SEPARATING SEMIGROUP OF GENUS 4 CURVES

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Abstract. A rational function on a real algebraic curve C is called separating if it takes real values only at real points. Such a function defines a covering $\mathbb{R}C \to \mathbb{RP}^1$. Let $c_1, \dots, c_r$ be connected components of $\mathbb{R}C$. M. Kummer and K. Shaw defined the separating semigroup of C as the set of all sequences $(d_1(f), \dots, d_r(f))$ where f is a separating function and $d_i(f)$ is the degree of the restriction of f to c_i.

In the present paper we describe the separating semigroups of all genus 4 curves. For the proofs we consider the canonical embedding of C into a quadric X in $\mathbb{P}^3$ and apply Abel's theorem to 1-forms on C obtained as Poincaré residues of certain meromorphic 2-forms.

1. Introduction

By a *real algebraic curve* we mean a complex algebraic curve C endowed with an antiholomorphic involution $\mathrm{conj} : C \to C$ (the complex conjugation involution). In this case we denote the *real locus* $\{p \in C \mid \mathrm{conj}(p) = p\}$ by $\mathbb{R}C$. A real curve is called *dividing* or *separating* if $\mathbb{R}C$ divides C into two halves exchanged by the complex conjugation. All curves considered here are smooth and irreducible.

A necessary and sufficient condition for C to be separating is the existence of a *separating morphism* $f : C \to \mathbb{P}^1$, that is a morphism such that $f^{-1}(\mathbb{RP}^1) = \mathbb{R}C$. The restriction of a separating morphism to $\mathbb{R}C$ is a covering over $\mathbb{RP}^1$. If we fix a numbering of the connected components $c_1, \dots, c_r$ of $\mathbb{R}C$, we may consider the sequence $d(f) = (d_1, \dots, d_r)$ where d_i is the covering degree of f restricted to c_i. Kummer and Shaw [2] defined the *separating semigroup* of C as

$$\mathrm{Sep}(C) = \{d(f) \mid f : C \to \mathbb{P}^1 \text{ is a separating morphism}\}.$$

It is easy to check that this is indeed a semigroup (see [2, Prop. 2.1]). We denote:

$$\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 1\}, \qquad \mathbb{N}_0 = \{n \in \mathbb{Z} \mid n \geq 0\}.$$

It is shown in [2] that $\mathrm{Sep}(C) = \mathbb{N}^{g+1}$ if C is an *M-curve* of genus g (i.e. $\mathbb{R}C$ has $g+1$ connected components) and $\mathrm{Sep}(C)$ is $\mathbb{N}^2$ (resp. $\mathbb{N}^3$ or $2 + \mathbb{N}_0$) if C is a separating curve of genus 1 (resp. 2). The separating semigroups of hyperelliptic curves of any genus and of curves of genus 3 are computed in [3]. A much simpler proof for curves of genus 3 is given in [4, Remark 3.3], and in §4 we also give a proof for hyperelliptic curve, which is essentially the same as in [3] but exposed from the point of view proposed in [4].

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In the present paper we compute the separating semigroup of all genus 4 curves (this result was announced in [4, Remark 3.6]). Let C be a separating curve of genus 4. If C is an M-curve, then $\mathrm{Sep}(C) = \mathbb{N}^5$ (see above). If C is hyperelliptic but not an M-curve, then $\mathrm{Sep}(C) = \{2\} \cup (4 + \mathbb{N}_0)$ (see [3] and §4).

Assume now that C is not hyperelliptic. It is well-known that the image of C under the canonical embedding is a degree 6 curve on an irreducible quadric surface X in $\mathbb{P}^3$. Since C is real and separating, the real structure on $\mathbb{P}^3$ can be chosen so that C is a real curve on an irreducible real surface X such that $\dim \mathbb{R}X = 2$, thus $\mathbb{R}X$ is an ellipsoid, a hyperboloid, or a quadratic cone.

When X is an ellipsoid or a hyperboloid, all *rigid isotopy classes* of smooth real sextic curves C of genus 4 on X (i.e. the connected components of the space of such curves) are described in [1]. The same arguments can be easily adapted to the case when X is a quadratic cone (see also the footnote in [5, p. 14]). Representatives of all the rigid isotopy classes of separating non-maximal curves up to automorphisms of X are depicted in Figure 1. Four of them are realizable as a small perturbation of three plane sections. The other two are perturbations of the union of a plane section and a section by a thin cylinder around a line. The arrows in Figure 1 represent *complex orientations*, i.e., the boundary orientations induced from one of the halves of $C \setminus \mathbb{R}C$.

