ON RIGID RATIONAL CUSPIDAL PLANE CURVES

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Let Y be a smooth affine algebraic surface over $\mathbb{C}$. Suppose that it is $\mathbb{Q}$-acyclic, i.e. $H_i(Y;\mathbb{Q}) = 0$, $i > 0$, and that it is of log-general type, i.e. $\bar{k}(Y) = 2$ where $\bar{k}$ is the logarithmic Kodaira dimension (see [4], [5], [7]). In [2], it is asked if such a surface is necessarily rigid. The rigidity means that $h^1(\Theta) = 0$ where X is the minimal smooth completing of Y by a simple normal crossing divisor (SNC-divisor) D, and $\Theta = \Theta_X\langle D\rangle$ is the logarithmic tangent bundle of X along D. If $\bar{k} = 2$ then all known $\mathbb{Q}$-acyclic surfaces are rigid. For $\chi = h^0 - h^1 + h^2$, one has $\chi(\Theta) = (K_X + D)^2 + 2$ (see [2, Lemma 1.3(5)]).

1. Statement of the result. Consider a plane irreducible curve D. It is easy to see that $Y = \mathbb{P}^2 \setminus D$ is $\mathbb{Q}$-acyclic if and only if D is rational and cuspidal (we say that a curve is *cuspidal*, if all its singularities are *cusps*, i.e. analytically irreducible). If D has at least three cusps then $\bar{k}(Y) = 2$ (see [9]). The rigidity of Y is equivalent to the projective rigidity of D: any embedded equisingular deformation is projectively equivalent to D [3, (2.1)]. The rigidity has place in all known examples [3]. This note is devoted to a proof of the following fact.

Proposition (1.1). (see [8]). *A projectively rigid rational cuspidal curve in $\mathbb{P}^2$ has at most 9 cusps.*

2. Logarithmic Bogomolov-Miyaoka-Yau (log-BMY) inequality. Let D be an SNC-curve on a smooth projective surface X, and let $Y = X \setminus D$. If $\bar{k}(Y) \geq 0$ then there exists the *Zariski decomposition* $K + D = H + N$ where H, N are $\mathbb{Q}$-divisors on X such that (i) the intersection form is negative definite on the subspace V_N generated by irreducible components of N (in particular, $N^2 \leq 0$); (ii) $HC \geq 0$ for each irreducible curve $C \subset X$; (iii) H is orthogonal to V_N (hence, $(K + D)^2 = H^2 + N^2$).

Theorem (2.1). [7], [5]. *If $\bar{k}(Y) = 2$ then $H^2 \leq 3e(Y)$ where e is the Euler characteristic.*

3. Dual graph. Let E be an SNC-curve on a smooth surface whose irreducible components are $E_1, \dots, E_k$. Let $A_E = (E_i \cdot E_j)_{ij}$ be its intersection matrix. This is the same as the incidence matrix of the *dual graph* Γ_E of E. Its vertices correspond to the irreducible components of E and the edges correspond to their intersection points; the weight of a vertex is defined as the self-intersection number of the corresponding component. Set $d(\Gamma_E) = \det(-A_E)$. An extremal linear branch of a graph will be called a *twig*. Let us denote the endpoint of a twig T by $\mathrm{tip}(T)$. The *inductance* of a twig T is $\mathrm{ind}(T) = d(T - \mathrm{tip}(T))/d(T)$. Applying Cramer's rule, we get the following lemma.

Lemma (3.1). *If Γ is a weighted tree with $d(\Gamma) \neq 0$ and $B = (b_{ij}) = A^{-1}$ where A is the incidence matrix then $b_{ij} = -d(\Gamma - [ij])/d(\Gamma)$ where $[ij]$ is the minimal subgraph containing the i-th and the j-th vertex.*

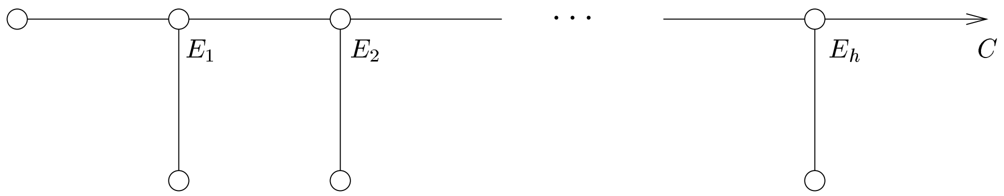
Combining (3.1) with Jacobi's formula for a minor of the inverse matrix, applied to the 2×2-minor corresponding to the vertices v and v_0, we get one more lemma:

Lemma (3.2). *Let Γ be a weighted tree, T its twig incident to $v_0 \in \Gamma - T$, and $v = \mathrm{tip}(T)$. Set $d_T(\Gamma) = d(\Gamma - T - v_0)$. Then $d_T(\Gamma) = d(\Gamma - v)d(T) - d(\Gamma)d(T - v)$.*

Corollary (3.3). *If* $d(\Gamma) = 1$ *and* $d(T) \neq 0$ *then* $\operatorname{ind}(T) =]d_T(\Gamma)/d(T)[$. (*Here* $]a[$ *denotes* $[a] - a$ *where* $[a] := \min\{n \in \mathbb{Z} \mid n \geq a\}$.)

