ALGORITHMIC RECOGNITION OF QUASIPOSITIVE 4-BRAIDS OF ALGEBRAIC LENGTH THREE

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Abstract. We give an algorithm to decide whether a given braid with four strings is a product of three factors which are conjugates of standard generators of the braid group. The algorithm is of polynomial time. It is based on the Garside theory. We give also a polynomial algorithm to decide if a given braid with any number of strings is a product of two factors which are conjugates of given powers of the standard generators (in my previous paper this problem was solved without polynomial estimates).

1. Introduction and statement of main results

In this paper we continue the study started in [18] and [20]. Let G be a Garside group with set of atoms $\mathcal{A}$, for example, $G = \mathrm{Br}_n$ – the braid group and $\mathcal{A} = \{\sigma_1, \dots, \sigma_{n-1}\}$ – the set of its standard generators (called also Artin generators). Recall that Br_n is generated by $\mathcal{A}$ subject to the relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j|>1; \qquad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ for } |i-j|=1.$$

If an element of G is a product of conjugates of atoms, we say that it is $\mathcal{A}$-*quasipositive* or just *quasipositive* when it is clear which $\mathcal{A}$ is meant. Note that for Artin-Tits groups (in particular, for braid groups) the notion of quasipositivity does not depend on the choice between the standard or the dual Garside structure. We are looking for a solution to the *Quasipositivity Problem* – the algorithmic problem to decide whether a given element of G is quasipositive or not. This problem arises in the study of plane complex algebraic or pseudoholomorphic curves, see, e. g., [22, 6, 15–17, 19].

Let $e : G \to \mathbb{Z}$ be the homomorphism which takes all atoms to 1. The value $e(X)$ is called the *algebraic length* or *exponent sum* of X. The quasipositivity problem for n-braids is solved in [18] for $n = 3$ and in [20] for any n but only for braids of algebraic length two. Note that the case $n < 3$ is trivial and the case $e(X) < 2$ is the simplest particular case of the conjugacy problem. The case $n = 4$, $e(X) = 3$ is done in the present paper, see Theorem 1.4.

In fact, a slightly more general problem is solved in [20]. We found an algorithm to decide whether a given braid X is a product of two conjugates of atom powers. The algorithm in [20] is rather efficient in practice but no polynomial time bounds are known for it. Here we give a polynomial time solution to this problem; in the case of braid groups, it is also polynomial with respect to the number of strings. Namely, Theorem 1.1 states that if X is a product of two conjugates of atom

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX powers, then each element of the super summit set SSS(X) for the Birman–Ko–Lee Garside structure satisfies a certain quickly checkable condition (see Corollary 1.2 and Proposition 3.10), and it is known [5] that an element of SSS(X) can be computed in polynomial time.

Theorem 1.1 also plays a central role in our proof of Theorem 1.4 (the main result of the paper) which states that if a 4-braid X with $e(X)=3$ is quasipositive, then SSS(X) contains an element of the form xY for an atom x and a quasipositive braid Y of algebraic length 2. So, Theorem 1.4 solves the quasipositivity problem for 4-braids X with $e(X)=3$. This solution is of polynomial time provided a polynomial upper bound for the size of SSS(X). Such a bound is given by S.-J. Lee [14; Corollary 4.5.4]. Note that recently Calvez and Wiest [7] independently obtained the main result of [14; Chapter 4] (a polynomial time solution to the conjugacy problem in Br_4) by similar methods.

Let us give precise statements of the main results. For elements a,b of a group G we set $b^a=a^{-1}ba$, $b^G=\{b^c\mid c\in G\}$, and we write $a\sim b$ if $a\in b^G$. When speaking of Garside groups, we use the terminology and notation from [20] which is mostly the same as in [13]; see Section 2.1 for a very brief summary.

Theorem 1.1. *Let $(G,\mathcal{P},\delta)$ be a homogeneous symmetric square free Garside structure of finite type* (for example, the Birman-Ko-Lee Garside structure on Br_n) *and let $\mathcal{A}$ be the set of atoms.*

Let $Z\in \mathrm{SSS}(Z)\cap\big((x^k)^G(y^l)^G\big)$ where $k,l\ge 1$ and $x,y\in\mathcal{A}$. Then, up to exchange of x^k and y^l, one of the following possibilities takes place:

(i) *$Z=XY$ where $X\sim x^k$, $Y\sim y^l$, and $\ell(Z)=\ell(X)+\ell(Y)$;*

(ii) *$Z=x_1^pYx_1^{k-p}$ where $Y\sim y^l$, $x_1\in x^G\cap\mathcal{A}$, $0\le p\le k$, and $\ell(Z)=k+\ell(Y)$;*

(iii) *$Z=x_1^py_1^lx_1^{k-p}$ where $x_1\in x^G\cap\mathcal{A}$, $y_1\in y^G\cap\mathcal{A}$, and $0\le p\le k$.*

Using the blocking property [20; Corollary 7.2] (see Theorem 3.3 below), Theorem 1.1 implies the following result.

Corollary 1.2. *Let the hypothesis of Theorem 1.1 hold and* $\inf Z<0$.

If Case (i) occurs, i. e., if $Z=(x_1^k)^P(y_1^l)^Q$ with $x_1\in x^G\cap\mathcal{A}$, $y_1\in y^G\cap\mathcal{A}$, and $\ell(P)+\ell(Q)\ge 1$ (we may assume also that $\inf P=\inf Q=0$ *and that $\|P\|$ and $\|Q\|$ are minimal possible) then the left normal form of Z is*

$$\delta^{-p-q}\cdot A_1\cdot\ldots\cdot A_p\cdot C_1\cdot\ldots\cdot C_{k+p+q}\cdot y_1^l\cdot B_1\cdot\ldots\cdot B_q \tag{1.1}$$

where $A_1\cdot\ldots\cdot A_p$, $C_1\cdot\ldots\cdot C_{k+p+q}$, and $B_1\cdot\ldots\cdot B_q$ are the left normal forms of $\delta^p\tau^{-q}(P^{-1})$, $\delta^q x_1^kPQ^{-1}$, and Q respectively.

If Case (ii) occurs, i. e., if $Z=x_1^p(y_1^l)^Qx_1^{k-p}$ with $x_1\in x^G\cap\mathcal{A}$, $y_1\in y^G\cap\mathcal{A}$, and $\ell(Q)=n\ge 1$ (we may assume also that $\inf Q=0$ *and that $\|Q\|$ is minimal possible) then the left normal of Z is*

$$\delta^{-n}\cdot C_1\cdot\ldots\cdot C_{p+n}\cdot y_1^l\cdot B_1\cdot\ldots\cdot B_n\cdot x_1^{k-p} \tag{1.2}$$

where $C_1\cdot\ldots\cdot C_{n+p}$ and $B_1\cdot\ldots\cdot B_n$ are the left normal forms of $\delta^nx_1^pQ^{-1}$ and Q respectively. $\square$

All possibilities for the left normal forms of Z in Case (iii) of Theorem 1.1 are listed in Proposition 3.10.

