

RIGID ISOTOPY OF MAXIMALLY WRITHED LINKS

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ABSTRACT. This is a sequel to the paper [4] which identified maximally writhed algebraic links in \mathbb{RP}^3 and classified them topologically. In this paper we prove that all maximally writhed links of the same topological type are rigidly isotopic, i.e. one can be deformed into another with a family of smooth real algebraic links of the same degree.

1. INTRODUCTION

A *real algebraic curve* $A \subset \mathbb{P}^3$ is a (complex) one-dimensional subvariety invariant with respect to the involution of complex conjugation $(z_0 : z_1 : z_2 : z_3) \mapsto (\bar{z}_0 : \bar{z}_1 : \bar{z}_2 : \bar{z}_3)$. We denote with $\mathbb{R}A$ the fixed point locus of $\text{conj}|_A$, note that $\mathbb{R}A = A \cap \mathbb{RP}^3$. Following the classical terminology by a *real branch* of A we mean the image $\nu(K)$ for a connected component K of $\mathbb{R}\tilde{A}$, where $\nu : \tilde{A} \rightarrow A$ is a normalization.

In the case of nonsingular A we call $L = \mathbb{R}A$ the *real algebraic link*, cf. [4], if every component of the normalization of A has non-empty real part, and if L is not contained in any proper projective subspace of \mathbb{RP}^3 . We say that L is irreducible if A is irreducible. In this paper real algebraic links as well as algebraic curves are assumed to be irreducible.

The degree $d \in \mathbb{Z}$ of a real algebraic link $L \subset \mathbb{RP}^3$ is a positive number such that $[A] = d[\mathbb{P}^1] \in H_2(\mathbb{P}^3) \cong \mathbb{Z}$, the genus of L is the genus of A . Given a point $p \in \mathbb{RP}^3 \setminus \mathbb{R}A$ we denote with $\pi_p : \mathbb{P}^3 \setminus \{p\} \rightarrow \mathbb{P}^2$ the linear projection from p so that $\pi_p(A) \subset \mathbb{P}^2$ is a plane curve of the same degree d . If p is chosen generically then $\pi_p(A)$ is a nodal curve. The set

$$(\pi_p(A) \cap \mathbb{RP}^2) \setminus \pi_p(\mathbb{R}A)$$

is not always empty. It is a finite set consisting of *solitary nodes* of $\mathbb{R}\pi_p(A) = \pi_p(A) \cap \mathbb{RP}^2$, i.e. the points of real intersection of pairs of complex conjugate local branches. It was shown in [6] that each node

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of $\mathbb{R}\pi_p(A)$ may be prescribed a sign ± 1 or 0 so that the sum $w(L)$ of the signs of all nodes does not depend on a choice of p and is an invariant of L .

Non-solitary nodes of $\mathbb{R}\pi_p(A)$ are intersection points of pairs of real local branches of A . If these local branches belong to different real branches of A then the sign of the corresponding node is defined as zero. Otherwise, the definition of sign at a non-solitary node agrees with the convention used in Knot Theory for the definition of the *writhe* of the knot diagram, see Figure 1 (we choose any orientations of the two local branches that can agree with an orientation of the real branch containing them).

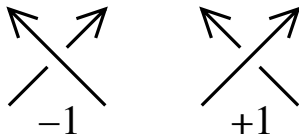


FIGURE 1. Writhe signs for a crossing point of two real branches on the knot diagram.

In the case of solitary nodes the signs were introduced in [6]. Accordingly, $w(L)$ is called the *encomplexed writhe* (as in [6]), or the *Viro invariant* of L . Note that the number of nodes of $\pi_p(A)$ is $N_d - g$, where $N_d = \frac{(d-1)(d-2)}{2}$. Thus we have the straightforward upper bound

$$(1) \quad |w(L)| \leq N_d - g \leq N_d$$

for any real algebraic link $L \subset \mathbb{RP}^3$ of degree d . Accordingly, an (irreducible) real algebraic link of degree d with $|w(L)| = N_d$ is called a *maximally writhed knot* or an *MW-knot* (since this properties implies that $g = 0$, the curve A is rational and thus L is connected).

The following extremal property of (1) was shown in [3].

Theorem 1 ([3]). *If $L, L' \subset \mathbb{RP}^3$ are MW-knots of the same degree then the pairs (\mathbb{RP}^3, L) and (\mathbb{RP}^3, L') are diffeomorphic.*

In other words, the topological type of an MW-knot of a given degree is unique.

Consider now the case of an arbitrary g . By Harnack's inequality the number of real branches of A is at most $g + 1$. The (irreducible) real curves A with $g + 1$ real branches are known as *M-curves*. In this case the complement $A \setminus \mathbb{R}A$ consists of two components bounded by $\mathbb{R}A$. The orientations of two component of $A \setminus \mathbb{R}A$ yield a pair of opposite orientations of L . Through these orientations, an orientation of one real branch of A determines the orientation of all other real branches, and

we define the sign $\sigma(q) = \pm 1$ also in the case when a node $q \in \mathbb{R}\pi_p(A)$ corresponds to a crossing point of different real branches of A according to Figure 1. We set

$$(2) \quad w_\lambda(L) = w(L) + \sum_q \sigma(q) = w(L) + 2 \sum_{K \neq K'} \text{lk}(K, K'),$$

see [4]. Here the first sum is taken over the nodes q corresponding to intersection points of different real branches of A while the second sum is taken over unordered pairs of distinct real branches K, K' .

Clearly, if $|w_\lambda(L)| = N_d$ then L consists of a single real branch. Furthermore, a stronger inequality holds

$$|w_\lambda(L)| \leq N_d - g.$$

It has a chance to be sharp even for a multicomponent L .

Definition 1.1. A *maximally writhed* real algebraic link (or an MW_λ -link) $L \subset \mathbb{R}\mathbb{P}^3$ is a real algebraic link in $g+1$ real branches, $g \geq 0$, such that the irreducible real algebraic curve A with $\mathbb{R}A = L$ is of genus g and that $|w_\lambda(L)| = N_d - g$.

Note that $N_d > g$ as, otherwise A must be planar. Therefore, $w_\lambda(L) \neq 0$ if L is an MW_λ -link. We refer to the sign of $w_\lambda(L)$ as the *chirality* of an MW_λ -link L .

Let $\alpha = (a_0, \dots, a_g)$ be a partition of the number $d-2$ into $l = g+1$ positive integer numbers. This means that $a_0 \geq \dots \geq a_g$ are positive integer numbers such that $\sum_{j=0}^g a_j = d-2$. In [4] to each α we have associated a (topological) link $W_g(\alpha)$ in $g+1$ components. Namely, $W_g(\alpha)$ is a link that sits on the boundary (the union of $g+1$ hyperboloids) of a regular neighborhood of the Hopf link H in $g+1$ component. Each of the $g+1$ hyperboloids contains a single component of $W_g(\alpha)$. Furthermore, the linking number of the j -th component K_j (enhanced with some orientation) and the j th component of the Hopf link is $a_j + 2$ while the linking number of K_j with any other component of H is a_j .

Theorem 2 ([4]). *For every MW_λ -link L of degree d there exists a partition $\alpha = (a_0, \dots, a_g)$ such that $(\mathbb{R}\mathbb{P}^3, L)$ and $(\mathbb{R}\mathbb{P}^3, W_g(\alpha))$ are diffeomorphic.*

Conversely, for every α there exists an MW_λ -link L of degree d such that the pairs $(\mathbb{R}\mathbb{P}^3, L)$ and $(\mathbb{R}\mathbb{P}^3, W_g(\alpha))$ are diffeomorphic.

We say that an MW_λ -link L corresponds to the partition α if $(\mathbb{R}\mathbb{P}^3, L)$ and $(\mathbb{R}\mathbb{P}^3, W_g(\alpha))$ are diffeomorphic.

The following result is the main theorem of this paper. It strengthens the second half of Theorem 2. Two real algebraic links are said *rigidly isotopic* if they can be connected with a one-parametric family of real algebraic links of the same degree.

Theorem 3. *If L and L' are MW_λ -links of the same chirality corresponding to the same partition α then L and L' are rigidly isotopic.*

This theorem is a particular case of Theorem 4 (see Section 8) where we describe rigid isotopy classes of nodal curves of any genus. The proof of Theorem 4 is based on the theory of divisors on nodal Riemann surfaces and on the properties of MW_λ -curves established in [4].