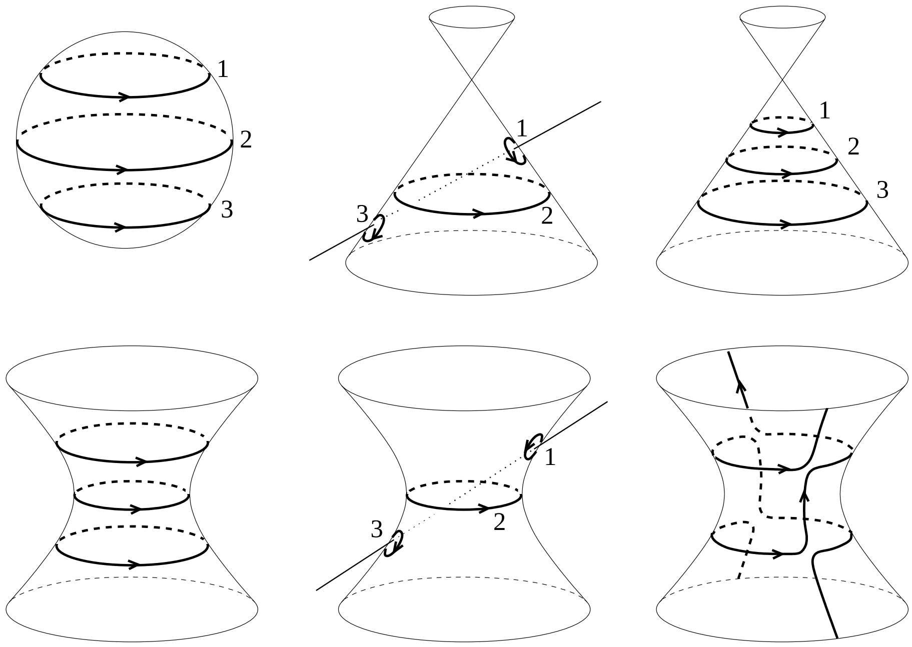

FIGURE 1. Rigid isotopy types of separating non-M-sextics on quadrics.

We see in Figure 1 that the rigid isotopy type (up to $\mathrm{Aut}(X)$) of a separating curve is determined by the topology of the pair $(\mathbb{R}X, \mathbb{R}C)$. Moreover, it is determined by the number of connected components of $\mathbb{R}C$ and the number of those of them which bound a smooth disk in $\mathbb{R}X$ (such components are called *ovals* of C). Though it is not evident a priori, it turns out that $\mathrm{Sep}(C)$ also depends on X and on these two numbers only. The main result of the paper is the following.

Theorem 1.1. *Let C be a separating curve of genus 4 on a real quadric X. We number its components according to Figure 1 (arbitrarily if C is an M-curve). Then* $\mathrm{Sep}(C)$ *is as in Table 1.*

X	$b_0(\mathbb{R}C)$	Number of ovals	$\mathrm{Sep}(C)$
Ellipsoid	3	3	$(1,2,1)+\mathbb{N}_0^3$
	5	5	$\mathbb{N}^5$
Quadratic cone	3	0	$\{(1,1,1)\}\cup\big((1,2,1)+\mathbb{N}_0^3\big)$
	3	2	$(1,2,1)+\mathbb{N}_0^3$
	5	4	$\mathbb{N}^5$
Hyperboloid	1	0	$3+\mathbb{N}_0$
	3	0	$\mathbb{N}^3$
	3	2	$(1,2,1)+\mathbb{N}_0^3$
	5	4	$\mathbb{N}^5$

Table 1. Separating semigroups of genus 4 curves on quadrics.

Sections 2 and 3 are devoted to the proof of Theorem 1.1. It is based on the techniques proposed in [4, §3]. In Section 4 we give a proof of [3, Theorem 2] similar to the proof of Theorem 1.1 of the present paper.

Till the end of Section 3, X and C are as in Theorem 1.1 and C is not an M-curve (Theorem 1.1 for M-curves is proven in [2, Thm. 1.7]). We denote the number of connected components of $\mathbb{R}C$ by r and the number of ovals of C by l. As in the definition of $\mathrm{Sep}(C)$, the connected components of $\mathbb{R}C$ are denoted by $c_1,\dots,c_r$ (numbered according to Figure 1).