Note that due to Corollary 1.2, it is very fast to check whether Z satisfies Conditions (i) or (ii): it is enough to recognize the pattern y_1^l in the left normal form of Z and to check (using Theorem 3.2) whether we obtain a conjugate of x^k after its removal; then, of course, the same should be done with x^k and y^l swapped. If $\inf Z \ge 0$, then Condition (iii) can be checked for all pairs of atoms (x_1, y_1) from $(x^G) \times (y^G)$ (Proposition 3.10 can be used to reduce the number of tests).

Corollary 1.3. *Let the hypothesis of Theorem 1.1 holds and* $\inf Z < 0$. *Then any cycling orbit of* $\mathrm{USS}(Z)$ *and any decycling orbit of* $\mathrm{USS}(Z^{-1})^{-1}$ *contains an element whose left normal form is as in [20; Theorem 1b], i. e., of the form (1.2) with* $p = 0$.

This fact was conjectured in [20; Remark (4) on p. 1083]. In particular, it gives a proof of [20; Theorem 1b] independent of the transport properties of cyclic sliding. Theorem 1.1 and Corollary 1.3 are proven in Section 3. An important ingredient of the proof is the blocking property of square free homogeneous symmetric Garside structures [20; Section 7] (see Theorem 3.3).

Theorem 1.4. *Let* $(G, \mathcal{P}, \delta)$ *be a square free homogeneous symmetric Garside structure of finite type such that* $\|\delta\| = 3$ (for example, the Birman-Ko-Lee Garside structure on Br_4) *and let* $\mathcal{A}$ *be the set of atoms.*

Let $X \in a_1^G a_2^G a_3^G$ *with* $a_1, a_2, a_3 \in \mathcal{A}$. *Then there exists a permutation* (x, y, z) *of* (a_1, a_2, a_3) *such that* $\mathrm{SSS}(X)$ *contains an element of the form* $x_1 Y$ *with* $x_1 \in x^G \cap \mathcal{A}$, $Y \in y^G z^G$ *such that either* $\inf Y = \inf x_1 Y$ *or* $Y \in \mathcal{P}$.

So, this theorem reduces the quasipositivity problem for the case $e(X) = 3$ to the quasipositivity problem for the case $e(X) = 2$. Theorem 1.4 is an immediate consequence of Lemmas 5.1 – 5.4.

Remark 1.5. It seems plausible that Theorem 1.4 holds with minor changes for products of three conjugates of given powers of atoms.

Remark 1.6. The following example shows that $\mathrm{SSS}(X)$ cannot be replaced by $\mathrm{USS}(X)$ in Theorem 1.4. We consider the 4-braid

$$X = \sigma_2^{\sigma_1 \sigma_3^3} \sigma_2^{\sigma_1^2 \sigma_2^{-1}} \sigma_3^{\sigma_2}.$$

Then, for the Birman–Ko–Lee Garside structure on Br_4, we have: $\ell_s(X) = 12$, $\inf_s X = -5$, $\sup_s X = 7$, all elements of $\mathrm{USS}(X)$ are rigid, and $|\mathrm{USS}(X)| = 48$. A computation shows that $x^{-1}Z$ is not quasipositive for any $x \in \mathcal{A}$, $Z \in \mathrm{USS}(X)$.

In Section 6 we give a summary of those results from Lee's thesis [14] about the structure of $\mathrm{SSS}(X)$ which extend to any homogeneous Garside group with $\|\Delta\| = 3$. This section is independent of the rest of the paper.

2. Garside groups

2.1. Notation and some definitions. Given two elements a, b of a group G, we set $b^a = a^{-1}ba$ and $b^G = \{b^c \mid c \in G\}$.

Garside groups were introduced in [10, 9] as a class of groups to which Garside's methods [12] extend. We use the definitions and notation for Garside structures introduced in [13] and reproduced almost without changes in [20]. So, a Garside structure on a group G is $(G, \mathcal{P}, \Delta)$ where Δ is the Garside element and $\mathcal{P} =$

$\{X \mid X \succcurlyeq 1\}$; we set $\tau(X) = X^\Delta$; we denote the infimum, supremum, canonical length, and (when $X \in \mathcal{P}$) letter length of $X \in G$ by $\inf X$, $\sup X$, $\ell(X)$, and $\|X\|$ respectively; we denote the minimal values of $\inf Y$, $\sup Y$, and $\ell(Y)$ over all $Y \in X^G$ by $\inf_s X$, $\sup_s X$, and $\ell_s(X)$ (see details in [13, 20]).

The only difference between the notation in [13] and in [20] is that we denote the set of simple elements by $[1,\Delta]$ instead of the commonly used notation $[0,1]$. We set also $]1,\Delta] = [1,\Delta] \setminus \{1\}$, $[1,\Delta[= [1,\Delta] \setminus \{\Delta\}$, $]1,\Delta[= [1,\Delta[\setminus \{1\}$.

The only new terminology introduced in [20] is the following. We say that a Garside structure is **homogeneous** if $\|XY\| = \|X\| + \|Y\|$ for any $X, Y \in \mathcal{P}$. In this case we define a group homomorphism $e : G \to \mathbb{Z}$ by setting $e(X) = \|X\|$ for $X \in \mathcal{P}$. A Garside structure is called **symmetric** if $A \preccurlyeq B \Leftrightarrow B \succcurlyeq A$ for any simple elements A, B and it is called **square free** if $x^2 \not\preccurlyeq \Delta$ for any atom x. The main example of symmetric homogeneous square free Garside structures are the dual Garside structures on Artin-Tits groups of spherical type introduced by Bessis [1], in particular, the Birman-Ko-Lee Garside structure [4] on Br_n. Another example is the Garside structure on the braid extension of the complex reflection group $G(e,e,r)$ introduced in [2].

In this paper we denote the Garside element by Δ when we speak of an arbitrary Garside structure, but we denote it by δ (as in [4]) if the Garside structure under consideration is supposed to be homogeneous and symmetric.