Note that in the case of rational curves Theorem 2 corresponds to [3, Theorem 1] with a simpler proof specific for genus 0. In contrast to that, the only known to us way to prove Theorem 3 is to deduce it from Theorem 4, even in the case of rational curves. In particular, this proof requires [4, Theorem 2] in whole generality. Note that Theorem 4 provides another proof of [4, Theorem 3], i.e. existence of MW_λ -links isotopic to $W_g(\alpha)$ for any partition $\alpha = (a_0, \dots, a_g)$.

2. ALGEBRAIC HOPF LINKS

Definition 2.1. A real algebraic link L with l real branches is called *an irreducible algebraic Hopf link* if it is irreducible, its degree is $l + 2$, and each real branch of A is non-contractible in \mathbb{RP}^3 .

An irreducible algebraic Hopf link has minimal possible degree among all irreducible real algebraic links with l real branches such that each real branch is non-contractible in \mathbb{RP}^3 by Proposition 2.2. By Proposition 2.3 it is always a Hopf link topologically.

Proposition 2.2. *Let $L \subset \mathbb{RP}^3$ be an irreducible real algebraic link of degree $d > 1$ such that each component of L is non-contractible in \mathbb{RP}^3 . Then $d \geq l + 2$.*

Proof. Take a pair of conjugate points on $A \setminus \mathbb{R}A \subset \mathbb{P}^3$ and consider a real plane through this pair. Since any plane in \mathbb{RP}^3 must intersect each non-contractible component of L we get at least $l + 2$ intersection points. \square

Proposition 2.3. *An irreducible algebraic Hopf link L or its mirror image is an MW_λ -link $W_{l-1}(1, \dots, 1)$, and thus is topologically isotopic to a Hopf link, i.e. the union of l fibers of the (positive) Hopf fibration $\mathbb{RP}^3 \rightarrow S^2$.*

Proof. The link L satisfies to condition (iii^t) of Theorem 2 of [4]. Namely, any plane section in \mathbb{RP}^3 intersects L in at least $d - 2 = l$ points (one for each component). By Theorem 2 of [4] L or its mirror image is topologically isotopic to $W_{l-1}(1, \dots, 1)$ as $(1, \dots, 1)$ is the only partition of l into the sum of l positive integers. \square

Lemma 2.4 (cf. Theorem 2.5 of [2]). *An effective real divisor $D = \sum_{j=1}^n a_j p_j$, $a_j > 0$, $p_j \in S$, on a real curve S is non-special if at least g distinct real branches of A intersect $\{p_1, \dots, p_n\}$.*

Proof. By the Riemann-Roch Theorem, a divisor D is special if and only if the difference $K - D$ of the canonical class K and the divisor D is representable with an effective divisor D' . Suppose that there exists a holomorphic form whose zero divisor is $D + D'$. But the zero divisor of a form on each real branch of a curve must be even (if counted with multiplicities). Thus the degree of $D + D'$ is at least $2g$ while the degree of the canonical class is $2g - 2$. \square

By a *real Riemann surface* we mean an (irreducible) Riemann surface S enhanced with an antiholomorphic involution conj . Its *real locus* $\mathbb{R}S$ is the fixed locus of conj . By Harnack's inequality, $g + 1$ is the maximal possible number of real branches of S if g is the genus of S . Real curves with this maximality property are known as *M-curves*.

Recall that all effective divisors linearly equivalent to an effective divisor D of degree d form a projective space $|D| \approx \mathbb{P}^r$ whose dimension is called the rank $r(D)$ of the divisor. If the divisor is base point free, i.e. every point of S is *not* contained in some divisor from $|D|$ then in addition we have the map

$$\iota_D : S \rightarrow |D|^\vee \approx \mathbb{P}^r$$

to the space $|D|^\vee$ projectively dual to $|D|$. The point $x \in S$ is mapped to the hyperplane in $|D|$ consisting of divisors containing x . If S is a real Riemann surface and D is a divisor invariant with respect to the conjugation then we say that D is a *real divisor*. In this case the map ι_D is also real, i.e. equivariant with respect to the conjugation and the image $A = \iota_D(S) \subset \mathbb{P}^r$ is a real projective curve.

Proposition 2.5. *Let S be a real Riemann surface of genus $g = l - 1$ with l real branches (i.e. S is an M-curve), and D be a real divisor of degree $l + 2$ such that every real branch of S contains odd number of points from D (counted with multiplicity). Then the algebraic curve $A = \iota_D(S) \subset \mathbb{P}^3$ corresponding to (S, D) is an irreducible algebraic Hopf link. In particular, A is non-singular, and $r(D) = 3$.*

Proof. The proposition is the special case of Corollary 2.8 of [2] for $r = 3$. \square

Corollary 2.6. *Irreducible algebraic Hopf links of the same chirality are rigidly isotopic.*

Proof. Let $L, L' \subset \mathbb{RP}^3$ be two irreducible algebraic Hopf links represented as $\iota_D(S)$ and $\iota_{D'}(S')$ for real Riemann surfaces S and S' and real divisor D and D' on them. By Proposition 2.3 the Riemann surfaces S and S' are M -curves. Therefore their quotients by the complex conjugation are diffeomorphic to a disk with g holes. Thus (see e.g. [5, Theorem 2.2]) S and S' can be deformed into each other through a family of real curves.

Without loss of generality (by taking an appropriate real plane section of the corresponding irreducible algebraic Hopf links) we may assume that the real divisors D and D' contain a single point on each real branch of S and S' as well as a pair of complex conjugate points. Thus we may enhance the deformation of S to S' with a deformation of D into D' in the space of real divisors on real Riemann surfaces. By Proposition 2.5 this deformation corresponds to a deformation in the class of irreducible algebraic Hopf links, i.e. a rigid isotopy.

Therefore, L and L' are rigidly isotopic up to a projective equivalence. Since the group $PGL_4(\mathbb{R})$ consists of two connected components, L and L' are rigidly isotopic if and only if their invariant w_λ (which cannot vanish for a MW_λ -link) is of the same sign. \square

Remark 2.7. Note that Conjecture 3.4 of [2] is false. Not only algebraic Hopf links, but all MW_λ -links are unramified in the sense of [2] by the condition (iii)^t of [4]. Taking a real algebraic link $L \subset \mathbb{RP}^3$ isotopic $W_g(a_0, \dots, a_g)$ with even a_j we get an unramified curve whose real branches are contractible in \mathbb{P}^3 . Existence of such link is ensured by Theorem 3 of [4].

3. NODAL LINKS

Recall that a nodal projective curve $A \subset \mathbb{P}^n$, $n \geq 2$, is a (complex) algebraic curve such that all of its singularities are simple nodes, i.e. points of crossings of two non-singular local branches with distinct tangent lines. As before, A is real if it is invariant with respect to the involution of complex conjugation $\mathbb{P}^n \rightarrow \mathbb{P}^n$. The real locus $\mathbb{R}A = \mathbb{RP}^n \cap A$ of a nodal curve is a disjoint union of immersed circles and a finite set $\mathbb{R}E \subset \mathbb{R}A$. The points of $\mathbb{R}E$ are called *solitary nodes of $\mathbb{R}A$* .

Definition 3.1. A subset $L \subset \mathbb{RP}^3$ is called an irreducible *nodal real algebraic link* if it is infinite and there exists an irreducible nodal real

algebraic curve $A \subset \mathbb{RP}^3$ such that $L = \mathbb{R}A$, and L is not contained in any proper projective subspace of \mathbb{RP}^3 . In particular, $L \neq \emptyset$. Two nodal real algebraic links are *rigidly isotopic* if they can be connected with a 1-parametric isotopy in the class of nodal real algebraic links of the same degree and with the same number of nodes.

A *node* of L is a node of the curve A .

Nodal real algebraic links provide a generalization of real algebraic links. As in the case of real algebraic links, an irreducible nodal real algebraic link $L \subset \mathbb{RP}^3$ uniquely determines $A \subset \mathbb{P}^3$.

Definition 3.2. We say that a real algebraic link L_1 *degenerates to a real nodal algebraic link* L_0 if there exists a continuous family of real algebraic links $L_t \subset \mathbb{RP}^3$, $t \in [0, 1]$, of constant degree such that L_t for $t > 0$ is a (non-nodal) real algebraic link. In this case we also say that the nodal link L_0 *can be perturbed* to the smooth link L_1 . The perturbation (resp. degeneration) is called *equigeneric* if the genus stays constant.

A smooth perturbation of a nodal curve is equigeneric if each node is perturbed as in Figure 2 (left) but not as in Figure 2 (right). All degenerations and perturbations considered in this paper are assumed to be equigeneric.