2. Main lemmas

Let $D=D_0+D_1$ be a real plane section of X where D_0 is the component of D of an even multiplicity (in our case D_0 is non-empty only when X is a quadratic cone and $D=D_0$ is a double generator). The divisor $D-C$ belongs to the canonical class of X, thus it is the divisor of some meromorphic 2-form Ω_D on X. It defines a "chess-board" orientation on $\mathbb{R}X\setminus(\mathbb{R}C\cup\mathbb{R}D_1)$, i.e., an orientation which changes when crossing $\mathbb{R}C\cup\mathbb{R}D_1$. We define the *$D$-orientation* of $\mathbb{R}C\setminus\mathbb{R}D$ as the boundary orientation induced by the "chess-board" orientation of $\mathbb{R}X\setminus(\mathbb{R}C\cup\mathbb{R}D_1)$. Let ω_D be the Poincaré residue of Ω_D. Then the D-orientation coincides with the orientation defined by ω_D in the sense that $\omega_D(v)>0$ for $v\in T(\mathbb{R}C)$ if and only if the D-orientation is positive on v. Similarly to the complex orientations, the D-orientation is defined up to simultaneous reversing on all components of $\mathbb{R}C$.

Lemma 2.1. (See [4, Thm. 3.2].) *Let $f: C\to\mathbb{P}^1$ be a separating morphism and let $P=f^{-1}(x)$, $x\in\mathbb{RP}^1$. If $P\not\subset D$, then the D-orientation cannot coincide with the complex orientation at all points of $P\setminus D$.* □

Lemma 2.2. *Let $p_1,\dots,p_n$ and $q_1,\dots,q_n$ be pairwise distinct points on C. Suppose that the divisors $P=p_1+\dots+p_n$ and $Q=q_1+\dots+q_n$ are linearly equivalent. Then there exist smooth paths $p_j:[0,t_0]\to C$, $p_j(0)=p_j$, $t_0>0$, such that $P_t:=\sum p_j(t)\in|P|$ for each $t\in[0,t_0]$, and the derivative $p_j'(0)$ is nonzero for each $j=1,\dots,n$.*

Proof. Let f be a meromorphic function on C with simple zeros at P and simple poles at Q. Let P_t be the divisor of zeros of $f-t$. Then the result follows from the implicit function theorem. □

Lemma 2.3. *Let D be an irreducible real plane section of X. We fix a complex orientation on C (the arrows in Figure 2).*

If $r = 3$ and $\mathbb{R}D$ crosses the components of $\mathbb{R}C$ as shown in Figure 2 on the left or in the middle, then

$$\big((1,3,1)+\mathbb{N}_0^3\big)\cup\big((1,2,2)+\mathbb{N}_0^3\big)\subset \operatorname{Sep}(C). \tag{1}$$

If $r = 1$ and $\mathbb{R}D$ crosses the components of $\mathbb{R}C$ as shown in Figure 2 on the right, then $5+\mathbb{N}_0 \subset \operatorname{Sep}(C)$.

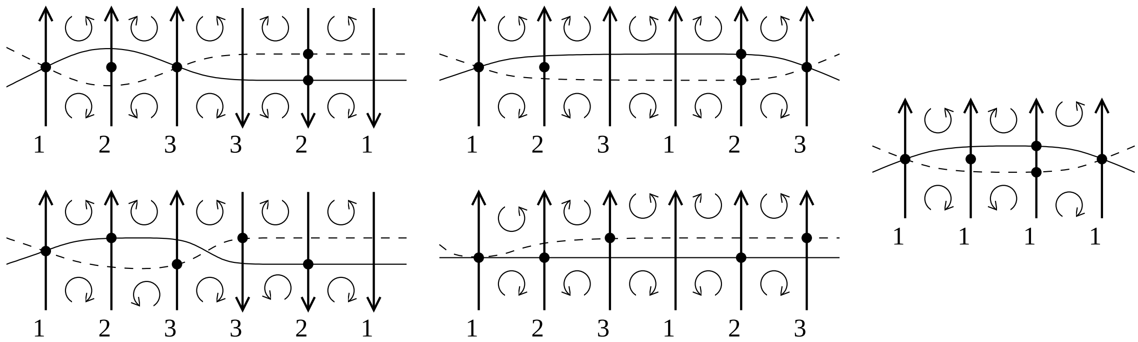

FIGURE 2. To Lemma 2.3. $\mathbb{R}D$ is shown by the horizontal solid line.

Proof. Let D' be a real plane section of X, close to D and intersecting $D\cup C$ as shown in Figure 2 by the dashed line. Consider the divisor $P = p_1+\dots+p_5$ on C, where the five points $p_1,\dots,p_5$ are placed as shown in the respective cases in Figure 2. Choosing D' sufficiently close to D, we may assume that no line lying on X passes through two points of P.

By Riemann–Roch Theorem the dimension of the linear system $|P|$ is at least 1. Let us show that $|P|$ does not have base points, which means that $\dim|P-p_j| = 0$ for each $j = 1,\dots,5$. By Riemann–Roch Theorem it is enough to show that the divisor $P-p_j$ is non-special, i.e. $|K_C-(P-p_j)| = \varnothing$. The linear system $|K_C|$ is cut on C by plane sections (recall that C is canonically embedded in $\mathbb{P}^3$), hence we have to show that no plane section passes through all points of $P-p_j$. Suppose that a plane section D'' passes through all points of $P-p_j$. We consider the following three cases.