We denote the left (resp. right) gcd and lcm of X and Y by $X \wedge Y$ and $X \vee Y$ (resp. by $X \wedge^{\!\urcorner} Y$ and $X \vee^{\!\urcorner} Y$). We denote the usual (i. e., left) cycling, decycling, and cyclic sliding operators by $\mathbf{c}$, $\mathbf{d}$, and $\mathfrak{s}$ respectively. We denote the initial factor, final factor, and preferred prefix of X by $\iota(X)$, $\varphi(X)$, and $\mathfrak{p}(X)$. So, $\mathbf{c}(X) = X^{\iota(X)}$, $\mathbf{d}(X) = X^{\varphi(X)^{-1}}$, $\mathfrak{s}(X) = X^{\mathfrak{p}(X)}$. We denote the right counterparts of $\mathbf{c}$, $\mathbf{d}$, ι, φ by $\mathbf{c}^{\urcorner}$, $\mathbf{d}^{\urcorner}$, $\iota^{\urcorner}$, $\varphi^{\urcorner}$, i. e., if $A_1 \cdot \ldots \cdot A_r \cdot \Delta^p$, $r \geq 1$, is the right normal form of X, then

$$\iota^{\urcorner}(X) = \tau^p(A_r), \quad \varphi^{\urcorner}(X) = A_1, \quad \mathbf{c}^{\urcorner}(X) = X^{\iota^{\urcorner}(X)^{-1}}, \quad \mathbf{d}^{\urcorner}(X) = X^{\varphi^{\urcorner}(X)}.$$

2.2. Some facts about general Garside groups. Let $(G, \mathcal{P}, \Delta)$ be any Garside structure of finite type.

Lemma 2.1. *Let $X, Y \in G$. Then:*
(a). $\inf XY > \inf X + \inf Y$ *if and only if* $\Delta \preccurlyeq \iota^{\urcorner}(X)\iota(Y)$.
(b). $\sup XY < \sup X + \sup Y$ *if and only if* $\varphi(X)\varphi^{\urcorner}(Y) \preccurlyeq \Delta$.

Proof. (a). See [20; Lemma 2.4].

(b). Follows from (a) applied to Y^{-1} and X^{-1}. Indeed, suppose that $\sup XY < \sup X + \sup Y$. Then $\inf Y^{-1}X^{-1} = \inf(XY)^{-1} = -\sup XY > -\sup X - \sup Y = \inf X^{-1} + \inf Y^{-1}$. Hence $\Delta \preccurlyeq \iota^{\urcorner}(Y^{-1})\iota(X^{-1})$ by (a). Note that $\varphi(X)\iota(X^{-1}) = \iota^{\urcorner}(Y^{-1})\varphi^{\urcorner}(Y) = \Delta$, thus $\Delta \preccurlyeq \iota^{\urcorner}(Y^{-1})\iota(X^{-1}) = (\Delta\varphi^{\urcorner}(Y)^{-1})(\varphi(X)^{-1}\Delta)$ whence $1 \preccurlyeq \varphi^{\urcorner}(Y)^{-1}\varphi(X)^{-1}\Delta$ and, finally, $\varphi(X)\varphi^{\urcorner}(Y) \preccurlyeq \Delta$. □

Lemma 2.2. *Let* $\sup XsY \leq \sup X + \sup Y$ *where* $X, Y \in G$, $s \in [1,\Delta]$. *Then there exist* $u, v \in [1,\Delta]$ *such that* $s = uv$, $\sup Xu = \sup X$, *and* $\sup vY = \sup Y$.

Proof. If $\sup Xs \leq \sup X$, then we just set $u = s$, $v = 1$ and we are done. So, assume that $\sup Xs = \sup X + 1$. Then, by Lemma 2.1b, we have $\varphi(Xs)\varphi^{\urcorner}(Y) \preccurlyeq \Delta$. Let $v = \varphi(Xs)$. Then $s \succcurlyeq v$ by Lemma 2.5, i. e., $s = uv$ for some $u \in [1,\Delta]$. Since $v = \varphi(Xuv)$, we have $\sup Xuv = \sup Xu + \sup v$, hence $\sup Xu = \sup Xs - \sup v = \sup Xs - 1 = \sup X$. Since $v\varphi^{\urcorner}(Y) \preccurlyeq \Delta$, we have $\sup vY = \sup Y$. □

Lemma 2.3. [8; Prop. 3.1]. *Suppose that* $X = A_1 \cdot A_2 \cdot \ldots \cdot A_r$ *is in left normal form* ($A_i \in \,]1, \Delta[$, $i = 1, \ldots, r$), *and let* A_0 *be a simple element. Then the decomposition* $A_0 X = A'_0 \cdot A'_1 \cdot \ldots \cdot A'_r$ *is left weighted where the* A'_i*'s are defined recursively together with simple elements* $t_0, \ldots, t_r$ *by the conditions that* $t_0 = A_0$, $A'_{i-1} \cdot t_i$ *is the left normal form of* $t_{i-1} A_i$ *for* $i = 1, \ldots, r$, *and* $A'_r = t_r$. *We have* $A'_i \neq \Delta$ *for* $i > 0$ *and* $A'_i \neq 1$ *for* $i < r$ (*but it is possible that* $A'_0 = \Delta$ *or* $A'_r = 1$). □

Corollary 2.4. *Under the hypothesis of Lemma 2.3, suppose that* $\sup A_0 X = \sup A_0 + \sup X$ *and* $\|A_i\| = 1$ *for some* $i \in \{1, \ldots, r\}$. *Then* $\varphi(A_0 X) = \varphi(X)$. □

Lemma 2.5. [8; Prop. 3.3]. *Suppose that* $X = A_1 \cdot A_2 \cdot \ldots \cdot A_r$ *is in left normal form with* $A_i \in \,]1, \Delta[$ *and* $i = 1, \ldots, r$. *Let* A_{r+1} *be a simple element. Then the decomposition* $X A_{r+1} = A''_1 \cdot \ldots \cdot A''_{r+1}$ *is left weighted where the* A''_i *are defined recursively together with simple elements* $A'_1, \ldots, A'_r$ *by the conditions that* $A'_{r+1} = A_{r+1}$, $A'_i \cdot A''_{i+1}$ *is the left normal form of* $A_i A'_{i+1}$ *for* $i = r, \ldots, 1$, *and* $A''_1 = A'_1$. *We have* $A''_i \neq \Delta$ *for* $i > 1$ *and* $A''_i \neq 1$ *for* $i \leq r$ (*but it is possible that* $A''_1 = \Delta$ *or* $A''_{r+1} = 1$). □

3. Super summit set of a product of two conjugates of atom powers in square-free homogeneous symmetric Garside groups

In this section we prove Theorem 1.1 and Corollary 1.3. Throughout this section $(G, \mathcal{P}, \delta)$ is a square free symmetric homogeneous Garside structure with set of atoms $\mathcal{A}$.

3.1. Preliminaries.

Lemma 3.1. [20; Lemma 3.1]. *Let* $x \in \mathcal{A}$ *and* $A \in \mathcal{P}$. *If* $xA \preccurlyeq \delta$, (*resp.* $Ax \succcurlyeq \delta$), *then there exists* $x_1 \in x^G \cap \mathcal{A}$ *such that* $xA = Ax_1$ (*resp.* $Ax = x_1 A$).

Proof. Immediately follows from the fact that the Garside structure is symmetric and homogeneous. □

The following three results are proven in [20].

