MARKOV MOVES FOR QUASIPOSITIVE BRAIDS

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Abstract. Let $B_m = \langle \sigma_1, \dots, \sigma_m \,|\, \sigma_j\sigma_{j+1}\sigma_j = \sigma_{j+1}\sigma_j\sigma_{j+1},\ [\sigma_j, \sigma_k] = 1$ for $|k - j| > 1\rangle$ be the braid group. A braid b is called *quasipositive* if it has the form $b = (a_1\sigma_1 a_1^{-1}) \dots (a_k\sigma_1 a_k^{-1})$. Using Gromov's theory of pseudo holomorphic curves, we prove that $b \in B_m$ *is quasipositive if and only if* $b\sigma_m \in B_{m+1}$ *is quasipositive.*

Let B_m denote the group of braids with m strings (m-braids). Recall that it is defined by generators $\sigma_1, \dots, \sigma_{m-1}$ and relations $[\sigma_j, \sigma_k] = 1$ for $|k - j| > 1$ and $\sigma_j\sigma_k\sigma_j = \sigma_k\sigma_j\sigma_k$ for $|k - j| = 1$. An m-braid b is called *quasipositive* (see [6]) if $b = \prod_{j=1}^{k} a_j\sigma_1 a_j^{-1}$ for some braids $a_j \in B_m$ (recall that all standard generators are conjugated).

One says that $b' \in B_{m+1}$ is obtained from $b \in B_m$ by a *Markov move* if $b' = b\sigma_m^{\varepsilon}$ for $\varepsilon = \pm 1$ (we identify here B_m with the subgroup of B_{m+1} generated by $\sigma_1, \dots, \sigma_{m-1}$). Say that the Markov move is *positive* if $\varepsilon = 1$ and *negative* otherwise. It follows immediately from the definitions that a braid is quasipositive if it is obtained from a quasipositive braid by a positive Markov move. Here we prove the converse.

Theorem 1. *Let $b' \in B_{m+1}$ be obtained from b by a positive Markov move. If b' is quasipositive then b is also quasipositive.*

This is a pure existence theorem. Our proof is based on Gromov's theory of pseudo holomorphic curves [2] and it is absolutely non-constructive: we do not know an algorithm to find a quasipositive presentation of b starting with a quasipositive presentation of b'.

Theorem 1 can be applied to the study of the topology of plane real algebraic curves because the realisability of an arrangement of ovals by an algebraic curve of a given degree implies that a certain braid is quasipositive (see [5]). In another paper we shall give examples of such applications.

The idea of the proof also came from the topology of real algebraic curves. It was the result of the following observation. Let $C \subset \mathbf{RP}^2$ be an algebraic curve and A is its small convex arc whose convex hull contains a point p and does not contain other parts of C. Then the braid corresponding to the projection from p is obtained by a positive Markov move from the braid corresponding to the projection from a point on A. In the proof of Theorem 1 we simulate this situation by pseudo-holomorphic curves.

Remark. If b' is obtained from any braid b by a negative Markov move then b' is never quasipositive. This is an immediate consequence of the following result of

Burckel [1] and Laver [3] (a geometric proof was given by Wiest [7]): *any conjugate of a quasipositive braid is positive in Dehornoy's right-invariant order.*

Say that an oriented link in 3-sphere is a *quasipositive link* if it is isotopic to the closure of a quasipositive braid.

Question 1. Let L be a quasipositive link and b a braid representing L with the minimal possible number of strings. Is it true that b is quasipositive?

In the case when b is a 2-braid (i.e. the braid index of L is 2), the affirmative answer follows immediately from the fact that for a non-slice link L, at most one of L and its mirror image can be quasipositive (see [6]).

Question 2. Let b_1 and b_2 be two quasipositive braids representing the same link. Is it always possible to pass from b_1 to b_2 using only conjugations and positive Markov move?