Case 1. $p_j \notin D\cap D'$. Then D'' has three common points with D or D', hence it coincides with D or D' which is impossible because each of D, D' contains only three points of P.

Case 2. $p_j\in D\cap D'$ and $P\not\subset D\cup D'$. Then the single point of $P\setminus(D\cup D')$ can be moved out of the union of all the plane sections passing through three-point subsets of $P-p_j$.

Case 3. $p_j\in D\cap D'$ and $P\subset D\cup D'$ (see the two bottom pictures in Figure 2). Each of D, D' passes through two points of $P-p_j$, hence D'' crosses $D\cup D'$ only at points of P because $D\cdot D'' = D'\cdot D'' = 2$. Choosing D' sufficiently close to D, we may ensure that both D' and D'' are $\mathcal{C}^1$-close to D, in particular, D'' goes monotonically from the left to the right (according to Figure 2). Easy to see that this is impossible under condition that D'' passes through all points of $P - p_j$ and crosses $D \cup D'$ only at the points of P.

We have proven that the linear system $|P|$ does not have base points. Then by Lemma 2.2 there exists a smooth deformation $P_t = p_1(t) + \cdots + p_5(t) \in |P|$, $t \in [0, t_0]$ such that $p_j(0) = p_j$ and $p'_j(0) \neq 0$ for each $j = 1, \ldots, 5$.

In the rest of the proof we use the arguments as in the proof of [4, Thm. 3.2]. We set $\omega = \omega_D$ and consider the D-orientation on $\mathbb{R}C \setminus \mathbb{R}D$ (see the beginning of this section). It is the boundary orientation induced by the "chess-board" orientation, which is shown in Figure 2 by circular arrows. Let $v_j = p'_j(0) \in T_{p_j}(\mathbb{R}C)$, $j = 1, \ldots, 5$. By Abel's theorem we have $\omega(v_1) + \cdots + \omega(v_5) = 0$ and $\omega(v_j) = 0$ when $p_j \in D$. There are only two points p_{j_1}, p_{j_2} of P not belonging to D, hence $\omega(v_{j_1}) = -\omega(v_{j_2})$. We see in Figure 2 that the D-orientation and the complex orientation coincide at one of the points p_{j_1}, p_{j_2} and they are opposite at the other point. Hence the complex orientations are of the same sign on v_{j_1} and v_{j_2}.

By applying the same arguments with D' instead of D we conclude that in all cases with $r = 3$, the complex orientation on the vectors v_j have the same sign whenever the corresponding points p_j belong to the same connected component of $\mathbb{R}C$. In the case $r = 1$ (the rightmost case in Figure 2), these arguments give only that the complex orientation has the same sign on the vectors v_2, v_3, v_4 (here we number the points p_j from the left to the right). However, by applying the same arguments to the plane section through p_2, p_4, p_5 (resp. p_1, p_2, p_3), we obtain that the complex orientation has the same sign on v_1 and v_3 (resp. on v_4 and v_5).

Therefore, for $0 < t \ll t_0$, the divisors P and P_t are *interlacing* (see [2, §2.1]), i.e., each component of $\mathbb{R}C \setminus P$ contains exactly one point of P_t. Then (see [2, Prop. 2.11]) the meromorphic function on C whose divisor is $P - P_t$ defines a separating morphism $C \to \mathbb{P}^1$. Hence we have $\{(1,3,1), (1,2,2)\} \subset \mathrm{Sep}(C)$ when $r = 3$, and we have $5 \in \mathrm{Sep}(C)$ when $r = 1$. Since the divisor P is non-special, the result follows from [2, Prop. 3.2]. □

Lemma 2.4. *Suppose that X is a quadratic cone or hyperboloid, $l = 2$, and L is a real line on X. Then L cannot have non-empty intersection with both ovals of C.*

Proof. Any real curve on X intersects an oval at an even number of points counting multiplicities. Thus the result follows from the fact that $L \cdot C = 3$. □

3. Proof of Theorem 1.1

3.1. Proof for ellipsoids. Let X be an ellipsoid and $r = 3$. If $(d_1, d_2, d_3) \in \mathrm{Sep}(C)$, then $d_2 \geq 2$; see [4, Example 3.5].

The pencil of planes through a point of c_1 and a point of c_3 defines a separating morphism which realizes $(1, 2, 1) \in \mathrm{Sep}(C)$.

Let us choose a point in each component of $\mathbb{R}X \setminus \mathbb{R}C$ which is homeomorphic to a disk. Lemma 2.3 applied to a plane section D passing through these two points completes the proof of Theorem 1.1 for ellipsoids.