Theorem 3.2. [20; Theorem 1a]. *Let* $X \sim x^k$ *where* $x \in \mathcal{A}$, $k \geq 1$. *Then the left normal form of* X *is* $\delta^{-n} \cdot A_n \cdot \ldots \cdot A_1 \cdot x_1^k \cdot B_1 \cdot \ldots \cdot B_n$ *where* $n \geq 0$, $x_1 \in x^G \cap \mathcal{A}$, *and* $A_i \delta^{i-1} B_i = \delta^i$ *for* $i = 1, \ldots, n$. *In particular,* $\ell(X) = k + 2n = k - 2 \inf X$. □

Theorem 3.3. (Blocking property [20; Corollary 7.2]). *Let* $X \sim x^k$ *where* $x \in \mathcal{A}$, $X \notin \mathcal{P}$, $k \geq 1$. *Let* $U \in G$ *be such that* $\inf XU = \inf X + \inf U$. *Then* $\iota(XU) = \iota(X)$. □

Lemma 3.4. [20; Lemma 7.5]. *Let* $A \in [1, \delta]$ *and* $P \in \mathcal{P}$. *Then* $\delta \wedge (AP) = \delta \wedge (A^2 P)$. *In particular, if* $X \in G$ *is such that* $\inf AX = \inf X$, *then* $\iota(A^2 X) = \iota(AX)$ *and* $\inf A^2 X = \inf AX = \inf X$. □

Remark 3.5. The conclusion of [20; Lemma 7.5] was erroneously stated in the form $\iota(AP) = \iota(A^2 P)$. This is wrong in general without the assumption $\delta \not\preccurlyeq AP$ as one can see in the example $G = \mathrm{Br}_4$ (with the Birman–Ko–Lee Garside structure, thus $\delta = \sigma_3 \sigma_2 \sigma_1$), $A = \sigma_2 \sigma_1$, $P = \tau^2(A)$, and hence $\iota(AP) = \sigma_2$, $\iota(A^2 P) = A$. The statement and the proof of [20; Lemma 7.5] become correct if one replaces all $\iota(\ldots)$ by $\delta \wedge (\ldots)$. This mistake does not affect the usage of the lemma in the proof of the blocking property.

Lemma 3.6. *Let $x \in \mathcal{A}$, $k \geq 1$, $X \in (x^k)^G$, $s \in [1,\delta]$. If $\ell(Xs) \leq \ell(X)$ or $\ell(s^{-1}X) \leq \ell(X)$, then $\ell(X^s) \leq \ell(X)$.*

Proof. If $\ell(Xs) \leq \ell(X)$, then $\ell(X^s) = \ell(s^{-1}Xs) \leq \ell(s^{-1}) + \ell(Xs) \leq 1 + \ell(Xs) \leq 1 + \ell(X)$. We have also $\ell(X^s) \equiv k \equiv \ell(X) \mod 2$ by Theorem 3.2. Hence $\ell(X^s) \leq \ell(X)$. The case $\ell(s^{-1}X) \leq \ell(X)$ is similar. □

Lemma 3.7. *Let $x \in \mathcal{A}$, $k \geq 1$, $X \in (x^k)^G$, $U \in G$, $s \in [1,\delta]$. Suppose that*

$$\sup UXs \leq \sup UX = \sup U + \sup X. \tag{3.1}$$

Then $\ell(X^s) \leq \ell(X)$.

Proof. The case $s \in \{1,\delta\}$ is trivial, so we assume that $s \in \,]1,\delta[$. By Lemma 3.6, it is enough to show that $\sup Xs \leq \sup X$. Suppose the contrary:

$$\sup Xs = \sup X + \sup s. \tag{3.2}$$

The inequality in (3.1) can be rewritten as $\sup UXs < \sup UX + \sup s$. By combining it with (3.2) and the equality in (3.1), we obtain

$$\sup UXs < \sup UX + \sup s = \sup U + \sup X + \sup s = \sup U + \sup Xs.$$

By Lemma 2.1b, this implies $\varphi(U)\varphi^{\urcorner}(Xs) \preccurlyeq \delta$. By Corollary 2.4 combined with (3.2) and Theorem 3.2, we have $\varphi^{\urcorner}(Xs) = \varphi^{\urcorner}(X)$. Hence $\varphi(U)\varphi^{\urcorner}(X) \preccurlyeq \delta$ which contradicts the equality in (3.1). □

3.2. Products of two atoms. Normal forms in Case (iii) of Theorem 1.1. Recall that $(G,\mathcal{P},\delta)$ is a square free symmetric homogeneous Garside structure with set of atoms $\mathcal{A}$.

Proposition 3.8. *Let x and y be two atoms such that $xy \preccurlyeq \delta$. Then there exist $m \geq 2$ and pairwise distinct atoms $a_1,\dots,a_m$* (we assume that the indices are defined mod m) *such that:*

(i) *$x = a_1$, $y = a_2$, and $a_i a_{i+1} = xy$ for any i;*
(ii) *$a_{i+2} = a_i^{xy}$ for any i;*
(iii) *the product $a_i \cdot a_j$ is left weighted unless $j \equiv i+1 \mod m$.*

Proof. We define $a_1, a_2, \dots$ recursively by $a_1 = x$, $a_2 = y$, $a_i a_{i+1} = a_{i-1}a_i$. Then all a_i are atoms by Lemma 3.1 and (i) holds; (ii) follows from (i). Let us prove (iii). Suppose that $a_i \cdot a_j$ is not left weighted, i.e., $a_i a_j \preccurlyeq \delta$. Note that $a_i \vee a_j = xy$. Since the Garside structure is symmetric, we have $a_i \prec a_i a_j$ and $a_j \prec a_i a_j$. Hence $xy = a_i \vee a_j \preccurlyeq a_i a_j$. Since $\|xy\| = \|a_i a_j\|$, it follows that $a_i a_j = xy = a_i a_{i+1}$ whence $a_j = a_{i+1}$. □

For $x, y \in \mathcal{A}$, we set

$$\mu_{x,y} = \begin{cases} 0, & \text{if } x \cdot y \text{ is left weighted,} \\ 1, & \text{if } x = y, \\ m, & \text{if } xy \preccurlyeq \delta \text{ and } m \text{ is as in Proposition 3.8.} \end{cases}$$

Remark 3.9. It follows from Proposition 3.8 that the submonoid of G generated by any pair of atoms is either free or isomorphic to the positive monoid of the dual Garside structure in an Artin-Tits group of type $I_2(m)$ (see [21; Proposition 1.2]). It is interesting to study if the same is true for the subgroup of G generated by a pair of atoms. Note that the subgroup generated by a submonoid M of a group is not necessarily isomorphic to the group of fractions of M. For example, the submonoid M of Br_3 generated by σ_1 and σ_2^{-1} is free whereas the subgroup generated by M is the whole Br_3 which is not a free group.

Proposition 3.10. *(a). Let $Z = x^k y^l$ where $k,l \ge 1$ and $x,y \in \mathcal{A}$, $x \ne y$. Then $Z \notin \mathrm{SSS}(Z)$ if and only if one of the following conditions holds:*

(i) $\mu_{y,x} \ge 3$;
(ii) $\mu_{x,y} = 3$, $k = 1$, *and* $l \ge 3$;
(iii) $\mu_{x,y} = 3$, $l = 1$, *and* $k \ge 3$.