Recall that a smooth oriented 2-surface F in a smooth symplectic 4-manifold (X,ω) is called *symplectic* if $j^*(\omega)$ is positive on F where j is the embedding $F\subset X$. Let (z,w), $z=x+iy$, $w=u+iv$, $i=\sqrt{-1}$, be coordinates in $\mathbf{C}^2$ and ω_0 the standard symplectic form $\omega_0=dx\wedge dy+du\wedge dv$.

Lemma. *Let $F\in\mathbf{C}^2$ be the graph of a smooth function $w=f(z)=u(z)+iv(z)$ defined in a domain $D\subset\mathbf{C}$. If $|u'_x|,|u'_y|,|v'_x|,|v'_y|<1/\sqrt{2}$ then F is symplectic.*

Proof. Let $p(z,w)=z$. Then $((p|_F)^{-1})^*(\omega_0)=(1+u'_xv'_y-u'_yv'_x)dx\wedge dy$. $\square$

We shall consider m-braids as isotopy classes of m-valued functions $f:[0,1]\to\mathbf{C}$ such that each of $f(0)$ and $f(1)$ is a set of m points with distinct real parts. The generator $\sigma_k\in B_m$ is represented by $t\mapsto\{1,\dots,k-1,k+(1\pm e^{\pi it})/2,k+1,\dots,m\}$.

Proof of Theorem 1. Denote the conic $w^2-2w=z^2$ by H and let $p=(0,2)\in H$. Denote the cylinder $\{(z,w)\mid \operatorname{Im}z\ge 0,\ |z|\le 3\}$ by C_∞. Let D be the half-disk $C_\infty\cap\{w=0\}$ and $C_p=\{(z,w)\mid \operatorname{Im}\big(2z/(2-w)\big)\ge 0,\ |2z/(2-w)|\le 3\}$ the cone over D with the vertex p (the union of all complex lines (pp'), $p'\in D$).

H is the graph of the 2-valued function $w=1\pm\sqrt{1+z^2}$. It has two points of ramification $z=\pm i$. Let h_- and h_+ be the single-valued branches in the band $\{|\operatorname{Im}z|<1\}$ such that $h_-(0)=0$ and $h_+(0)=2$. Denote also the graphs of h_- and h_+ by H_- and H_+.

Denote the cylinder $\{(z,w)\mid \operatorname{Im}z=0\,,|z|\le 1,\ |w|\le 1\}$ by U. Let us fix a geometric realization $B\subset U$ of the braid $\sigma_1^{-1}\sigma_2^{-1}\dots\sigma_{m-1}^{-1}b\,\sigma_{m-1}^{-1}\dots\sigma_2^{-1}\sigma_1^{-1}$ such that $B\cap\big(\{\pm1\}\times\mathbf{C}\big)=\{\pm1\}\times(\{-1\}\cup X)$ where X is a finite set $\{x_2,\dots,x_m\}\subset[-1/2,1/2]$. Thus, the "lower corners" $(-1,-1)$ and $(1,-1)$ belong to B (they correspond to the first string of the braid).

For a real $\varepsilon>0$, let U_ε, and B_ε be the images of U, and B under the linear transformation $(z,w)\mapsto\big(\varepsilon z/2\,,\ -h_-(\varepsilon/2)\,w\big)$, and let $X_\varepsilon=\{x_{2,\varepsilon},\dots,x_{m,\varepsilon}\}$ where $x_{j,\varepsilon}=-h_-(\varepsilon/2)x_j$ (so, we place the "lower corners" of U_ε onto the lower branch of H). B_ε is the graph of an m-valued function $w=f(z)$ defined on the segment $[-\varepsilon/2,\varepsilon/2]$. Let us continue f to the rectangle $R=\{|\operatorname{Re}z|\le\varepsilon,\ |\operatorname{Im}z|\le\varepsilon\}$. If $|\operatorname{Re}z|\le\varepsilon/2$, we put $f(z)=f(\operatorname{Re}z)$. If $|\operatorname{Re}z|\ge\varepsilon/2$, we put $f(z)=\{f_1(z),x_{2,\varepsilon},\dots,x_{m,\varepsilon}\}$. where $f_1(z)=h_-(z)$ for $\operatorname{Re}z=\pm\varepsilon$, $f_1(z)=h_-(\varepsilon/2)$ for $\operatorname{Re}z=\pm\varepsilon/2$, and f_1 is linear on each segment $[\pm\varepsilon+yi,\ \pm\varepsilon/2+yi]$ with $|y|\le\varepsilon$. For small z, we have $h_-(z)=-z^2+o(z^2)$. Hence, for $\varepsilon\ll 1$, the branches of f do not meet each other (see Figure 1).