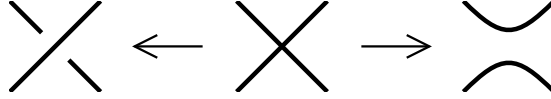


FIGURE 2. Equigeneric (on the left) and non-equigeneric (on the right) perturbations of a spatial nodal curve.

Proposition 3.3. *Any irreducible real algebraic link $L = \mathbb{R}A \subset \mathbb{RP}^3$ of genus g and degree d degenerates to a nodal link with at least $d - g - 3$ nodes.*

Proof. By induction it is enough to assume that if $L = \mathbb{R}A$ is a nodal link of genus g and degree d with $n < d - g - 3$ nodes, then it degenerates to a nodal link with at least one node more.

Let $p \in \mathbb{RP}^3$ be a generic point so that the image $B = \pi_p(A) \subset \mathbb{P}^2$ under the projection $\pi_p : \mathbb{P}^3 \setminus \{p\} \rightarrow \mathbb{P}^2$ is a real nodal planar curve. By choosing p on a line passing through two distinct real points of A , we may assume that B has a real node with two real branches which is not a projection of a node of A . This means that there are distinct points $x, y \in \mathbb{R}A$ such that

$$u = \pi_p(x) = \pi_p(y).$$

We choose the coordinates so that $p = (0 : 0 : 0 : 1)$.

Let $\iota : S \rightarrow A$ be the normalization and $D = \iota^{-1}(A \cap H)$ be the divisor on S obtained as the intersection of A and a generic real plane $H \subset \mathbb{P}^3$. The divisor D is a real divisor of rank $r \geq d - g$. The immersions of S given by $A \subset \mathbb{P}^3$ and $B \subset \mathbb{P}^2$ correspond to a real projective 3-dimensional subspace of $|D|$ and a real projective plane inside it.

Consider the vector space $\Gamma(S, D)$ formed by the sections of the line bundle defined by D . In the coordinates the curve $A \subset \mathbb{P}^3$ is given by four linearly independent sections $s_0, s_1, s_2, s_3 \in \Gamma(S, D)$ such that s_0, s_1, s_2 define the curve $B \subset \mathbb{P}^2$. The condition that s_0, s_1, s_2, s define an immersion with n prescribed nodes imposes n linear conditions on $s \in \Gamma(S, D)$. Let $V \subset \Gamma(S, D)$ be the space of all sections satisfying these conditions, in particular, we have $s_3 \in V$. Since $r(D) - n > 3$, it follows that there exists $s_4 \in V$ not contained in the linear span of s_0, s_1, s_2, s_3 . Taking s_4 close to s_3 we may assume that s_0, s_1, s_2, s_4 define a curve $A' \subset \mathbb{P}^3$ without a node at $\pi_p^{-1}(u)$.

With the help of the plane sections of the curves A, A' passing through one point of intersection with $\pi_p^{-1}(u)$ but not the other we can choose $s'_j \in \Gamma(D, S)$, $j = 3, 4$, with the following properties:

- (1) s_j is a linear combination of s_0, s_1, s_2, s'_j (in particular, the curve defined by s_0, s_1, s_2, s'_3 is projectively equivalent to A);
- (2) $s'_3(\tilde{x}) = 0 = s'_4(\tilde{y})$;
- (3) $s'_3(\tilde{y}) \neq 0 \neq s'_4(\tilde{x})$.

Here $\tilde{x} = \iota^{-1}(x), \tilde{y} = \iota^{-1}(y) \in S$. Note that (1) implies linear independence of $s_0, s_1, s_2, s'_3, s'_4$.

The image of S in \mathbb{P}^3 under $[s_0 : s_1 : s_2 : s'_3 + ts'_4]$, $t \in \mathbb{R}$, is a real curve $A_t \subset \mathbb{P}^3$ with $A_0 = A$. Since the first three sections define a nodal immersion to \mathbb{P}^2 , all singularities of $A_t \subset \mathbb{P}^3$ (if any) are nodes. By linear independence of $s_0, s_1, s_2, s'_3, s'_4$, the curve A_t cannot be contained in a plane in \mathbb{P}^3 .

By (2) and (3) there exists $t_u \in \mathbb{R}$ such that A_{t_u} has a node over u while A_0 does not have a node over u . Thus A can be degenerated to a curve with at least one node (though not necessarily over u since another node may appear during the deformation of t from 0 to t_u , in particular the new node could be solitary, or even a pair of complex conjugate nodes). \square

4. NODAL MW_λ -LINKS AND CHORD DIAGRAMS

Definition 4.1. An irreducible nodal algebraic link is called a nodal MW_λ -link if it can be (equigenerically) perturbed to a (smooth) MW_λ -link.

Lemma 4.2. *All nodes of a nodal MW_λ -link are real and non-solitary.*

Proof. Let L_0 be a nodal MW_λ -link and L_t , $t \in (0, 1]$ be smooth MW_λ -links degenerating to L_0 . If L_0 has a non-real node ν then the planar curve $\pi_p(L_\epsilon) \subset \mathbb{P}^2$ must have a non-real node near $\pi_p(\nu)$ for small $\epsilon > 0$. Thus L_ϵ can not be maximally writhed.

Suppose that $L_0 = \mathbb{R}A_0$ has a solitary real node ν . Let $p \in \mathbb{R}\mathbb{P}^3 \setminus \mathbb{R}A_0$ be a point on a line ℓ connecting a pair of complex conjugate points of A_0 . The projection $\pi_p(A_\epsilon) \subset \mathbb{P}^2$ has two solitary nodal points: one corresponding to ν , and one corresponding to ℓ . We get a contradiction to Condition (iv^a) of [4, Theorem 2]. \square

Many of the properties of MW_λ -links found in [4] hold also for any nodal MW_λ -link $L_0 = \mathbb{R}A_0 \subset \mathbb{R}\mathbb{P}^3$.

Let $q \in A_0 \setminus \mathbb{R}A_0$. Denote with $\ell_{q\bar{q}} \subset \mathbb{R}\mathbb{P}^3$ the real line whose complexification in \mathbb{P}^3 passes through q (and thus also through \bar{q}).

Lemma 4.3. *For any $q \in A_0 \setminus \mathbb{R}A_0$ we have $\ell_{q\bar{q}} \cap L_0 = \emptyset$.*

Proof. By Lemma 4.2 L_0 has no solitary nodes. Thus at a point of $\ell_{q\bar{q}} \cap L_0$ there exists a tangent plane $H \subset \mathbb{R}\mathbb{P}^3$ containing $\ell_{q\bar{q}}$. Existence of such plane contradicts to the property [4, Theorem 2 (ii)] of MW_λ -links which is clearly conserved under passing to the limiting nodal links. \square

Corollary 4.4. *For each real branch K the linking number*

$$(3) \quad a(K) = 2|\text{lk}(\ell_{q\bar{q}}, K)|$$

(taken twice) in $\mathbb{R}\mathbb{P}^3$ is independent of $q \in A_0 \setminus \mathbb{R}A_0$.

Proof. Lemma 4.3 implies the corollary since the curve A_0 is irreducible and thus the family of lines of the form $\ell_{q\bar{q}}$, $q \in A_0 \setminus \mathbb{R}A_0$, is connected. \square

We have

$$(4) \quad \sum_K a(K) = d - 2$$

by [4, equation (3)] as the integer numbers $a(K)$ agree with the numbers introduced in [4, Lemma 4.12] under the degeneration of L_t to L_0 .

When speaking of nodal curves, we shall use the language of chord diagrams.

Definition 4.5. A *chord diagram* (X, Λ) is a pair consisting of the disjoint union Λ of l oriented circles, and the topological space X obtained as the result of attachment of $\delta \geq 0$ intervals (chords) along their endpoints to Λ so that distinct endpoints are glued to distinct points of Λ . We consider (X, Λ) up to homeomorphisms of the pair preserving the orientation of Λ . We denote the complement of the chord endpoints in Λ with $\Lambda^\circ \subset \Lambda$.

Definition 4.6. A chord diagram (X, Λ) is called *planar* if it can be embedded to the disjoint union Δ of l copies of 2-disks so that Λ is mapped to the boundary of Δ .

Clearly no chord of a planar chord diagram can connect points from different components of Λ . Thus a planar diagram for arbitrary l is a disjoint union of l connected planar chord diagrams.

