$.

¹Note that in [KS20] and [Ore19] the order of components of hyperbolic curves is reversed.

Proof. (1) Let P be a separating divisor with degree partition $(n, 1, m)$, let p be its point on the outer oval. Consider a line ℓ passing through p and such that $P \not\subset \ell$. Set $D = 2\ell \in |X + K_{\mathbb{P}^2}|$. Fix a complex orientation on $\mathbb{R}X$ and choose a D -orientation so that it coincides with the complex orientation on the pseudoline. Then $P \setminus D \neq \emptyset$ and the two orientations coincide at all points of this set, which contradicts Theorem 2.2.

(2) Let P be a separating divisor with degree partition $(n, 2, m)$. Consider a doubled line $D = 2\ell \in |X + K_{\mathbb{P}^2}|$ passing through two points of P on the outer oval and choose a D -orientation so that it coincides with the complex orientation on the pseudoline. As P is separating, by Theorem 2.2 we have $P \subset \ell$. Hence if $n + m \geq 4$ we obtain a contradiction with Bézout's theorem.

Let us prove that $(2, 2, 1) \notin \text{Sep}(X)$. As $\deg_{X_1} P = 2$, the line ℓ intersects the pseudoline X_1 in at least two points. Hence $\ell \cdot X_1 \geq 3$, as the intersection number of ℓ and X_1 is odd, and we obtain a contradiction with Bézout's theorem.

(3) Let P be a separating divisor with degree partition $(1, n, 1)$. Once again, consider a doubled line $D = 2\ell \in |X + K_{\mathbb{P}^2}|$ passing through two points of P — on the inner oval and on the pseudoline. Choose the D -orientation so that it is opposite to the complex orientation on the pseudoline. Since P is separating, by Theorem 2.2 we have $P \subset \ell$ and hence, if $n \geq 4$, we obtain a contradiction with Bézout's theorem.

Let us prove that $(1, 3, 1) \notin \text{Sep}(X)$. As $\deg_{X_2} P = 3$, we have $\ell \cdot X_2 \geq 3$ and, since their intersection number is even, we have $\ell \cdot X_2 \geq 4$. Hence the line ℓ intersects X in at least $1 + 4 + 1 = 6$ points, which leads us to a contradiction. $\square$

In the following Lemma we construct separating divisors realizing all degree partitions except those prohibited by Lemma 3.1. It finishes the calculation of $\text{Sep}(X)$ for the hyperbolic quintics.

Lemma 3.2. $((2, 3, 1) + \mathbb{N}_0^3) \cup ((1, 3, 2) + \mathbb{N}_0^3) \subset \text{Sep}(X)$.

Proof. Consider a divisor $P = p_1 + \dots + p_n$ lying on the degenerate conic C_1 (shown by a solid line in Fig. 3) intersected by another degenerate conic C_2 (shown by a dashed line in Fig. 4). Clearly, P has degree partition $(1, 3, 2)$.

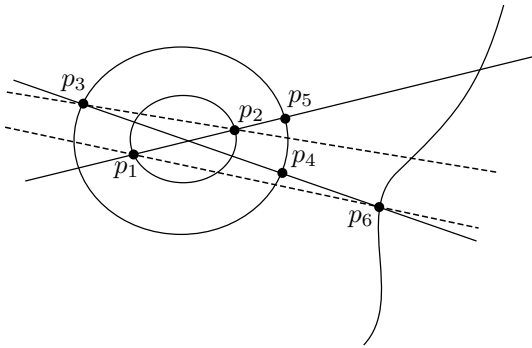


Figure 3. Divisor P with degree partition $(1, 3, 2)$ and auxiliary conics C_1 and C_2 .

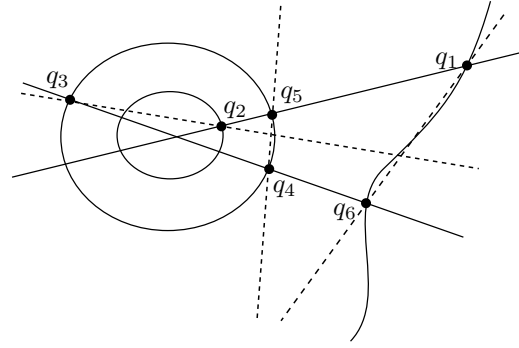


Figure 4. Divisor Q with degree partition $(2, 3, 1)$ and auxiliary conics.