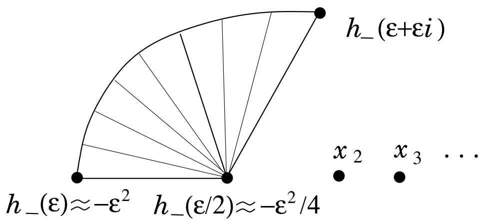

FIGURE 1

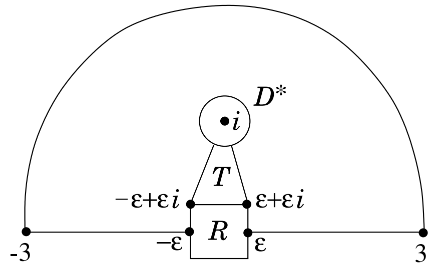

FIGURE 2

Let $E_\varepsilon = R \times \{|w| \leq \varepsilon\}$ and let $F_\varepsilon \subset E_\varepsilon$ be the graph of f. When $\varepsilon \to 0$, the height of U_ε decreases faster than the width. Hence, $\max_{z\in R} |f'(z)| \to 0$, and one can choose ε so small that the complexifications of real lines passing through p meet F_ε transversally and cut from it a braid B^p_ε which is equivalent to B. For such ε, the graph of the restriction of f onto the upper side of R is contained in C_p.

Put $L = \mathbf{C} \times X_\varepsilon$ and $\bar{B}_\varepsilon = B_\varepsilon \cup \big(((H \cup L) \cap \partial C_\infty) \setminus U_\varepsilon\big)$. In other words, $\bar{B}$ is the graph of an $(m+1)$-valued function on the closed curve ∂D. This function is defined as $\{f, h_+\}$ on $[-\varepsilon, \varepsilon]$ and $\{h_+, h_-, x_{2,\varepsilon}, \dots, x_{m,\varepsilon}\}$ outside $[-\varepsilon, \varepsilon]$. It is easy to check that the braid represented by $\bar{B}$ is conjugated to $b' = b\sigma_m$ (see Figure 3).

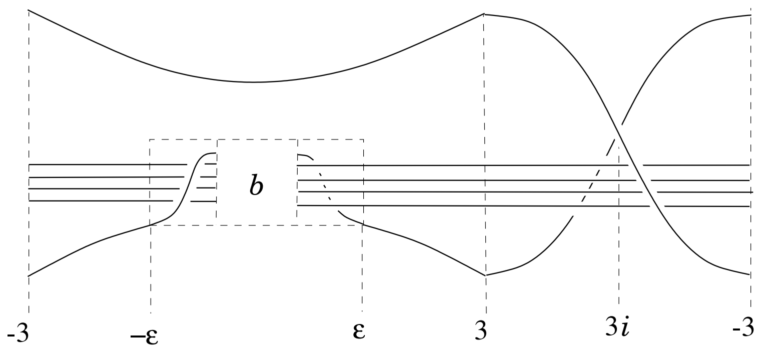

FIGURE 3

Now, let us construct a symplectic surface F in $C_\infty \cup C_p$ such that $F \cap \partial C_\infty = \bar{B}$ and $F \cap \partial C_p$ represents b.