By a loop λ in a chord diagram (X, Λ) we mean a simple closed loop in X . If (X, Λ) is planar and $j : X \rightarrow \Delta$ is an embedding from Definition 4.6 then for each component of $\Delta \setminus j(X)$ there is a loop in X around this component, we refer to these loops as *planar loops*.

Definition 4.7. The chord diagram $X(L_0)$ of a nodal link L_0 is the chord diagram obtained by attaching a chord to $\mathbb{R}\tilde{A}_0$ connecting the respective points of the normalization for every node of $\mathbb{R}A_0$.

For a loop $\lambda \subset L_0$ the number

$$(5) \quad a(\lambda) = 2|\text{lk}(\lambda, \ell_{q\bar{q}})|$$

is independent of the choice of $q \in A_0 \setminus \mathbb{R}A_0$ by Lemma 4.3 and irreducibility of A_0 (cf. Corollary 4.4).

Proposition 4.8. *The chord diagram of a nodal MW_λ -link L_0 is planar.*

Proof. Let $L_0 = \mathbb{R}A_0$ be a nodal MW_λ -link and $\nu : \tilde{A}_0 \rightarrow A_0$ be the normalization of A_0 . Let x be a node of L_0 and H be the real plane tangent to both local branches of A_0 at x . Consider the intersection number $n(K)$ of H and A_0 along a real branch $K \ni x$. Namely, $n(K)$ is the intersection number in \mathbb{P}^3 of the complexification of H and $\nu(U)$, where U is a small neighborhood of \tilde{K} in \tilde{A}_0 (here \tilde{K} is the connected component of $\mathbb{R}\tilde{A}_0$ such that $\nu(\tilde{K}) = K$). Consider a point $q \in \nu(U) \setminus K$ close to x . Then the line $\ell_{q\bar{q}}$ is close to the tangent line at x to K . The plane H may be perturbed to a real plane $H' \supset \ell_{q\bar{q}}$. We have $\#(H' \cap K) \geq 2|\text{lk}(\ell_{q\bar{q}}, K)| = a(K)$ since $H' \supset \ell_{q\bar{q}}$. But in addition the complexification of H' intersects A_0 at q and \bar{q} . Thus $n(K) \geq \#(H' \cap K) + 2 \geq a(K) + 2$.

If two distinct real branches K_1, K_2 meet at x then taking the sum over all real branches we get the inequality

$$\sum n(K) \geq \sum a(K) + 4 = d + 2$$

contradicting to the Bezout theorem for the intersection of H and A_0 .

A node $y \in L_0$ where both local branches correspond to the same real branch K defines a subdivision of K into two loops $\lambda_j \subset K$, $j = 1, 2$. Each loop λ_j is the closure of the image of an arc of $\tilde{K} \setminus \nu^{-1}(y)$ under ν . If the chord diagram $X(L_0)$ is not planar then we may assume that K, x, y where chosen so that λ_1 and λ_2 pass through x . We have $a(\lambda_1) + a(\lambda_2) = a(K)$; see (5).

Denote with $n(\lambda_j)$ the intersection number of the complexification of H and $\nu(U_j)$ for a small neighborhood U_j of $\nu^{-1}(\lambda_j)$ in \tilde{A}_0 . We get $n(\lambda_j) \geq a(\lambda_j) + 2$ as above. If $y \notin H$ then $\nu(U_1 \cap U_2)$ is disjoint from H , thus $n(K) = n(\lambda_1) + n(\lambda_2) \geq a(K) + 4$, and we get a contradiction to the Bezout Theorem as before.

If $y \in H$ then $n(K) = n(\lambda_1) + n(\lambda_2) - n(y)$, where $n(y)$ is the local intersection number of A_0 and the complexification of H at y . But in this case we have $n(\lambda_1) + n(\lambda_2) \geq a(K) + 4 + n(y)$ so we get a contradiction to the Bezout Theorem anyway. \square

Definition 4.9. The *degree-chord* diagram of a nodal MW_λ -link L_0 is a chord diagram enhanced with a number $a(\lambda)$ defined for any planar loop of L_0 . The number $a(\lambda)$ is the *degree* of the planar loop λ of the planar chord diagram $X(L_0)$.

5. DIVISORS ON SINGULAR RIEMANN SURFACES

Let S be a (closed connected irreducible) Riemann surface.

Definition 5.1. A *chord diagram* X_S on S is a topological space obtained by attaching to S a finite number of 1-cells (*chords*) $I \approx [0, 1]$ in an acyclic way, i.e. so that the union of all chords is a forest (in particular no chord is a loop in X_S). Let δ be the number of chords. The *singular surface* $\Sigma = \Sigma(X_S)$ is obtained by contracting each chord to a point.

If the boundaries of the chords are disjoint in S , we say that X_S is *nodal*. Then we may consider Σ as an abstract nodal curve. Otherwise Σ has k -fold points with $k > 2$.

The theory of divisors and their linear systems on Σ is very similar to its counterpart on ordinary Riemann surfaces. Denote with $N \subset \Sigma$ the singular locus, i.e. the image of the union of chords under the contraction map $c : X_S \rightarrow \Sigma$. Let $\tilde{N} = c^{-1}(N) \cap S$. As usual, for a

meromorphic function $f : S \rightarrow \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ we denote with (f) its divisor on S composed of its zeroes and poles with multiplicities. Let $\Sigma^* = \Sigma \setminus N = S \setminus \tilde{N}$.

Let us fix an effective divisor $D = \sum a_j p_j$, $a_j \in \mathbb{Z}_{>0}$, with $p_j \in S \setminus \tilde{N}$. The number $n = \sum a_j$ is the degree of D . Denote with V_D the set of meromorphic functions $f : S \rightarrow \mathbb{P}^1$ such that $(f) \geq -D$ (i.e. f has, at worst, a pole of order a_j at p_j , and holomorphic elsewhere) and

$$c(x) = c(y) \implies f(x) = f(y), \quad \forall x, y \in S.$$

The space V_D is a vector space. We say that a divisor D' on S is Σ -equivalent to D , and write $D \sim D'$, if $D - D' = (f)$ for $f \in V_D$. Recall that we require D to be disjoint from \tilde{N} while we do not impose this requirement on D' . In particular, the relation \sim is not an equivalence relation (cf. Remark 5.5). Let $|D|_\Sigma$ be the set of effective divisors Σ -equivalent to D . It can be naturally identified with the projectivization of V_D .

The Riemann-Roch theorem implies that for each chord $I \subset X_S$ there exists a meromorphic form ω_I on S , holomorphic on $S \setminus \partial I$ and having a pole of the first order at each point of ∂I . Clearly, the residues of ω_I at the two points of ∂I must be opposite. Multiplying this form by a constant, we may assume that the residues of ω_I at ∂I are ± 1 . Furthermore, for a choice of a system of a -cycles, i.e. of g simple loops α_j , $j = 1, \dots, g$, in S such that $S \setminus \bigcup_{j=1}^g \alpha_j$ is a planar domain, we may assume that

$$(6) \quad \int_{\alpha_j} \omega_I = 0$$

by adding a holomorphic form on S .

Denote with $\Omega(\Sigma)$ the vector space generated by the holomorphic forms on S and the forms ω_I for the chords $I \subset X_S$. This is the space of meromorphic forms on S holomorphic on $S \setminus \tilde{N}$, having at worst poles of the first order at the points of \tilde{N} , and such that the sum of residues at all points of \tilde{N} corresponding to the same point of N vanishes. Thus $\Omega(\Sigma)$ depends only on Σ , and not on a particular representation of Σ by X_S . We have

$$\dim \Omega(\Sigma) = g + \delta.$$

We refer to the elements of $\Omega(\Sigma)$ as Σ -holomorphic form.

Any element of $H_1(\Sigma^*)$ defines a functional on $\Omega(\Sigma)$ through integration. The *Jacobian* of Σ is defined as the quotient

$$\text{Jac}(\Sigma) = \Omega^*(\Sigma)/H_1(\Sigma^*)$$

of the dual vector space to $\Omega(\Sigma)$ by the image of $H_1(\Sigma^*)$. It is a $(g + \delta)$ -dimensional complex variety homeomorphic to $(S^1)^{2g} \times (\mathbb{C}^\times)^\delta$. The divisor D defines the *Abel-Jacobi map*

$$\alpha_D : \text{Sym}^n(\Sigma^*) \rightarrow \text{Jac}(\Sigma)$$

by associating to D' the functional $\omega \mapsto \int_{\Gamma} \omega$, where Γ is a 1-chain in Σ^* with $\partial\Gamma = D' - D$.