4. Puiseux expansion and dual graph of resolution. Let C be a germ of a plane irreducible analytic curve at a singular point p, and let $E = \bigcup E_i$ be the exceptional curve of a minimal resolution of the singularity. Let Γ be the dual graph of $E \cup C$. In suitable analytic coordinates, C has the form $x = t^n$, $y = a_m t^m + a_{m+1} t^{m+1} + \dots$. Set $d_1 = n$, $m_i = \min\{j \mid a_j \neq 0$ and $j \neq 0 \mod d_i\}$, $d_{i+1} = \gcd(d_i, m_i)$. Let h be such that $d_h \neq 1$, $d_{h+1} = 1$. Set $r_1 = m_1$; $r_i = r_{i-1} d_{i-1}/d_i + m_i - m_{i-1}$ for $i > 1$.

Proposition (4.1). [1] (a). *The graph* Γ *has the form*



where edges depict linear chains of vertices.

(b) Let R_i, D_i, *and* S_i *be the connected components of* $\Gamma - E_i$ *respectively to the left, to the bottom, and to the right of* E_i. *Then* $d(R_i) = r_i/d_{i+1}$, $d(D_i) = d_i/d_{i+1}$, $d(S_i) = 1$.

Denote by n_p the sum of the inductances of all twigs of Γ, not containing C.

Corollary (4.2). $n_p =]d_1/r_1[+ \sum_{i=1}^{h}]r_i/d_i[> 1/2$.

Proof. Since $d(\Gamma) = 1$, the required equality follows from (3.3). Hence, $n_p \geq]d_1/r_1[+]r_1/d_1[$. It is clear that if $0 < x < 1$, $x \neq 1/2$ then $]x[+]1/x[> 1/2$.

5. Let D be a rational cuspidal curve in $\mathbb{P}^2$, and $\sigma : X \to \mathbb{P}^2$ be the minimal resolution of singularities of D, i.e. $\tilde{D} = \sigma^{-1}(D)$ is an SNC-divisor and $X \setminus \tilde{D} = Y$. Let $K + \tilde{D} = H + N$ be the Zariski decomposition. Denote: $S = \operatorname{Sing}(D)$, $s = \#S$.

Lemma (5.1). *If* $s \geq 3$ *then* $-N^2 = \sum_{p \in S} n_p$.

Proof. The surface Y is $\mathbb{Q}$-acyclic, moreover, by [9] we have $\bar{k}(Y) = 2$. Therefore, Y does not contain any simply connected curve [10], [6]. Since $s \geq 3$, the graph $\Gamma_{\tilde{D}}$ has at least three brancings. Under these conditions, the statement of the lemma is proved in [4, (6.20)–(6.24)].

Proof of (1.1). Since $\bar{k}(Y) = 2$ (see [9]), log-BMY inequality (2.1) implies $H^2 \leq 3$, hence, by (5.1) and (4.2), we have $(K + \tilde{D})^2 = H^2 - \sum n_p < 3 - s/2$. Let $h^i = h^i(\Theta_X \langle \tilde{D} \rangle)$. Since D is supposed to be rigid, i.e. $h^1 = 0$, we have $(K + \tilde{D})^2 + 2 = \chi(\Theta_X \langle \tilde{D} \rangle) = h^0 + h^2 \geq 0$. Therefore, $s < 6 - 2(K + \tilde{D})^2 \leq 10$.

REFRERENCES

[**1**] *Eisenbud D., Neumann W.D.* // Ann. Math. Stud. V. 110. Princeton: Princeton Univ. Press, 1985. [**2**] *Flenner H., Zaidenberg M.* // Contemporary Math. 1994. V. 162. P. 143–208. [**3**] *Flenner H., Zaidenberg M.* // On a class of rational cuspidal plane curves // Preprint, 1995. P. 1–28. [**4**] *Fujita T.* // J. Fac. Sci. Univ. Tokyo (Ser. 1A). 1982. V. 29. P. 505–566. [**5**] *Kobayashi R., Nakamura S., Sakai F.* // Proc. Japan Acad. 1989. V. 65(A). P. 238–241. [**6**] *Miyanishi M., Tsunoda S.* // J. Math. Kyoto Univ. 1992. V. 32. P. 443–450. [**7**] *Miyaoka Y.* // Math. Ann. 1984. V. 268. P. 159–171. [**8**] *Orevkov S.Yu., Zaidenberg M.G.* // Algebraic Geometry. Proc. Conf., Santama Univ., March 15–17, 1995 (to appear). [**9**] *Wakabayashi I.* // Proc. Japan Acad. 1978. V. 54(A). P. 157–162. [**10**] *Zaidenberg M.G.* // Izv. AN SSSR, Ser. Matem. 1987. V. 51. P. 534–567; 1991. V. 55. P. 444–446. (in Russian); *English transl.*: Math. USSR-Izv. 1988. V. 30. p. 503–532; 1992. V. 38. p. 435–437.

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