For $z \in \mathbf{C}$, let $C_p(z) = \{w \,|\, (z, w) \in C_p\}$ be the fiber of C_p over z. We have

$$\begin{aligned} C_p(z) &= \{w \,|\; \operatorname{Im} \bar{z}(w-2) \geq 0,\ |2z| \leq 3 \cdot |2-w|\} \\ &= \{w \,|\; \operatorname{Arg}(w-2) \in [\theta, \pi+\theta],\ |w-2| \geq 2r/3\}, \qquad z = re^{\theta i}. \end{aligned}$$

In Figure 4, the domain $C_p(z)$ is shaded.

The vertical line $\{z = i\}$ is tangent to H at $q = (i, 1)$. It is easy to check (see Figure 4) that there exist r^*, $1 < r^* < 2$ and a small disk $D^* \subset \mathbf{C}$ centred at i such that $(D^* \times \mathbf{C}) \cap (H \cup L) \subset E^* \subset C_p$ where $E^* = D^* \times \{|w| \leq r^*\}$. Let $T \subset \mathbf{C}$ be the triangle with vertices $-\varepsilon + \varepsilon i$, $\varepsilon + \varepsilon i$, i (see Figure 2). Denote $E_T = (T \setminus D^*) \times \{|w| <\leq 1\}$ and $E = E_\varepsilon \cup E_T \cup E^*$. One can check (see Figure 4) that $E_T \cup E^* \subset C_p$.

Define F as $F_\varepsilon \cup F_T \cup F^* \cup \big((H \cup L) \setminus E\big)$ where $F_\varepsilon \subset E_\varepsilon$ was constructed above and the surfaces $F_T \subset E_T$ and $F^* \subset E^*$ are as follows.

It is easy to check that $|h_-(z)| < 1$ for $z \in T$ (see Figure 5). Hence, the part of $F \setminus H_+$ which is already constructed, defines pairwise disjoint sections over $(\partial T) \setminus D^*$ of the trivial fibration $E^* \to T \setminus D^*$, $(z, w) \mapsto z$. They can be extended to pairwise non-intersecting global sections. Let us define F_T as the union of their graphs. Note that $F_T \cap H_+ = \varnothing$ because H_+ is outside E_T (see Figure 5).

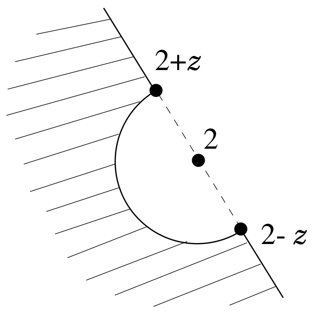

FIGURE 4

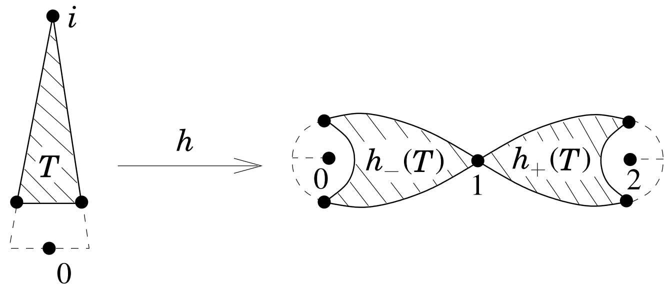

FIGURE 5

Let us construct F^*. Since $F \cap \big((D \setminus D^*) \times \mathbf{C}\big)$ is unbranched over $D \setminus D^*$, its boundary braid over ∂D^* is b'. Let $\prod_{j=1}^{k} a_j \sigma_1 a_j^{-1}$ be a quasipositive presentation of b' (with respect to some base point z_0). Inside D^*, choose distinct points $z_1, \dots, z_k$ and paths α_j connecting z_0 to some $z_j' \in \partial D_j$ where $D_j = \{|z - z_j| \le \delta^2\}$. Define $F_j = F \cap (D_j \times \mathbf{C})$ as the graph of the $(m+1)$-valued function $z \mapsto \{\pm\sqrt{z - z_j}\,,\ 2\delta, \dots, m\delta\}$ on D_j. When $\delta \ll 1$, we have $F_j \subset E^*$.