The following theorem and corollary are a variant of the classical Abel-Jacobi and Riemann-Roch theorems for the case of singular curves. Their different versions can also be found in textbooks.

Theorem 5.2 (the Abel-Jacobi Theorem for Σ). *For an effective divisor D' supported on Σ^* we have $D' \in |D|_{\Sigma}$ if and only if $\alpha_D(D') = 0 \in \text{Jac}(\Sigma)$.*

Proof. Suppose that $D' \in |D|_{\Sigma}$, i.e. $(f) = D' - D$ with $f \in V_D$. By the conventional Abel-Jacobi theorem we have $\int_{\Gamma} \omega = 0$ for any holomorphic

1-form ω in S . Suppose that I is a chord in X_S and $x_+, x_- \in S$ are the endpoints of I . Let ω_I be the meromorphic form on S , holomorphic on $S \setminus \{x_+, x_-\}$, with the simple poles of residue ± 1 at x_{\pm} , and satisfying to (6). Multiplying f by a scalar we may assume that $f(x_+), f(x_-) \in \mathbb{C}$ are non-real numbers. The following equality is known as a reciprocity law for the Abelian differentials of the third kind (cf. e.g. [1, Lecture 6, Lemma 3])

$$(7) \quad \int_{\Gamma} \omega_I = \int_{\Gamma_I} \frac{df}{f}.$$

Here $\Gamma = f^{-1}(\mathbb{R}_{\geq 0} \cup \{\infty\})$ is oriented so that $\partial\Gamma = D' - D$ while Γ_I is a path in S from x_- to x_+ disjoint from Γ . To prove (7) it suffices to integrate the 1-form $\text{Log}(f)\omega_I$ over a small contour γ going around Γ for a holomorphic branch of $\text{Log}(f)$ in $S \setminus \Gamma$. Since the values of $\text{Log}(f)$ at the two sides of Γ differ by $2\pi i$, we see that $\int_{\gamma} \text{Log}(f)\omega_I$ equals to $2\pi i$ times the left-hand side of (7). By Cauchy's residue formula $\int_{\gamma} \text{Log}(f)\omega_I$ equals to $2\pi i$ times the right-hand side of (7).

Since $f \in V_D$, the right-hand side of (7) is zero modulo $2\pi i\mathbb{Z}$, and thus $\alpha_D(D') = 0$. Conversely, suppose that $\alpha_D(D') = 0$. By the conventional Abel-Jacobi theorem there exists a meromorphic function $f : S \rightarrow \mathbb{P}^1$ with $(f) = D' - D$. By (7) the integral $\int_{\Gamma_I} \frac{df}{f}$ vanishes (modulo $2\pi i\mathbb{Z}$) for any chord I . Thus $f \in V_D$. \square

Denote with $\Omega_{-D}(\Sigma) \subset \Omega(\Sigma)$ the vector space formed by the Σ -holomorphic forms vanishing in all points of D .

Corollary 5.3 (The Riemann-Roch Theorem for Σ). *We have*

$$\dim |D|_{\Sigma} = n - g - \delta + \dim \Omega_{-D}(\Sigma).$$

Proof. Consider the differential $d\alpha$ of the Abel-Jacobi map at $D \in \text{Sym}^n(\Sigma^*)$. It is a linear map from the n -dimensional vector space composed as the direct sum of the tangent spaces of S at the points of D (with multiplicities) to the $(g + \delta)$ -dimensional space $\Omega^*(\Sigma)$. The cokernel of this differential coincides with the space V_D . According to Theorem 5.2 the kernel is V_D . \square

We say that D is *non-special* in Σ if $\dim |D|_{\Sigma} = n - g - \delta$, i.e., if $\Omega_{-D}(\Sigma) = 0$ (recall that $\text{supp } D \subset \Sigma^*$).

Consider the chord diagram X_S^I obtained from X_S by removing a chord I and the corresponding singular surface Σ^I .

Lemma 5.4. *If a divisor $D = \sum_{j=1}^n a_j p_j$, $p_j \in \Sigma^*$, is non-special in Σ then it is also non-special in Σ^I .*

Proof. By definition we have $\Omega(\Sigma^I) \subset \Omega(\Sigma)$ whence $\Omega_{-D}(\Sigma^I) \subset \Omega_{-D}(\Sigma)$. Thus the hypothesis of $\Omega_{-D}(\Sigma) = 0$ implies the conclusion $\Omega_{-D}(\Sigma^I) = 0$. \square

With each effective divisor D such that $\text{supp } D \subset \Sigma^*$ we associate the map

$$(8) \quad \iota_D : S \rightarrow |D|_{\Sigma}^{\vee},$$

sending x to the hyperplane of V_D corresponding to divisors $D' \in |D|_{\Sigma}$ with $x \in D'$. This mapping evidently factors through a mapping $\Sigma \rightarrow |D|_{\Sigma}^{\vee}$.

As in the classical theory, any holomorphic mapping $f : S \rightarrow \mathbb{P}^k$ which factors through Σ is a composition of ι_D with a linear projection $\pi : |D|_{\Sigma}^{\vee} \dashrightarrow \mathbb{P}^k$ where D is the pull-back of a generic plane section of $f(S)$. The image of π is the minimal subspace of \mathbb{P}^k containing $f(S)$.

Remark 5.5. When a smooth curve is embedded to a complete linear system $|D|$, the embedded curve can be completely recovered (up to projective transformation) by any hyperplane section. Nevertheless, the situation is very different in the case of nodal curves. If a hyperplane section passes through some nodes, then it does not determine the embedding. This is the reason why we have supposed that $\text{supp } D \subset \Sigma^*$ in this section.

6. REAL SINGULAR RIEMANN SURFACES

A chord diagram X_S on a Riemann surface S is *real algebraic* if the Riemann surface S is real (i.e., endowed with an antiholomorphic involution $\text{conj} : S \rightarrow S$), and the boundaries of all chords are contained in $\mathbb{R}S = \text{Fix}(S)$. Then we may consider

$$\mathbb{R}X_S = X_S \setminus (S \setminus \mathbb{R}S)$$

(the union of $\mathbb{R}S$ with all the chords). If X_S is nodal then the pair $(\mathbb{R}X_S, \mathbb{R}S)$ is the chord diagram in the sense of Definition 4.5. Otherwise, it can be considered as a degeneration of such diagram.

We say that a real algebraic chord diagram X_S is planar if $\mathbb{R}X$ can be embedded into the disjoint union of disks in such a way that $\mathbb{R}S$ is mapped homeomorphically onto the boundary. Note that in the case when the chord boundaries are disjoint, this definition agrees with Definition 4.6. We have $l + \delta$ planar loops in $\mathbb{R}X_S$ (here $l = b_0(\mathbb{R}S)$) that correspond to the components of the complement of the image of this embedding.

Lemma 6.1. *Suppose that X_S is a real planar chord diagram and $\omega \in \Omega(\Sigma)$ is a real form on S whose zeroes are disjoint from \tilde{N} . Then every planar loop in $\mathbb{R}X_S$ contains an even number of zeroes of ω (counted with multiplicities).*

Proof. In the complement of its zeroes, a real form ω defines an orientation of the underlying curve. Since the residues of ω at the endpoints of each chord are opposite, this orientation agrees with an orientation of the planar loop near the chord. Thus the lemma follows from the orientability of a circle. \square

Corollary 6.2. *An effective real divisor $D = \sum_{j=1}^n a_j p_j$, $a_j \in \mathbb{Z}_{>0}$, $p_j \in \Sigma^*$, is non-special, if X_S is planar and at least $g + \delta$ distinct planar loops of X_S intersect $\text{supp}(D) = \bigcup_{j=1}^n \{p_j\}$.*

Proof. Suppose that $\omega \in \Omega_{-D}(\Sigma)$. If ω does not have poles at the boundary of a chord in X_S then we may remove this chord and the statement follows by induction from the corresponding statement for the resulting diagram in $(\delta - 1)$ chords.

Otherwise ω has the total of $2g + 2\delta - 2$ zeroes in Σ^* (counted with multiplicities). But Lemma 6.1 implies existence of at least $2g + 2\delta$ zeroes contained in the given $g + \delta$ planar loops. \square

Consider a real algebraic nodal curve $A_0 \subset \mathbb{P}^3$ endowed with a fixed orientation of its real branches and its real *equigeneric perturbation*

$A_t \subset \mathbb{P}^3$, (cf. Definition 4.1 but now A_t is not required to be smooth) for small $t > 0$. We say that A_t is *positive* (resp. *negative*) at a node $x \in A_0$ if the perturbed local branches form a positive (resp. negative) crossing according to Figure 1 under a generic projection.