3.2. Proof for quadratic cones. Let X be a quadratic cone. Theorem 1.1 for this case is a combination of the following propositions.

Proposition 3.1. *If $(r, l) = (3, 0)$ and $d = (d_1, 1, d_3) \in \mathrm{Sep}(C)$, then $d = (1, 1, 1)$.*

Proof. Let $f : X \to \mathbb{P}^1$ be a separating morphism realizing $(d_1, 1, d_3)$ and P be a fiber of f. The result follows from Lemma 2.1 where D is a double generator passing through the point of $P \cap c_2$. □

Proposition 3.2. *If* $(r,l)=(3,0)$, *then* $\{(1,1,1),(1,2,1)\}\subset \mathrm{Sep}(C)$.

Proof. A pencil of plane sections passing through a real line L defines a morphism $f: X\to\mathbb{P}^1$. If L is a generator of X, then f realizes $(1,1,1)\in\mathrm{Sep}(C)$. If L meets c_1 and c_3 but not c_2, then f realizes $(1,2,1)\in\mathrm{Sep}(C)$. □

Proposition 3.3. *If* $(r,l)=(3,0)$, *then*

$$\big\{(d_1,d_2,d_3)\in(1,2,1)+\mathbb{N}_0^3 \;\big|\; d_1+d_2+d_3\ge 5\big\}\subset\mathrm{Sep}(C). \tag{2}$$

Proof. Let L be a real line close to the rotation axis of X and not passing through the apex. Let D be the section of X by a real plane passing through L. Then (1) holds by Lemma 2.3, and we also have $(2,2,1)+\mathbb{N}_0^3\subset\mathrm{Sep}(C)$ by symmetry. □

Proposition 3.4. *If* $(r,l)=(3,2)$ *and* $(d_1,d_2,d_3)\in\mathrm{Sep}(C)$, *then* $d_2\ge 2$.

Proof. Suppose that there exists a separating morphism $f: X\to\mathbb{P}^1$ realizing $(d_1,1,d_3)$. Let P be a fiber of f and let $D=2L$ where L is the generator passing through the point of $P\cap c_2$. By Lemma 2.4 L cannot pass through all points of P. Hence we obtain a contradiction with Lemma 2.1. □

Proposition 3.5. *If* $(r,l)=(3,2)$, *then* $(1,2,1)\in\mathrm{Sep}(C)$.

Proof. Suppose that C is as in Figure 1, i.e. it is a perturbation of a plane section and a thin cylinder whose axis is linked with c_2. Then the pencil of plane sections passing through a line L intersecting c_1 and c_3 realizes $(1,2,1)\in\mathrm{Sep}(C)$. As we pointed out in the introduction, any other curve with $(r,l)=(3,2)$ on X is obtained from this model curve by a continuous deformation. Such a deformation can be followed by a simultaneous continuous deformation of the line L intersecting c_1 and c_3. During the deformation, L cannot become a generator of the cone by Lemma 2.4. Thus the line L remains to be linked with c_2, hence the pencil of planes through L always defines the same element of $\mathrm{Sep}(C)$. □

Proposition 3.6. *If* $(r,l)=(3,2)$, *then the inclusion (2) holds.*

Proof. Let D be section of X by a real plane avoiding the apex and passing through the line shown in Figure 1. Then Lemma 2.3 implies (1) and, by symmetry, (2). As in the proof of Proposition 3.5, the order of crossings of D with C cannot change during a continuous deformation. □

3.3. Proof for hyperboloids. Let X be a hyperboloid and let A and B be real lines on X from different rulings. Fix some orientations on $\mathbb{R}A$ and $\mathbb{R}B$ and denote the corresponding homology classes in $H_1(\mathbb{R}X)$ by a and b respectively. Then there are two rigid isotopy classes of irreducible plane sections determined by their homology classes, which are $a+b$ and $a-b$. We assume that A, B, and the orientations of $\mathbb{R}X$, $\mathbb{R}A$, $\mathbb{R}B$ are chosen so that the horizontal and vertical plane sections in Figure 1 (oriented according to the arrows) belong to the classes $a+b$ and $a-b$ respectively and $ab=-ba=1$. Thus the class of $\mathbb{R}C$ in $H_1(\mathbb{R}X)$ is

$$\begin{cases} 3a+3b, & \text{if } (r,l)=(3,0),\\ 3a+b, & \text{if } (r,l)=(1,0),\\ a+b, & \text{if } (r,l)=(3,2).\end{cases} \tag{3}$$

The following two lemmas are easy and we omit the proofs.