If $Z \in \mathrm{SSS}(Z)$, then the left normal form of Z is

$$\begin{cases} x^k \cdot y^l & \text{if } \mu_{x,y} = \mu_{y,x} = 0, \\ (xy)^k \cdot y^{l-k} & \text{if } \mu_{x,y} = 2 \text{ and } k \le l \text{ (the case } l \le k \text{ is similar)}, \\ xy \cdot (x^y)^{k-1} \cdot y^{l-1} & \text{if } \mu_{x,y} \ge 3. \end{cases}$$

(b). Let $Z = x^p y^l x^q$ where $p,q,l \ge 1$ and $x,y \in \mathcal{A}$, $xy \ne yx$. Then $Z \notin \mathrm{SSS}(Z)$ if and only if one of the following conditions holds:

(i) $\mu_{x,y} = 3$, $p = l = 1$, *and* $q \ge 2$;
(ii) $\mu_{y,x} = 3$, $q = l = 1$, *and* $p \ge 2$.

If $Z \in \mathrm{SSS}(Z)$, then the left normal form of Z, is

$$\begin{cases} x^p \cdot y^l \cdot x^q & \text{if } \mu_{x,y} = \mu_{y,x} = 0, \\ xy \cdot x_1^{p-1} \cdot y^{l-1} \cdot x^q & \text{if either } \mu_{x,y} \ge 4, \text{ or } \mu_{x,y} = 3 \text{ and } l \ge 2, \\ yx \cdot x_2^{p} \cdot y_2^{l-1} \cdot x^{q-1} & \text{if either } \mu_{y,x} \ge 4, \text{ or } \mu_{y,x} = 3 \text{ and } l \ge 2, \\ (xy)^2 \cdot y^{p-2} \cdot x^{q-1} & \text{if } \mu_{x,y} = 3 \text{ and } l = 1, \\ (yx)^2 \cdot y_2^{p-1} \cdot x^{q-2} & \text{if } \mu_{y,x} = 3 \text{ and } l = 1 \end{cases}$$

where x_1, x_2, and y_2 are defined by $xy = yx_1$ and $yx = xy_2 = y_2x_2$.

Proof. A straightforward computation using Proposition 3.8. To see that the listed elements Z are in the super summit set, it is enough to check that in each case $\mathfrak{s}(Z)$ belongs to the same list and $\ell(\mathfrak{s}(Z)) = \ell(Z)$. Thus $\ell(\mathfrak{s}^m(Z)) = \ell(Z)$ for any m whence $Z \in \mathrm{SSS}(Z)$ by [13]. $\square$

3.3. Proof of Theorem 1.1 and Corollary 1.3. Recall that $(G, \mathcal{P}, \delta)$ is a square free symmetric homogeneous Garside structure with set of atoms $\mathcal{A}$.

For $x, y \in \mathcal{A}$ and $k, l \ge 1$, we set:

$$\begin{aligned} \vec{\mathcal{G}}'_{p,q}(x^k, y^l) &= \{XY \mid X \sim x^k,\, Y \sim y^l,\, \ell(X) = 2p + k,\, \ell(Y) = 2q + l, \\ &\qquad \ell_s(XY) = \ell(X) + \ell(Y)\}, \\ \vec{\mathcal{G}}''_{p,n}(x^k, y^l) &= \{Z = x_1^p Y x_1^{k-p} \mid Y \sim y^l,\, x_1 \in x^G \cap \mathcal{A},\, \ell(Y) = 2n + l, \\ &\qquad \ell_s(Z) = k + \ell(Y)\}, \\ \vec{\mathcal{G}}'''_{p}(x^k, y^l) &= \{Z = x_1^p y_1^l x_1^{k-p} \mid x_1 \in x^G \cap \mathcal{A},\, y_1 \in y^G \cap \mathcal{A},\, Z \in \mathrm{SSS}(Z)\} \end{aligned}$$

and $\mathcal{G}(x^k, y^l) = \mathcal{G}'(x^k, y^l) \cup \mathcal{G}''(x^k, y^l) \cup \mathcal{G}'''(x^k, y^l)$ where

$$\vec{\mathcal{G}}'(\,) = \bigcup_{p,q \ge 0} \vec{\mathcal{G}}'_{p,q}(\,), \quad \vec{\mathcal{G}}''(\,) = \bigcup_{0 \le p \le k;\, n \ge 0} \vec{\mathcal{G}}''_{p,n}(\,), \quad \vec{\mathcal{G}}'''(\,) = \bigcup_{0 \le p \le k} \vec{\mathcal{G}}'''_{p}(\,),$$

$$\mathcal{G}^*(x^k, y^l) = \vec{\mathcal{G}}^*(x^k, y^l) \cup \vec{\mathcal{G}}^*(y^l, x^k) \qquad \text{where } {}^* \text{ stands for } {}' \text{ or } {}'' \text{ or } {}'''.$$

It is clear that $Z \in \mathcal{G}(x^k, y^l)$ implies $Z \in \mathrm{SSS}(Z)$. In this notation, the conclusion of Theorem 1.1 reads as $\mathrm{SSS}(Z) \subset \mathcal{G}(x^k, y^l)$. Let us fix $k, l \ge 1$ and $x, y \in \mathcal{A}$.

Lemma 3.11. *Let $Z \in \vec{\mathcal{G}}'(x^k, y^l)$ and let s be a simple element such that $Z^s \in \mathrm{SSS}(Z)$. Then $Z^s \in \mathcal{G}(x^k, y^l)$.*

Proof. Let $Z = XY$, $X \sim x^k$, $Y \sim y^l$, $\ell(Z) = \ell(X) + \ell(Y)$. Since $Z, Z^s \in \mathrm{SSS}(Z)$, we have $\ell(Z) = \ell(Z^s)$, hence $\ell(X^s) + \ell(Y^s) \ge \ell(X^sY^s) = \ell(Z^s) = \ell(Z)$. On the other hand, we have $\ell(X^s) \le \ell(s^{-1}) + \ell(X) + \ell(s) = \ell(X) + 2$ and, similarly, $\ell(Y^s) \le \ell(Y) + 2$. We have also $\ell(X^s) \equiv k \equiv \ell(X)$ and $\ell(Y^s) \equiv l \equiv \ell(Y) \mod 2$ by Theorem 3.2. Hence

$$\ell(Z) \le \ell(X^s) + \ell(Y^s) \le \ell(Z) + 4, \qquad \ell(X^s) + \ell(Y^s) \equiv \ell(Z) \mod 2.$$

Thus $\ell(X^s) + \ell(Y^s)$ may take only three values: $\ell(Z)$, $\ell(Z) + 2$, and $\ell(Z) + 4$. We consider separately these three cases.

Case 1. $\ell(X^s) + \ell(Y^s) = \ell(Z)$. The result immediately follows.