Note that the signs of A_t at all nodes are invariant if we reverse the orientation of A_0 . Thus we may choose any orientation in the case if A_0 is rational. In the case when A_0 is of type I (the case of this paper) we take a complex orientation of $\mathbb{R}A_0$.

Lemma 6.3. *Let $\varphi : S \rightarrow \mathbb{P}^3$ be an analytic mapping which factors through an injective mapping $\Sigma \rightarrow \mathbb{P}^3$ and D be the pull-back of a real plane section. Suppose that $A_0 = \varphi(S)$ is a real nodal curve, $\text{supp}(D) \subset \Sigma^*$, D is non-special in Σ , and $\dim |D|_\Sigma \geq 3$. Let I_1, \dots, I_k be some chords of X_S .*

Then for any orientation on $\mathbb{R}S$ there exists a real equigeneric perturbation A_t of A_0 which has any prescribed signs at the nodes corresponding to I_1, \dots, I_k and which keeps all the other nodes.

Proof. Inductively with the help of Lemma 5.4, it suffices to prove this lemma for $k = 1$, i.e. that we may perturb a single node of A_0 equigenerically, keeping all the other nodes, and choosing any prescribed sign for the perturbation of this node. Since $A_0 \subset \mathbb{P}^3$ is nodal, its projection onto \mathbb{P}^2 from a generic point of \mathbb{P}^3 is a planar nodal curve $B \subset \mathbb{P}^2$. Let $s_0, s_1, s_2, s_3 \in V_D(\Sigma)$ the sections defining the curve A_0 , and such that s_0, s_1, s_2 define the curve B (cf. the proof of Proposition 3.3).

By Corollary 5.3,

$$\dim V_D(\Sigma') > \dim V_D(\Sigma),$$

where Σ' is the singular surface corresponding to the diagram X^{I_1} , i.e. when the chord I_1 is removed. Let $s \in V_D(\Sigma') \setminus V_D(\Sigma)$. Then s_0, s_1, s_2, s define an analytic mapping $S \rightarrow \mathbb{P}^3$ that factorize through Σ' but not through Σ . Thus $s_0, s_1, s_2, s_3 + ts, t \in \mathbb{R}$, define the required perturbation of A_0 where different signs of t correspond to different signs of the node perturbation. \square

7. NODAL HOPF LINKS

Definition 7.1. A nodal MW_λ -link is called a *nodal Hopf link* if it has $d - 3 - g$ nodes.

By (4) the degree of any planar loop in a nodal Hopf link is 1. Conversely, as the number of planar loops is $d - 2$, we get the following characterization of nodal Hopf links.

Proposition 7.2. *A nodal MW_λ -link $L_0 \subset \mathbb{RP}^3$ is a nodal Hopf link if and only if the degree of each planar loop is one.*

The following proposition is an immediate consequence of Proposition 3.3.

Proposition 7.3. *Any nodal MW_λ -link (in particular, a smooth MW_λ -link) can be degenerated to a nodal Hopf link.*

Lemma 7.4. *Let A be a real algebraic curve in \mathbb{P}^3 . Let P be a real plane such that $A \cap P$ is finite. Let $F \subset A \cap P$ be a finite set and let $\lambda_1, \dots, \lambda_s$ be loops contained in $\mathbb{R}A \setminus F$ such that all pairwise intersections $\lambda_i \cap \lambda_j$ are finite, and each loop λ_i is non-trivial in $H_1(\mathbb{RP}^3)$. Then*

$$\deg A \geq s + \sum_{x \in F} (A \cdot P)_x,$$

where $(A \cdot P)_x$ is the local intersection number of A and P at x .

Proof. It is enough to consider a perturbation P' of P in the class of conj-invariant smooth 4-dimensional submanifolds of \mathbb{P}^3 . We may choose P' to coincide with P near F . In addition we may ensure that $P' \cap \mathbb{RP}^3$ is transverse to each λ_j and all local intersections of P' with A are positive. \square

The following proposition generalizes Proposition 2.5. We use the notations of Section 5. Let V_D^\vee be the dual space of V_D and $|D|_\Sigma^\vee$ be its projectivization.

Let X_S be a planar nodal real algebraic chord diagram in $l = g + 1$ circles and δ chords.

Definition 7.5. A *Hopf divisor* in X_S is a real divisor $D \subset \Sigma^*$ of degree $l + \delta + 2$ such that the part of D contained in every planar loop of X_S has odd degree.

In other words, D has a single point at each of $l + \delta$ planar loop of X_S , and also two more points which either form a complex-conjugate pair, or both belong to the same planar loop of X_S . Proposition 7.2 assures that a real plane section of a nodal Hopf link is a Hopf divisor. The next proposition assures the converse.

Recall that a real curve $A_0 \subset \mathbb{P}^3$ is said to be *hyperbolic* with respect to a line $\ell \subset \mathbb{RP}^3$ if for any real plane $P \subset \mathbb{P}^3$ passing through ℓ each intersection point of $P \setminus \ell$ and A_0 is real.

Proposition 7.6. *For a Hopf divisor $D \subset \Sigma^*$ the map*

$$\iota_D : S \rightarrow |D|_\Sigma^\vee \approx \mathbb{P}^3,$$

from (8) is a well-defined immersion whose image is a nodal Hopf link. Furthermore, ι_D factors through an embedding of Σ to $|D|_\Sigma^\vee$.

Proof.

Step 1: ι_D is an immersion to \mathbb{P}^3 . We have $|D|_\Sigma \approx \mathbb{P}^3$ by Corollary 6.2. The same corollary implies that $\dim |D|_\Sigma > |D - x|_\Sigma$ for any $x \in \text{supp } D \cap \Sigma^*$. Also we have

$$\dim |D - \partial I|_{\Sigma'} = \dim |D|_{\Sigma'} - 2 < |D|_\Sigma.$$

Here the equality follows, once again, from Corollary 6.2 (recall that X_S is planar, so both points of ∂I belong to the same component of $\mathbb{R}S$) while the inequality follows from the observation that $|D|_\Sigma$ has only one additional linear constraint in addition to those of $|D|_{\Sigma'}$. Thus ι_D is well-defined, and $A = \iota_D(S) \subset \mathbb{P}^3$ is a real algebraic curve of degree $l + \delta + 2$ with plane sections parameterized by $|D|_\Sigma$.

If ι_D is not an immersion at $x \in S$ then any plane section containing x has multiplicity greater than one at x . Taking the plane through x and a pair of complex conjugate points of A we get a contradiction with Lemma 7.4.

Step 2: A is a nodal curve with exactly δ nodes. If $x \in \mathbb{R}A$ is such that $\iota_D^{-1}(x) \setminus \mathbb{R}S \neq \emptyset$ then the plane through x and a pair of conjugate points of A provides a contradiction with Lemma 7.4. If $x \in A \setminus \mathbb{R}\mathbb{P}^3$ is such that $\#(\iota_D^{-1}(x)) > 1$ then we get a similar contradiction by considering a real plane containing x and $\text{conj}(x)$. This means that $\iota_D|_{S \setminus \mathbb{R}S}$ is an embedding.

Suppose that we have $\iota_D(x) = \iota_D(y)$ for $x \neq y \in \mathbb{R}S$. Consider the plane H through $\iota_D(x)$ tangent to the local branches of S at x and y . The corresponding divisor has multiplicity at least 2 at x and y . If $x, y \in \mathbb{R}S \setminus \tilde{N}$ then by the parity reasons there must be another intersection point of this plane with the planar cycles containing x or y which gives a contradiction. Suppose that $y \in \tilde{N}$, and $y' \neq x$ is the other endpoint of the same chord. Then the divisor cut by H also contains y' . Let us perturb H to a generic plane section with two imaginary points near x . By the degree count no other imaginary points may appear under this perturbation. Thus y and y' produce at least three real intersection points which should be repartitioned among the two planar cycles adjacent to I . Thus at least one planar cycle of $\mathbb{R}X_S$ contains more than a single point of the divisor. Since the divisor has a pair of complex conjugate points, we have a contradiction once again. Thus ι_D identifies only the endpoints of the same chord.

To prove that A is nodal it remains to show that the two branches at each of its singular points are not tangent to the same direction. But

if they were, we could find a plane tangent to one of the branch with order at least 3 and, simultaneously, tangent to the other branch with order of at least 2, and obtain a contradiction with Lemma 7.4.