Note that each $F_j \cap (\partial D_j \times \mathbf{C})$ represents the braid $\sigma_1 \in B_{m+1}$. Let us define the sets $A_j = F \cup (\alpha_j \times \mathbf{C})$ so that they geometrically represent the braids a_j and define the part of F over $D^* \setminus \bigcup (D_j \cup \alpha_j)$ as the graph of an isotopy between b' and $\prod_{j=1}^{k} a_j \sigma_1 a_j^{-1}$. Since both braids are inside E^*, the isotopy can be chosen also inside E^*.

Let $z_* = 2z/(2-w)$, $w_* = w/(2-w)$ be the affine coordinates in $\mathbf{CP}^2$ where C_p is the cylinder $\{(z_*, w_*) \mid \operatorname{Im} z_* \ge 0,\ |z_*| \le 3\}$ over D. Then F defines an m-valued function $w_*(z_*)$ on ∂D. The corresponding braid is b. Indeed, the branches of $F \cap \partial C_p$ close to ∂D are isotopic to those of $F \cap \partial C_\infty$, and H becomes a parabola $w_* = -z_*^2/4$, hence, the braid looks as in Figure 6.

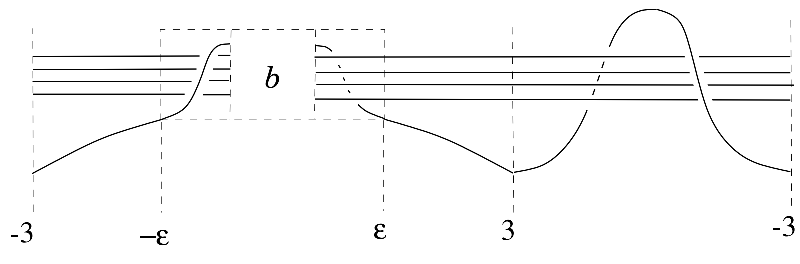

FIGURE 6

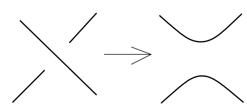

FIGURE 7

Let us describe what happens with the braid when the projection point p_t moves continuously from $(0:1:0)$ to p along the segment $z = \operatorname{Re} w = 0$, $\operatorname{Im} w \ge 2$ (the segment $z_* = \operatorname{Re} w_* = 0$, $\operatorname{Im} z_* \le 1$ in coordinates $(z_*\,, w_*)$). The two branches of H over the point $q_0 = (3i, 0)$ approache each other and at the moment when the line (p_t, q_0) is tangent to H, they bifurcate as in Figure 7. After the bifurcation, σ_m disappears and the $(m+1)$-th string turns into a circle non-linked with the rest of the braid. Then this circle mounts and goes away to infinity.

Let us smooth the constructed surface F in small neighbourhoods of its non-smooth points. The surface $F \subset (C_\infty \cap C_p)$ contains a complex analytic part F_{an} such that $F \setminus F_{an}$ is compact and unbranched with respect to the projection $(z, w) \mapsto z$. Hence, $A(F)$ is symplectic by Lemma where $A(z, w) = (z, aw)$, $0 < a \ll 1$, and $B(F)$ is symplectic with respect to Fubini-Studi symplectic form ω_{FS} on $\mathbf{CP}^2$ where $B(z, w) = (bz, aw)$, $0 < a \ll b \ll 1$ (we suppose that $\mathbf{C}^2$ is embedded to $\mathbf{CP}^2$ by $(z, w) \mapsto (z : w : 1)$). Thus, the closure of F in $\mathbf{CP}^2$ is symplectic with respect to $\omega = B^*(\omega_{FS})$.

Let us choose an almost complex structure J, tamed by ω (see [2]), such that F and all lines (pp') for $p' \in \partial D$ are J-holomorphic. This is possible because all these lines meet F transversally and the intersections are positive. By the results of Gromov [2], $C_p \setminus \{p\}$ is fibered over D by J-holomorphic lines passing through p. Since F is J-holomorphic, its projection onto D along the fibers is a branched covering which has only positive ramifications (see [4][1]). Hence, b is quasipositive. Theorem 1 is proven.

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