Lemma 3.7. *Let $p, q : [0,1] \to \mathbb{R}X$, $t \mapsto p_t$, $t \mapsto q_t$, be two continuous paths such that the line $p_t q_t$ is not contained in X for each $t \in [0,1]$. Let D_0 be an irreducible real plane section of X passing through p_0 and q_0. Then there exists a continuous family of irreducible real plane sections $\{D_t\}_{t\in[0,1]}$ such that $\mathbb{R}D_t$ is homologous to $\mathbb{R}D_0$ and passes through p_t and q_t for each $t \in [0,1]$.* □

Lemma 3.8. *Let D be a real irreducible plane section of X such that $[\mathbb{R}D] = a - b$. Let Γ be an oriented simple closed curve on $\mathbb{R}X$ which belongs to the homology class $a+b$ and has two intersection points with $\mathbb{R}D$. Let L be a real line passing through two points $p, q \in \mathbb{R}D \setminus \Gamma$. Then $\mathbb{R}L$ is linked with Γ in $\mathbb{RP}^3$ if and only if $\mathbb{R}L \cap \mathbb{R}D$ and $\Gamma \cap \mathbb{R}D$ are not interlacing on $\mathbb{R}D$, i.e., if and only if p and q belong to the same component of $\mathbb{R}D \setminus \Gamma$.* □

We split Theorem 1.1 for hyperboloids into four Propositions 3.9–3.12 below. It is well-known that X is biregularly equivalent to $A \times B$. Let $\pi_A : X \to A$ and $\pi_B : X \to B$ be the projections coming from this equivalence.

Proposition 3.9. *If $(r, l) = (1, 0)$, then* $\mathrm{Sep}(C) = 3 + \mathbb{N}_0$.

Proof. $\pi_A|_{\mathbb{R}C} : \mathbb{R}C \to \mathbb{R}A$ is a 3-fold covering (see (3)), hence it realizes $3 \in \mathrm{Sep}(C)$.

In contrary, $\pi_B|_{\mathbb{R}C} : \mathbb{R}C \to \mathbb{R}B$ is not a 3-fold covering. Indeed, otherwise $\pi_B|_{\mathbb{R}C}$ would be also a separating morphism, which contradicts (3). Hence there exists a fiber of π_B which has only one real intersection with C. Without loss of generality we may assume that it is A. Let D be a small real perturbation of $A \cup B$ such that $[\mathbb{R}D] = a - b \in H_1(\mathbb{R}X)$. Then D and C have 4 real intersection points. Let p and $\bar{p}$ be the remaining imaginary intersections. Consider the pencil $\mathcal{D}$ of plane sections passing through p and $\bar{p}$. The real loci of its members are pairwise disjoint. Hence any real member of $\mathcal{D}$ is irreducible because D has real intersections with any real line in X. Therefore for each $E \in \mathcal{D}$ we have $[\mathbb{R}E] = [\mathbb{R}D] = a - b \in H_1(\mathbb{R}X)$ and hence E has 4 real intersections with C by (3) because

$$\mathbb{R}E \cdot \mathbb{R}C = (a-b)(3a+b) = ab - 3ba = 4ab = 4. \tag{4}$$

Thus $\mathcal{D}$ realizes $4 \in \mathrm{Sep}(C)$.

Finally, $5 + \mathbb{N}_0 \subset \mathrm{Sep}(C)$ by Lemma 2.3 applied to any element of $\mathcal{D}$, because all the four intersections are positive by (4), and hence the rightmost case in Figure 2 takes place. □

Proposition 3.10. *If $(r, l) = (3, 0)$, then* $\mathrm{Sep}(C) = \mathbb{N}^3$.

Proof. The projection $\pi_A : X \to A$ realizes $(1,1,1) \in \mathrm{Sep}(C)$.

Let D be a real irreducible plane section of X such that $[\mathbb{R}D] = a - b$. By (3) we have $\mathbb{R}D \cdot \mathbb{R}C = 3(a-b)(a+b) = 6ab$, hence $\mathbb{R}D$ intersects $\mathbb{R}C$ transversally at six points. Moreover, $\mathbb{R}D$ crosses the components of $\mathbb{R}C$ in the order shown in Figure 2 in the middle. Indeed, the order cannot change during a rigid isotopy, hence it is enough to check this fact for the model curve in Figure 1. Thus (1) follows from Lemma 2.3. Since this result is invariant under renumbering of components of $\mathbb{R}C$, we conclude that $(d_1, d_2, d_3) \in \mathrm{Sep}(C)$ whenever $d_1 + d_2 + d_3 \geq 5$.