It is easy to see that the conic passing through the points of P is unique, so we have $\dim |K_X - P| = 0$. Hence P is special with $h^0(K_X - P) = 1$. By Riemann-Roch theorem we have $h^0(P) = 2$.

Let us show that the complete linear system $|P|$ is basepoint-free, i.e. that $\dim |P - p_j| = 0$ for all $j = 1, \dots, 6$. Indeed, by Riemann-Roch theorem

$$\dim |P - p_j| = \dim |K_X - (P - p_j)| + 5 - 6 + 1 = \dim |K_X - (P - p_j)| = 0,$$

as, the conic, passing through the five points of the divisor $P - p_j$ is unique.

Thus, the conditions of the setup 2.3 are satisfied and can apply Lemma 2.4. In the notation of 2.3, we prove that a properly chosen complex orientation is positive on the vectors v_1, v_2, v_3, v_4, v_5 .

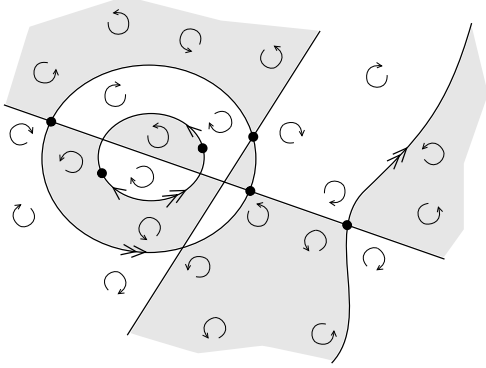


Figure 5. Orientations at points p_1 and p_2 .

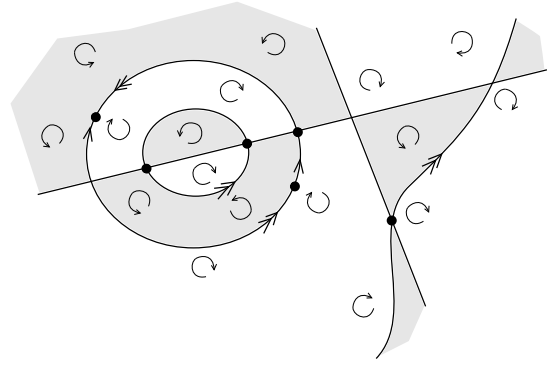


Figure 6. Orientations at points p_3 and p_4 .

Denote the line passing through p_1, p_2 and p_5 by ℓ_1 and the line passing through p_3, p_4 and p_6 by ℓ_2 . Let ℓ be a real line passing through the point p_5 as in Fig. 5. Consider a conic $D = \ell \cup \ell_2$ and the corresponding D -orientation. Then $P \setminus D = \{p_1, p_2\}$ and $\text{sign}_D(p_1) = -\text{sign}_D(p_2)$ (see Fig. 5). Hence by Lemma 2.4 the complex orientation has the same sign on v_1 and v_2 .

Similar arguments show that the complex orientation has the same sign on the vectors v_3 and v_4 , see Fig. 6. Finally, from the construction of the divisor P we have $\text{sign}_{C_2}(p_4) = -\text{sign}_{C_2}(p_5)$, so by Lemma 2.4 the complex orientation has the same sign on v_4 and v_5 . Therefore, by Lemma 2.5 P is separating and $(1, 3, 2) \in \text{Sep}(X)$.

To prove that $(2, 3, 1) \in \text{Sep}(X)$, it suffices to consider the divisor $Q = q_1 + \dots + q_6$ taken as in Fig. 4 and apply similar arguments. The conics required for this can be easily seen in Fig. 4.

Note now that $\deg P = \deg Q = 6$ (which equals to the genus of a plane quintic), and also that we have already shown above that $h^1(P) = h^1(Q) = 1$. Therefore, by Theorem 2.6, the inclusion

$$((2, 3, 1) + \mathbb{N}_0^3) \cup ((1, 3, 2) + \mathbb{N}_0^3) \subset \text{Sep}(X)$$

holds. □

4. SEPARATING SEMIGROUP OF (M-2)-QUINTICS

The following lemma is a classical fact in real algebraic geometry.

Lemma 4.1. A plane $(M-2)$ quintic X is separating if and only if its ovals are in non-convex position, see Fig. 1 (right).