Case 2. $\ell(X^s) + \ell(Y^s) = \ell(Z) + 2$. Then, for $(U, V) = (X, Y)$ or (Y, X), we have $\ell(U^s) = \ell(U)$ and $\ell(V^s) = \ell(V) + 2$, hence $\inf U^s = \inf U$, $\sup U^s = \sup U$, $\inf V^s = \inf V - 1$, $\sup V^s = \sup V + 1$ and we obtain

$$\inf X^s + \inf Y^s = \inf Z^s - 1 \qquad \text{and} \qquad \sup X^s + \sup Y^s = \sup Z^s + 1. \tag{3.3}$$

Case 2.1. $\inf X^s = 0$ or $\inf Y^s = 0$. Without loss of generality we may assume that $\inf X^s = 0$, i. e., $X^s = x_1^k$ where $x_1 \in x^G \cap \mathcal{A}$. In this case we have $\ell(X^s) = \ell(X)$ and $\ell(Y^s) = \ell(Y) + 2$. Let $(A, B) = (\iota(Y^s), \varphi(Y^s))$. Then, by Theorem 3.2, we have $Y^s = A\delta^{-1}Y_1B$ with $\ell(Y_1) = \ell(Y^s) - 2 = \ell(Y)$, $BA = \delta$, and hence, $Y^s = Y_1^B$. By (3.3) combined with Lemma 2.1b, we have $\delta \preccurlyeq \iota^{\urcorner}(X^s)\iota(Y^s)$. Since $\iota^{\urcorner}(X^s) = x_1$, we obtain $\delta \preccurlyeq x_1A$. Since, moreover, $\|\delta\| \ge \|x_1\| + \|A\|$, this yields $x_1A = \delta$. Since $BA = \delta$, we obtain $B = x_1$, hence

$$Z^s = x_1^k Y^s = x_1^k Y_1^B = x_1^{k-1} Y_1 x_1.$$

Since $Y_1 \sim y^l$ and $\ell(Y_1) = \ell(Y)$, we conclude that $Z^s \in \mathcal{G}(x^k, y^l)$.

Case 2.2. $\inf X^s < 0$ and $\inf Y^s < 0$. Let $(A, B) = (\varphi^{\urcorner}(X^s), \iota^{\urcorner}(X^s))$ and $(C, D) = (\iota(Y^s), \varphi(Y^s))$. Then, by Theorem 3.2, we have $X^s = A\delta^{-1}X_1B$ and $Y^s = C\delta^{-1}Y_1D$ where $BA = DC = \delta$, $X_1 \sim X$, $Y_1 \sim Y$, $\ell(X_1) = \ell(X^s) - 2$, and $\ell(Y_1) = \ell(Y^s) - 2$. By (3.3) combined with Lemma 2.1b we have $\iota^{\urcorner}(X^s)\iota(Y^s) = E\delta$ for some $E \in [1, \delta]$. Hence

$$Z^s = A\delta^{-1}X_1BC\delta^{-1}Y_1D = A\delta^{-1}X_1EY_1D = \delta^{-1}\tilde{A}X_1EY_1D$$

where $\tilde{A} = \tau^{-1}(A)$. Since $\tilde{A}B = C\tau(D) = \delta$, we have $\delta^2 = \tilde{A}BC\tau(D) = \tilde{A}E\delta\tau(D) = \tilde{A}ED\delta$ whence $\tilde{A}ED = \delta$.

Case 2.2.1. $\ell(\tilde{A}X_1) \le \ell(X_1)$ or $\ell(Y_1D) \le \ell(Y_1)$. By symmetry, it is enough to consider only the latter case. So, let $\ell(Y_1D) \le \ell(Y_1)$. Then, by Lemma 3.6, we have $\ell(Y_1^D) \le \ell(Y_1)$. Since

$$Z^s = \delta^{-1}\tilde{A}X_1EDY_1^D = X_1^{ED}Y_1^D$$

and

$$\ell(X_1^{ED}) + \ell(Y_1^D) \le (\ell(X_1) + 2) + \ell(Y_1) = \ell(X^s) + (\ell(Y^s) - 2) = \ell(Z^s),$$

we conclude that $Z^s \in \mathcal{G}(x^k, y^l)$.

Case 2.2.2. $\ell(\tilde{A}X_1) = \ell(X_1)+1$ and $\ell(Y_1D) = \ell(Y_1)+1$. Let us show that this is impossible. Indeed, in this case we have $\sup \tilde{A}X_1 = \sup \tilde{A} + \sup X_1 = \sup X_1 + 1 = \sup X^s$ and similarly $\sup Y_1D = \sup Y^s$. By (3.3), this yields

$$\sup \tilde{A}X_1 + \sup Y_1D = \sup X^s + \sup Y^s = \sup Z^s + 1 = \sup \tilde{A}X_1EY_1D.$$

By Lemma 2.2, this implies that there exist $u, v \in [1, \delta]$ such that $E = uv$, $\sup \tilde{A}X_1u = \sup \tilde{A}X_1$, and $\sup vY_1D = \sup Y_1D$. Then, by Lemma 3.7, we have $\ell(X_2) \le \ell(X_1)$ and $\ell(Y_2) \le \ell(Y_1)$ where $X_2 = u^{-1}X_1u$ and $Y_2 = vY_1v^{-1}$. Since

$$Z^s = \delta^{-1}\tilde{A}X_1uvY_1D = \delta^{-1}\tilde{A}uX_2Y_2vD = (X_2Y_2)^{vD},$$

we obtain $\ell_s(Z) \le \ell(X_2Y_2) \le \ell(X_2)+\ell(Y_2) \le \ell(X_1)+\ell(Y_1) = \ell(X^s)+\ell(Y^s)-4 = \ell(Z^s)-2$, a contradiction.

Case 3. $\ell(X^s)+\ell(Y^s) = \ell(Z)+4$. Let us show that this case is impossible. We have $\ell(s^{-1}Xs) = \ell(s^{-1})+\ell(X)+\ell(s)$ and $\ell(s^{-1}Ys) = \ell(s^{-1})+\ell(Y)+\ell(s)$, hence

$$\ell(s^{-1}X) = \ell(s^{-1}) + \ell(X) \qquad \text{and} \qquad \ell(Ys) = \ell(Y) + \ell(s) \tag{3.4}$$

whence

$$\sup s^{-1}X = \sup s^{-1} + \sup X = \sup X \quad \text{and} \quad \sup Ys = \sup Y + \sup s = \sup Y + 1.$$

Thus

$$\begin{aligned}\sup s^{-1}X + \sup Ys = \sup X + \sup Y + 1 &> \sup X + \sup Y\\ &= \sup Z = \sup Z^s = \sup s^{-1}XYs.\end{aligned}$$