Step 3: *A is hyperbolic with respect to any real tangent line (for definition, see just before Proposition 7.6).* Using Lemma 7.4 we conclude that a plane tangent to $\mathbb{R}A$ has only real intersections with $\mathbb{R}A$ (cf. the condition (ii) in Theorem 2 of [4]), and also that no plane may be tangent to a local branch of $\mathbb{R}A$ with order greater than 3. The latter condition implies that the (differential geometric) torsion of $\mathbb{R}A$ cannot change sign within the same real branch of A . Reflecting the orientation of $\mathbb{R}\mathbb{P}^3$ if needed, we may assume that $\mathbb{R}A$ has points of positive torsion.

Step 4: *Positivity of the linking number of a pair of oriented tangent lines to $\mathbb{R}A$.* Here we assume the orientation to be compatible with a fixed complex orientation of $\mathbb{R}A$. Due to the hyperbolicity of $\mathbb{R}A$ with respect to tangent lines, this fact follows from [4, Lemma 4.6 and Lemma 4.7] (see, in particular, [4, Figure 2]). In its turn this positivity implies that all points of $\mathbb{R}A$ have positive torsion, and also that all crossing points (in the knot-theoretic sense) of a plane projection of $\mathbb{R}A \subset \mathbb{R}\mathbb{P}^3$ are positive.

Step 5: *There exists a point $p \in \mathbb{R}\mathbb{P}^3$ such that all singularities of $\pi_p(A)$ are ordinary real nodes with real local branches.* The central projection $\pi_p : A \rightarrow \mathbb{P}^2$ from a generic real point p on a tangent line ℓ to $\mathbb{R}A$ at its non-nodal point has a cusp corresponding to ℓ . Furthermore, all singular points of the curve $\pi_p(\mathbb{R}A) \subset \mathbb{R}\mathbb{P}^2$ must be real singularities such that all of its branches are also real. Indeed, the curve $\pi_p(\mathbb{R}A)$ may not have a pair of complex conjugate singularities, as otherwise the inverse image under π_p of the real line through this pair would intersect A at least in 4 imaginary points producing a contradiction with Lemma 7.4. Also the curve $\pi_p(\mathbb{R}A)$ may not have a real singularity q with imaginary branches as otherwise we would get the contradiction to Lemma 7.4 by considering the plane passing through ℓ and $\pi_p^{-1}(q)$.

Consider the central projection $\pi_{p'} : A \rightarrow \mathbb{P}^2$ from a generic real point p' close to p . The image $\pi_{p'}(A)$ is a real nodal curves with the nodes of three types: the nodes of A , the perturbations of the nodes of $\pi_p(A)$ and the node resulting from the cusp of $\pi_p(A)$. Choosing p' in an appropriate way we ensure that the last node has real local branches. Recall that all the nodes of $\pi_p(\mathbb{R}A)$ are positive as knot-theoretical crossing points by the previous step.

Step 6: *A is a nodal Hopf link.* Lemma 6.3 allows us to perturb all the nodes of $\mathbb{R}A$ in a positive way. The result of perturbation has $N_d - g$ positive crossings, and thus is a MW_λ -link. \square

Proposition 7.7. *All nodal Hopf links of the same chirality (the sign of w_λ for a smooth MW_λ -perturbation), and with homeomorphic chord diagrams are rigidly isotopic, i.e. isotopic in the class of real nodal algebraic links of the same degree.*

Proof. A nodal Hopf link is determined by a Hopf divisor on a planar real algebraic chord diagram by Proposition 7.6. If a homeomorphism between the chord diagrams of two nodal Hopf links respects the corresponding Hopf divisors then there exists a 1-parametric family of Hopf divisors on planar real algebraic chord diagrams producing an isotopy between the two nodal Hopf link. Lemmas 7.8 and 7.9 reduce the general case to the case considered above. \square

Let (X, Λ) be a planar chord diagram in l circles with δ chords. We say that a triple (X, Λ, Δ) is a *Hopf triple* if $\Delta \subset \Lambda$ is a set of $\delta + l$ points disjoint from the chord endpoints, and such that each planar loop of (X, Λ) contains a single point of Δ .

Let $I \subset X$ be one of the chords of (X, Λ) . The chord I is adjacent to two plane loops, α_1^I and α_2^I . Let $\Delta' \subset X$ be a set of $l + \delta - 2$ points disjoint from the chord endpoints, and such that each planar loop of (X, Λ) except for α_1^I and α_2^I contains a single point of Δ' .

Let Δ_+ (resp. Δ_-) be the union of Δ' and the two-point set obtained by moving both points of ∂I in the direction (resp. contrary to the direction) of the orientation of Λ . Note that both (X, Λ, Δ_+) and (X, Λ, Δ_-) are Hopf triples. In this case we say that these triples are linked with a *chord move* in I .

Lemma 7.8. *For any two Hopf triples (X, Λ, Δ_\pm) linked with a chord move there exists a nodal Hopf link $A_0 \subset \mathbb{P}^3$ and two generic real planes $H_\pm \in \mathbb{P}^3$ such that $(\mathbb{R}X_{A_0}, \mathbb{R}S, S \cap \mathbb{R}H_\pm)$ is homeomorphic to (X, Λ, Δ_\pm) . Here X_{A_0} is the natural chord diagram on the normalization S of the nodal curve A_0 .*

Proof. Choose a real algebraic realization $(\mathbb{R}X_S, \mathbb{R}S, D')$ of (X, Λ, Δ') . Define D to be the divisor obtained as the union of $D' \cup \partial I$ with a pair Π of complex conjugate points in $S \setminus \mathbb{R}S$. Note that according to Lemma 6.2 the divisor D is non-special in the singular surface Σ^I corresponding to the removal of I from the chord diagram X_S . Thus there exists a real divisor E close to D and Σ^I -equivalent to D such that $E \cap \partial I = \emptyset$ and $(\mathbb{R}X_S, \mathbb{R}S, E \cap \mathbb{R}S)$ is a Hopf triple. Note that

$D \in |E|_\Sigma$ since $D \in |E|_{\Sigma_I}$ and the value of a meromorphic function $f : S \rightarrow \mathbb{P}^1$ with $(f) = D - E$ is zero (and thus is the same) on both points of ∂I .

By Proposition 7.6 $A_0 = \iota_E(S) \subset \mathbb{P}^3$ is a nodal Hopf link. Since $D \in |E|_\Sigma$ there exists a real plane $H \in \mathbb{P}^3$ with $\iota_E^{-1}(H) = D$. In particular, H passes through the node of A_0 corresponding to I . Let H_\pm be the real planes obtained by perturbing H to two different sides of the node. Since the pencil of real planes through Π defines a totally real map $S \setminus \Pi \rightarrow \mathbb{P}^1$ (i.e. the inverse image of $\mathbb{R}\mathbb{P}^1$ coincides with $\mathbb{R}S$), the triples $(\mathbb{R}X_{A_0}, \mathbb{R}S, S \cap \mathbb{R}H_\pm)$ and (X, Λ, Δ_\pm) are homeomorphic. \square

Lemma 7.9. *Any two Hopf triples (X, Λ, D_1) and (X, Λ, D_2) on the same planar chord diagram (X, Λ) can be linked with a sequence of chord moves.*

Proof. Inductively by δ we may assume that the lemma holds for all planar chord diagrams in less than δ chords. Since (X, Λ) is dual to a tree, there exists a chord I dual to a leaf edge of the tree.

This means that there exists a planar loop α adjacent to I and not adjacent to any other chord of (X, Λ) . We apply the induction hypothesis to the diagram obtained by removing I from (X, Λ) and the divisors obtained by removing $\alpha \cap D_j$ from D_j , $j = 1, 2$. \square

Corollary 7.10. *Two MW_λ -links of the same degree and chirality are rigidly isotopic if they can be degenerated to the nodal Hopf link with the same chord diagram.*

Proof. To deduce this statement from Proposition 7.7 it suffices to show that there exists an open neighborhood of the map $\iota_D : S \rightarrow \mathbb{P}^3$ (with δ nodes in the image) in the space of all holomorphic maps $S \rightarrow \mathbb{P}^3$ of the same degree such that the space inside this neighborhood formed by the MW_λ -links is open and connected. Corollary 6.2 implies such connectedness once we perturb only one coordinate of \mathbb{P}^3 leaving the other coordinates (responsible for the planar diagram in $\mathbb{R}\mathbb{P}^2$) unchanged. Combining this with local connectedness of the space of perturbations of the planar diagram we get the statement. \square

8. GENERALIZATION AND PROOF OF THEOREM 3

We say that a degree-chord diagram (X, Λ) , see Definition 4.9, is realized by an MW_λ -link $L_0 \subset \mathbb{R}\mathbb{P}^3$ if $X(L_0)$ is homeomorphic to (X, Λ) as the chord diagram and a generic real plane section of L_0 containing a pair of complex conjugate point has $a(\lambda)$ points on each planar loop λ in $X(L_0)$. Recall that $a(\lambda)$ does not depend on the choice of the plane section.