To prove Lemma 4.1, it suffices to apply the arguments from [Vir84, §3.2 and §3.3]. We will specifically need the following detail from this proof (cf. [Vir84, §3.3]):

Proposition 4.2. Let X be an $(M - 2)$ -quintic whose ovals are in non-convex position (see Fig. 1). Take one point on each oval. Then the pencil of conics passing through these four points defines a separating morphism $f: X \rightarrow \mathbb{P}^1$.

In her recent work [Man24], Manzarolli generalized this property (of non-convex arrangement of ovals) to the case of plane $(M - 2)$ -curves of arbitrary degree.

Let X be a separating $(M - 2)$ -quintic. Fix the numeration of the components of $\mathbb{R}X$ as in Fig. 1: let X_1 be a pseudoline, X_2 the oval located inside the triangle with vertices inside the other ovals, and X_3, X_4, X_5 the remaining ovals.

Separating morphism f from the proposition 4.2 has degree partition $(2, 1, 1, 1, 1)$. In [Man24, Example 2.6] Manzarolli, using Theorem 2.2, proves that a separating morphism of degree 6 from an $(M - 2)$ -quintic can only have degree partition $(2, 1, 1, 1, 1)$ and possibly $(1, 2, 1, 1, 1)$. In the following Lemma we prove that a separating morphism from an $(M - 2)$ -quintic cannot have degree 1 on the pseudoline.

Lemma 4.3. $(1, a, b, c, d) \notin \text{Sep}(X)$ for all $a, b, c, d \in \mathbb{N}$.

Proof. It is easy to check that there exist four bitangents such that their union with quintic is as in Fig. 7 (up to isotopy).

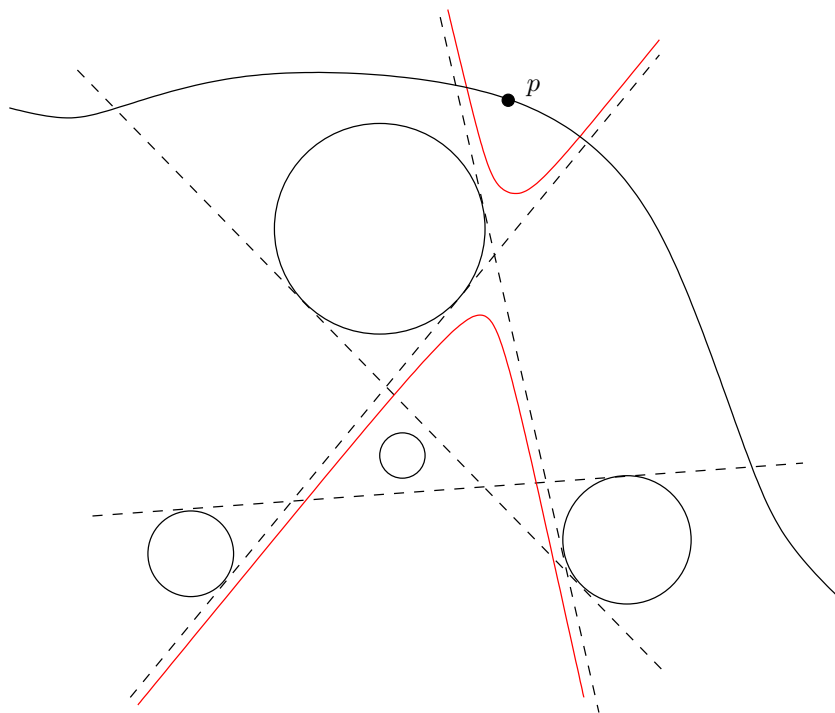


Figure 7. Proof of Lemma 4.3.

Suppose that f is a separating morphism with $d_1(f) = 1$. Let D be a small perturbation of two bitangents shown in Fig. 7. Then on $f^{-1}(f(p))$ the complex orientation coincides with the D -orientation, which contradicts Theorem 2.2. $\square$

The following Lemma finishes the calculation of the separating semigroup of $(M - 2)$ -quintics.

Lemma 4.4. Let X be a separating $(M - 2)$ -quintic. Then

$$(2, 1, 1, 1, 1) + \mathbb{N}_0^5 \subset \text{Sep}(X).$$

Proof. Let $f: X \rightarrow \mathbb{P}^1$ be a separating morphism from Proposition 4.2 and $P = f^{-1}(p_0)$ be the corresponding separating divisor. Then, since the points of the divisor P lie on a conic and, moreover, only one conic can pass through the points of P , we have $h^1(P) = h^0(K_X - P) = 1$. Therefore, the claim follows from Theorem 2.6. $\square$

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