By Lemma 2.1b, this implies $\varphi(s^{-1}X)\varphi^{\urcorner}(Ys) \preccurlyeq \delta$. We have $\varphi(s^{-1}X) = \varphi(X)$ by (3.4) combined with Corollary 2.4. Similarly, $\varphi^{\urcorner}(Ys) = \varphi^{\urcorner}(Y)$. Thus we obtain $\varphi(X)\varphi^{\urcorner}(Y) \preccurlyeq \delta$ which contradicts the condition $\ell(XY) = \ell(X) + \ell(Y)$. ☐

Lemma 3.12. *Let $Z \in \vec{\mathcal{G}}''(x^k, y^l)$ and let s be a simple element such that $Z^s \in \mathrm{SSS}(Z)$. Then $Z^s \in \mathcal{G}(x^k, y^l)$.*

Proof. Let $Z = x_1^pYx_1^q$ where $x_1 \in x^G \cap \mathcal{A}$, $Y \sim y^l$, $p+q = k$, $\ell(Z) = \ell(Y)+k$. If $p = 0$ or $q = 0$, then Lemma 3.11 applies. So, we assume that $p, q > 0$. Let us show that

$$\sup s^{-1}Z < \sup s^{-1} + \sup Z \qquad \text{or} \qquad \sup Zs < \sup Z + \sup s. \tag{3.5}$$

Indeed, suppose that the left inequality in (3.5) does not hold, i. e., $\sup s^{-1}Z = \sup s^{-1} + \sup Z = \sup Z$. Then

$$\sup s^{-1}Z + \sup s = \sup Z + 1 > \sup Z = \sup(s^{-1}Z \cdot s).$$

Hence $\varphi(s^{-1}Z)s \preccurlyeq \delta$ by Lemma 2.1b. Since $\varphi(s^{-1}Z) = \varphi(Z)$ by Corollary 2.4, this means that $\varphi(Z)s \preccurlyeq \delta$ which implies the right inequality in (3.5). Thus, (3.5) holds.

By symmetry, without loss of generality we may assume that the right inequality in (3.5) holds. Then $x_1 s = \varphi(Z)s \preccurlyeq \delta$ by Lemma 2.1b. Hence, by Lemma 3.1, we have $x_1 s = s x_2$ where $x_2 = x_1^s \in x^G \cap \mathcal{A}$, and we obtain $Z^s = x_2^p Y^s x_2^q$. If $\ell(Y^s) \le \ell(Y)$, then we are done. So, we suppose that $\ell(Y^s) = \ell(Y) + 2$. In this case we have also $\inf Y^s = \inf Y - 1$.

Let us show that

$$\inf x_2^p Y^s > \inf x_2^p + \inf Y^s \qquad \text{or} \qquad \inf Y^s x_2^q > \inf Y^s + \inf x_2^q. \tag{3.6}$$

Indeed, suppose that the right inequality in (3.6) does not hold, i. e., $\inf Y^s x_2^q = \inf Y^s + \inf x_2^q$, hence

$$\inf x_2^p + \inf Y^s x_2^q = \inf x_2^p + \inf Y^s + \inf x_2^q = \inf Y^s < \inf Y = \inf Z = \inf Z^s.$$

Then we have $\delta \preccurlyeq \iota^{\urcorner}(x_2^p)\iota(Y^s x_2^q)$ by Lemma 2.1a. By Theorem 3.3, we have $\iota(Y^s x_2^q) = \iota(Y^s)$. Hence $\delta \preccurlyeq \iota^{\urcorner}(x_2^p)\iota(Y^s)$ which implies the left inequality in (3.6). Thus, (3.6) holds.

By symmetry, without loss of generality we may assume that the left inequality in (3.6) holds. The rest of the proof is almost the same as in Case 2.1 of Lemma 3.11. Namely, let $(A, B) = (\iota(Y^s), \varphi(Y^s))$. Then, by Theorem 3.2, we have $Y^s = A\delta^{-1}Y_1 B$ with $\ell(Y_1) = \ell(Y^s) - 2 = \ell(Y)$, $BA = \delta$, and hence, $Y^s = Y_1^B$. Then we have $\delta \preccurlyeq \iota^{\urcorner}(x_2^p)\iota(Y^s) = x_2 A$ by Lemma 2.1a combined with the left inequality in (3.6). Since $BA = \delta$, we obtain $B = x_2$, hence

$$Z^s = x_2^p Y^s x_2^q = x_2^p Y_1^B x_2^q = x_2^{p-1} Y_1 x_2^{q+1}.$$

Since $Y_1 \sim y^l$ and $\ell(Y_1) = \ell(Y)$, we conclude that $Z^s \in \mathcal{G}(x^k, y^l)$. $\square$

Lemma 3.13. *Let $Z \in \vec{\mathcal{G}}'''(x^k, y^l)$ and let s be a simple element such that $Z^s \in$ SSS(Z). Then $Z^s \in \mathcal{G}'''(x^k, y^l)$.*

Proof. We shall assume that $\|\delta\| \ge 3$. In the case $\|\delta\| = 2$, the proof is the same but the notation should be slightly changed.

By the same arguments as in the proof of Lemma 3.13, we may assume that the right inequality in (3.5) holds. By Proposition 3.10, we have $\|\varphi(Z)\| = 1$ or 2.

Case 1. $\|\varphi(Z)\| = 1$. It follows from Proposition 3.10 that, up to exchange of the roles of x^k and y^l, we may assume that $Z = x_1^p Y x_1^q$ where $Y = y_1^l$, $x_1 \in x^G \cap \mathcal{A}$, $y_1 \in y^G \cap \mathcal{A}$, $p + q = k$, $q \ge 1$, and $\varphi(Z) = x_1$. The rest of the proof is the same as in Lemma 3.12.

Note that the presentation of Z in the form as in the definition of $\mathcal{G}'''(x^k, y^l)$ is not necessarily unique. For example, if $k = 4$, $l = 1$, and $Z = xyx^3$ where $xy = yz = zx$, $z \in \mathcal{A}$, then we work with $Z = x^1 y^1 x^3$, $\varphi(Z) = x$ when the right equality in (3.5) holds, but we work with $Z = y^4 z^1 y^0$, $\varphi^{\urcorner}(Z) = y$ when the left equality in (3.5) holds.