The following theorem generalizes Theorem 3 to nodal MW_λ -links. Thus its proof also provides a proof of Theorem 3.

Theorem 4. *The nodal MW_λ -links of degree d and genus g are classified up to rigid isotopy by the degree-chord diagrams.*

Namely, any planar degree-chord diagram (X, Λ) in $l = g + 1$ circles with δ chords and the degree function on the planar loops satisfying to (4) is realized by a nodal MW_λ -link of genus g and degree d . Any two nodal MW_λ -links with the same degree-chord diagram and the same chirality (i.e. the sign of w_λ for its smooth MW_λ -perturbation) are rigidly isotopic.

Suppose that (X, Λ) is a planar degree-chord diagram of nodal Hopf link (i.e. with the degree of each planar loop equal to 1). Consider the diagram (Y, Λ) obtained from (X, Λ) by removing some chords. Then (Y, Λ) is also a planar chord diagram. A planar loop λ of Y is composed of several planar loops of X . We set the degree of λ to be the number of such planar loops in X .

Lemma 8.1. *The degree-chord diagram (Y, Λ) is realizable as the result of perturbation of a nodal Hopf link with the chord diagram (X, Λ) .*

Proof. Consider a real algebraic chord diagram X_S corresponding to (X, Λ) , and a divisor D with a pair of complex conjugate points and a single non-nodal point at each of the planar loops of X_S . Let $\iota_D : S \rightarrow |D|_\Sigma^\vee \approx \mathbb{P}^3$ be the corresponding map (as in (8)). As in the proof of Proposition 7.6, by Corollary 6.2, Lemma 5.4, and Lemma 6.3 the nodes corresponding to the chords missing in (Y, Λ) may be perturbed in a positive way keeping the other nodes. The result is a nodal MW_λ -link L_Y as it could be further perturbed to a smooth MW_λ -link. The number of points of L_Y in its plane section near D agrees with our definition of $a(\lambda)$. \square

Suppose (X, Λ) is a planar chord diagram, and $x, y, z \in \Lambda$ be three distinct points different from the endpoints of the chords and contained in a single planar loop λ of X in the order agreeing with the cyclic orientation of λ . Consider a point y' on λ close to y and a point z' on λ close to z so that the cyclic order of x, y, y', z, z' agrees with that on λ .

Form a planar diagram (Y, Λ) by attaching two new chords to X : the one connecting x and y and the one connecting y' and z , see Figure 3. Also form a planar diagram (Z, Λ) by attaching two new chords to X : the one connecting x and z' and the one connecting y and z . In this case we say that the chord diagrams (Y, Λ) and (Z, Λ) differ by a *chord slide*.

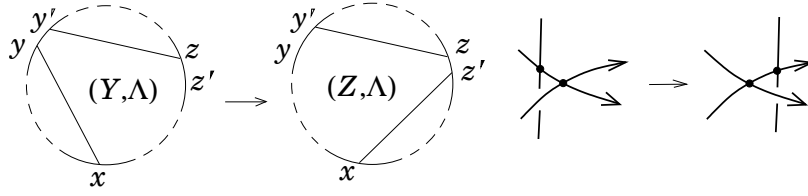


FIGURE 3. The chord diagrams (Y, Λ) , (Z, Λ) and the resulting modification of nodal links.

Let (X', Λ) be the plane diagram obtained from (Y, Λ) by removing the chord $[xy]$ from (Y, Λ) (which results in the same diagram as removing the chord $[xz']$ from (Z, Λ)). One of the planar loop of X' corresponds to two planar loops of Y (or of Z). We set its degree equal to 2. All other planar loops have degree 1. This turns (X', Λ) into a degree-chord diagram. We consider the diagrams (Y, Λ) and (Z, Λ) as the degree-chord diagrams with the degrees of all planar loops equal to 1.

Lemma 8.2. *Suppose that two nodal MW_λ -links $L_y, L_z \subset \mathbb{RP}^3$ both have the degree-chord diagram (X', Λ) , and degenerate to nodal Hopf links with the chord diagrams (Y, Λ) and (Z, Λ) . Then L_y and L_z are rigidly isotopic.*

Proof. Choose a real Riemann surface S of genus g with $l = g + 1$ real branches, and an orientation-preserving homeomorphism between $\mathbb{R}S$ and Λ . Consider the real algebraic chord diagram Y_S^0 on S obtained by attaching all chords of the diagram (X, Λ) , a chord connecting x and y , and a chord connecting y and z . Also consider the real algebraic chord diagram Z_S^0 on S obtained by attaching all chords of the diagram (X, Λ) , a chord connecting x and z , and a chord connecting y and z . Note that both Y_S^0 and Z_S^0 are planar diagrams which are not nodal. Also note that that Y_S^0 and Z_S^0 can be perturbed to nodal diagrams Y_S and Z_S from Figure 3.

Choose a divisor on D of degree $l + \delta + 2$ consisting of a pair of complex conjugate points on S and a single non-nodal point at each planar cycle of $\mathbb{R}Y_S^0$. Note that then the same divisor will have a single point at each planar cycle of $\mathbb{R}Z_S^0$.

Real algebraic chord diagrams Y_S^0 and Z_S^0 define the same map $\iota_D : S \rightarrow |D|_\Sigma \approx \mathbb{P}^3$ since the linear system $|D|_\Sigma$ is the same for both diagrams: the functions in V_D have the same values at x , y and z . The image $A = \iota_D(S)$ has a triple point with three real branches at $\tau = \iota_D(x) = \iota_D(y) = \iota_D(z)$, the rest of the singularities are nodes

corresponding to the chords of (X, Λ) . This can be seen in the same way as in the proof of Proposition 7.6.

We claim that all three local branches of A cannot be tangent to the same plane, as otherwise the corresponding plane section has a tangency of order 6 that must be repartitioned to the planar cycles. There are $l + g - 3$ planar cycles which are not adjacent to the points $x, y, z \in Y_S^0$. As the degree of each of them is 1, and the degree of A is $l + g + 2$ we get a contradiction. Thus we may choose a generic point $p \in \mathbb{RP}^3$ so that the local intersection sign of the y -branch and the z -branch with the x -branch of $\pi_p \circ \iota_D(\mathbb{R}S) \subset \mathbb{RP}^2$ coincide, see Figure 4. As in the proof of Proposition 7.6 we change the coordinates in \mathbb{RP}^3

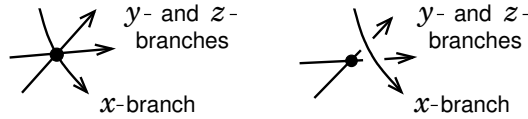


FIGURE 4. Perturbation of the triple point.

so that $p = (0 : 0 : 0 : 1)$.

Let Σ' be the singular surface obtained by contracting S along each chord of (X', Λ) . Using Corollary 6.2 and Lemma 5.4 (cf. Lemma 6.3) we may perturb the map ι_D keeping the image of the curve under π_p (the plane diagram) so that it factors through an embedding of Σ' , and so that the new crossing points resulting from the perturbation are positive, see Figure 4. Performing the same perturbation for a family of chord diagrams connecting Y_S^0 to Y_S and Z_S^0 to Z_S we obtain an isotopy between MW_λ -links degenerating to the nodal Hopf links with the chord diagrams (Y, Λ) and (Z, Λ) . \square

Proof of Theorem 4. If we have two nodal MW_λ -links corresponding to the same planar degree-chord diagram (X, Λ) then both of them degenerate to nodal Hopf links as in Proposition 3.3. The chord diagrams of these nodal Hopf links must contain (X, Λ) as a subdiagram so that the degree of a planar cycle in X equals to the number of the planar cycles that appear in the subdivisions of this cycle in the larger diagrams. But then the two larger chord diagrams can be connected with a sequence of chord slides. Theorem now follows from Lemma 8.2. \square

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