Let $p_k \in \mathbb{R}D \cap c_k$ and $p_{k+1} \in \mathbb{R}D \cap c_{k+1}$ be two consecutive points of $\mathbb{R}D \cap \mathbb{R}C$ with respect to the order along $\mathbb{R}D$ (the subscripts are considered mod 3). By Lemma 3.8 the line passing through p_k and p_{k+1} is linked with c_{k+2}. Hence any real plane containing this line has six real intersections with C. Thus the pencil of plane sections passing through p_k and p_{k+1} realizes the element (d_1, d_2, d_3) of $\mathrm{Sep}(C)$ such that $d_k = d_{k+1} = 1$ and $d_{k+2} = 2$. Thus $(d_1, d_2, d_3) \in \mathrm{Sep}(C)$ whenever $d_1 + d_2 + d_3 = 4$. □

Proposition 3.11. *If* $(r, l) = (3, 2)$ *and* $(d_1, d_2, d_3) \in \mathrm{Sep}(C)$, *then* $d_2 \geq 2$.

Proof. Suppose that there exists a separating morphism $f : X \to \mathbb{P}^1$ realizing $(d_1, 1, d_3)$. The projection $\pi_B : \mathbb{R}C \to \mathbb{R}B$ is not a covering. Hence there exists a fiber (we may assume that it is A) which has one real intersection point with C. Let $A \cap C = \{p\}$. Then $p \in c_2$, hence $A \cap (c_1 \cup c_3) = \varnothing$. Let $B', B'' \in |B|$ be distinct lines tangent to c_3. Let $D' = A + B'$, $D'' = A + B''$, and $P = f^{-1}(f(p))$. Then none of D', D'' cannot pass through all points of P by Lemma 2.4, and for one of them we obtain a contradiction with Lemma 2.1. Indeed, the D'- and D''-orientations coincide on c_1 and they are opposite on c_3. □

Proposition 3.12. *If* $(r, l) = (3, 2)$, *then* $(1, 2, 1) + \mathbb{N}_0^3 \subset \mathrm{Sep}(C)$.

Proof. We have $(1, 2, 1) \in \mathrm{Sep}(C)$ by the same arguments as in the proof of Proposition 3.5. Let us show that the inclusion (2) holds. The proof is also almost the same as for Proposition 3.6 but we also need Lemma 3.7. Namely, consider a rigid isotopy $\{C_t\}_{t\in[0,1]}$ such that C_0 is the model curve shown in Figure 1 and $C_1 = C$. We denote the components of C_t by $c_{t,1}$, $c_{t,2}$, $c_{t,3}$ according to Figure 1.

Let D_0 be the section of X by a real plane passing through the line shown in Figure 1 and such that $[\mathbb{R}D_0] = [c_2]$ in $H_1(\mathbb{R}X)$. Then Lemma 2.3 implies (1) and, by symmetry, (2) for the model curve C_0. Let us choose continuous paths $\{p_t\}$ and $\{q_t\}$ so that $p_0, q_0 \in \mathbb{R}D_0$ and $p_t \in c_{t,1}$, $q_t \in c_{t,3}$ for all t. By Lemma 2.4, for any t, the line $p_t q_t$ is not contained in X. Hence by Lemma 3.7 there exists a continuous family of irreducible real plane sections $\{D_t\}$ such that $p_t, q_t \in \mathbb{R}D_t$ and $[\mathbb{R}D_t] = [c_2]$ for each t. Then the mutual arrangement of $\mathbb{R}D_t$ and $\mathbb{R}C_t$ does not change during the deformation, hence we may apply Lemma 2.3 to C. □

4. Hyperelliptic curves

The main results of [3] can be also proved using the approach of the present paper. One of them (a description of $\mathrm{Sep}(C)$ when $\mathrm{genus}(C) = 3$) is already reproved in this way in [4]. Here we give a new proof of the other.

Theorem 4.1. ([3, Thm. 1].) *Let C be a non-maximal hyperelliptic curve of genus $g \geq 1$. Set $m = \lfloor (g+1)/2 \rfloor$. Then*

$$\mathrm{Sep}(C) = \begin{cases} \big((1,1)\mathbb{N}\big) \cup \big((m,m) + \mathbb{N}_0^2\big), & \text{if } g \text{ is odd}, \\ (2\mathbb{N}) \cup (g + \mathbb{N}_0), & \text{if } g \text{ is even}. \end{cases} \tag{5}$$

Proof. The curve C is defined by the equation $y^2 = F(x)$ where F is a real polynomial of degree $2g+2$ positive on $\mathbb{R}$. It can be embedded to a real Hirzebruch surface X of degree $g+1$ (the fiberwise compactification of the line bundle $\mathcal{O}(g+1)$) so that the hyperelliptic projection $\pi : C \to \mathbb{P}^1$ is the restriction of the fibration $X \to \mathbb{P}^1$. The restriction $\pi|_{\mathbb{R}C}$ is a two-fold covering over $\mathbb{RP}^1$. It is trivial if g is odd and non-trivial if g is even. We fix an affine chart $U \subset X$ with coordinates (x, y) such that C and π are given by $y^2 = F(x)$ and $(x, y) \mapsto x$ respectively. Then $\mathbb{R}C \cap U$ has two connected components c_1 and c_2. Each of them cuts transversally all fibers of π. We have $K_X + C \sim (g-1)F$ where F is a fiber of π. Denote the semigroup in the right hand side of (5) by S.

Let us show that $\mathrm{Sep}(C) \subset S$. Let $f : C \to \mathbb{P}^1$ be a separating morphism and P is its fiber contained in U. Let $P_i = P \cap c_i$ and $n_i = \#P_i$, $i = 1, 2$. Without loss of generality we may assume that $n_1 \le n_2$. Recall that $m = \lfloor (g+1)/2 \rfloor$. Suppose that $n_1 < m$. Let D_0 be the union of $m-1$ fibers of π such that $P_1 \subset D_0$ and $D_0 \setminus \pi^{-1}(P_1)$ is disjoint from P. If g is odd, we set $D = 2D_0$; if g is even, we set $D = 2D_0 + F$, where F is the fiber at infinity, i.e., the fiber of π not contained in U. In both cases we have $D - C \sim K_X$ and we may consider the corresponding meromorphic 2-form Ω_D on X and its Poincaré residue ω_D on C, which is a holomorphic 1-form; ω_D defines a D-orientation on $\mathbb{R}C \setminus \mathrm{supp} D$ (cf. §2). Then ω_D vanishes at the points of P_1 and the D-orientation coincides with the complex orientation on $c_2 \setminus \mathrm{supp} D$. Hence $P_2 \subset D_0$ by [4, Thm. 3.2], and hence $P_2 \subset \pi^{-1}(P_1)$. By symmetry we also have $P_1 \subset \pi^{-1}(P_2)$ and hence $n_1 = n_2$, which implies $\mathrm{Sep}(C) \subset S$.

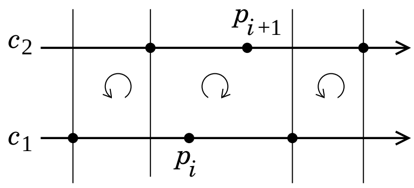

FIGURE 3. To the proof of Theorem 4.1.

Now let us prove the inverse inclusion $S \subset \mathrm{Sep}(C)$. The hyperelliptic projection realizes $2 \in \mathrm{Sep}(C)$ or $(1,1) \in \mathrm{Sep}(C)$ (depending on the parity of g), hence the semigroup generated by this element is contained in $\mathrm{Sep}(C)$. Thus, by [2, Prop. 3.2], it is enough to realize $g \in \mathrm{Sep}(C)$ or $(m,m) \in \mathrm{Sep}(C)$ (depending on the parity of g) by a separating morphism with non-special fibers. Let $p_i = (x_i, y_i)$, $i = 1, \dots, g$, be points on C such that $x_1 < x_2 < \dots < x_g$ and $\mathrm{sign}(y_i) = (-1)^i$, i.e., $p_1, p_3, \dots$ are on c_1 and $p_2, p_4, \dots$ are on c_2. Then the divisor $P = p_1 + \dots + p_g$ is non-special on C. Let us show that P is realizable as a fiber of a separating morphism. We proceed as in the proof of Lemma 2.3. By Riemann–Roch Theorem, $\dim |P| = 1$. Let $P_t = p_1(t) + \dots + p_g(t)$, $P_0 = P$, be a deformation of P in $|P|$. The linear system $|P|$ does not have base points. Indeed, if $p_i(t)$ is constant, then $P_t^* := P_t - p_i(t) \sim P^* := P - p_i$, hence $P_t^* + \tau(P^*) \sim P^* + \tau(P^*)$, where τ is the hyperelliptic involution, but this contradicts the fact that $P^* + \tau(P^*) \in |K_C|$ and that each element of $|K_C|$ is invariant under τ. Let D_i, $i = 1, \dots, g-1$, be the union of $g-2$ fibers of π passing through all points of P except p_i and p_{i+1}. Then $D_i \in |K_X + C|$ and we consider the D_i-orientation on $C \setminus \mathrm{supp}(D_i)$. It coincides with the complex orientation at one of the points p_i, p_{i+1} and it is opposite at the other point (see Figure 3). Hence (cf. the end of the proof of Lemma 2.3), the complex orientation has the same sign on the tangent vectors $p_i'(0)$ and $p_{i+1}'(0)$. This is true for all $i = 1, \dots, g-1$, hence the divisors P and P_t, $0 < t \ll 1$, are interlacing and the result follows from [2, Prop. 3.2].

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