Case 2. $\|\varphi(Z)\| = 2$. By Proposition 3.10, we may assume that $Z = x_0^p y_0^l x_0^q$ where $p + q = k$, $x_0 \in x^G \cap \mathcal{A}$, $y_0 \in y^G \cap \mathcal{A}$, and $\varphi(Z) = uv$ where (u, v) is (x_0, y_0) or (y_0, x_0). By the right inequality in (3.5) combined with Lemma 2.1b, we have $\varphi(Z)s \preccurlyeq \delta$, thus $uvs \preccurlyeq \delta$. Hence $vs \preccurlyeq \delta$ and $vs = sv_1$, $v_1 = v^s \in \mathcal{A}$ by Lemma 3.1. Then we have $usv_1 = uvs \preccurlyeq \delta$ whence $us \preccurlyeq \delta$ and $us = su_1$, $u_1 = u^s \in \mathcal{A}$. Thus $x_0^s = x_1$ and $y_0^s = y_1$ with $x_1, y_1 \in \mathcal{A}$, and we obtain $Z^s = x_1^p y_1^l x_1^q \in \mathcal{G}'''(x^k, y^l)$. $\square$

Proof of Theorem 1.1. As we already pointed out before Lemma 3.11, we need to prove that $\mathrm{SSS}(Z) \subset \mathcal{G}(x^k, y^l)$. We have $\mathrm{SSS}(Z) \cap \mathcal{G}(x^k, y^l) \neq \varnothing$. Indeed, if $Z \notin \mathcal{P}$, then $\mathrm{SSS}(Z) \cap \vec{\mathcal{G}}''(x^k, y^l) \neq \varnothing$ by [20; Theorem 1b] (in fact, only [20; Corollary 3.5] is needed here). If $Z \in \mathcal{P}$, then, again by [20; Theorem 1b], we have $Z \sim Z_1 = x_1^k y_1^l$ where $x_1 \in x^G \cap \mathcal{A}$, $y_1 \in y^G \cap \mathcal{A}$. By Proposition 3.10a, it follows that $Z_1 \in \mathrm{SSS}(Z)$, and hence $Z_1 \in \mathcal{G}'''(x^k, y^l)$, unless one of Cases (i)–(iii) of Proposition 3.10 occur. However, in each of these three cases, a cyclic permutation of the word $x_1^k y_1^l$ yeilds an element Z_2 of $\mathrm{SSS}(Z)$. Then we have $Z_2 \in \mathrm{SSS}(Z) \cap \mathcal{G}'''(x^k, y^l)$.

By the convexity theorem [11; Corollary 4.2], any element of $\mathrm{SSS}(Z)$ can be obtained from any other by successive conjugations by simple elements. Thus the result follows from Lemmas 3.11 – 3.13. □

The following proposition shows that the cycling operator acts on the sets $\vec{\mathcal{G}}'_{p,q}(x^k, y^l)$ and $\vec{\mathcal{G}}''_{p,n}(x^k, y^l)$ in the most natural and expected way.

Proposition 3.14. *If $p > 0$, then*

$$\mathbf{c}\big(\vec{\mathcal{G}}'_{p,q}(x^k, y^l)\big) \subset \vec{\mathcal{G}}'_{p-1,q+1}(x^k, y^l) \quad \text{and} \quad \mathbf{c}\big(\vec{\mathcal{G}}''_{p,n}(x^k, y^l)\big) \subset \vec{\mathcal{G}}''_{p-1,n}(x^k, y^l).$$

Note that $\vec{\mathcal{G}}'_{0,n}(x^k, y^l) = \vec{\mathcal{G}}''_{k,n}(x^k, y^l)$ and $\vec{\mathcal{G}}''_{0,n}(x^k, y^l) = \vec{\mathcal{G}}'_{n,0}(y^l, x^k)$.

Proof. The first inclusion follows from Corollary 1.2. Let us prove the second one. Let Z be as in the definition of $\vec{\mathcal{G}}''_{p,n}(x^k, y^l)$. We may suppose that the left normal form of Z is as in (1.2). We see from (1.2) that $\iota(Z) = \iota(x_1^p Y) = \tilde{C}_1 = \tau^n(C_1)$. By Lemma 3.4, we have $\iota(x_1^p Y) = \iota(x_1 Y)$. Hence $\tilde{C}_1 = x_1 s = s x_2$ where $s \preccurlyeq \iota(Y)$ and $x_2 \in x^G \cap \mathcal{A}$. Thus

$$Z = x_1^p s Y' x_1^{k-p} = s x_2^p Y' x_1^{k-p} = \tilde{C}_1 x_2^{p-1} Y' x_1^{k-p}$$

and

$$\mathbf{c}(Z) = x_2^{p-1} Y' x_1^{k-p} \tilde{C}_1 = x_2^{p-1} Y' x_1^{k-p+1} s = x_2^{p-1} Y' s x_2^{k-p+1} \in \vec{\mathcal{G}}'_{p-1,n}(x^k, y^l). \quad \square$$

Corollary 1.3 follows from Proposition 3.14.

4. Homogeneous symmetric Garside groups with $\|\delta\| = 3$

In this section we assume that $(G, \mathcal{P}, \delta)$ is a square free homogeneous symmetric Garside structure with set of atoms $\mathcal{A}$ and we assume that $\|\delta\| = 3$.

If $\delta^p \cdot A_1 \cdot \ldots \cdot A_n$ is the left normal form of X, then we denote:

$$\ell_1(X) = \mathrm{Card}\{i \mid \|A_i\| = 1\}, \qquad \ell_2(X) = \mathrm{Card}\{i \mid \|A_i\| = 2\}. \tag{4.1}$$

Lemma 4.1. *Let $X \in G$. Then*

$$\ell_1(X) = \inf X + 2 \sup X - e(X) \qquad \text{and} \qquad \ell_2(X) = -2 \inf X - \sup X + e(X).$$

Proof. Follows from $n_1 + n_2 = \ell(X)$ and $n_1 + 2n_2 = e(X) - 3 \inf X$, $n_i = \ell_i(X)$. □

Lemma 4.2. *Let $Y = \delta^p \cdot A_1 \cdot \dots \cdot A_n$ be in left normal form, $n \ge 3$. Suppose that $\inf_s Y > p$.*

(a). If $\iota(\mathbf{c}(Y)) = \tau^{-p}(A_2)$, then $\inf \mathbf{c}(Y) > p$.

(b). If $(\|A_2\|, \dots, \|A_n\|) \ne (1, \dots, 1)$, then $\inf \mathbf{c}(Y) > p$.

Proof. (a). If $\iota(\mathbf{c}(Y)) = \tilde{A}_2$, then $\mathbf{c}^2(Y) = \delta^p A_3 \dots A_n \tilde{A}_1 \tilde{A}_2$ where $\tilde{A}_j = \tau^{-p}(A_j)$. Since $\inf_s Y > p$, it follows from [5] that $\inf \mathbf{c}^2(Y) > p$. Hence $\delta \preccurlyeq A_3 \dots A_n \tilde{A}_1 \tilde{A}_2$. Then, by Lemma 2.1a, we have $\delta \preccurlyeq \iota^{\urcorner}(A_3 \dots A_n)\tilde{A}_1$, hence $\delta \preccurlyeq A_2 \dots A_n \tilde{A}_1$ which means that $\inf \mathbf{c}(Y) > p$.

(b). Suppose that $(\|A_2\|, \dots, \|A_n\|) \ne (1, \dots, 1)$. Let $i \ge 2$ be such that $\|A_i\| = 2$. Suppose that $\inf \mathbf{c}(Y) = p$. Then, by Lemma 2.5, the left normal form of $\mathbf{c}(Y)$ starts with $\delta^p \cdot A_2 \cdot \dots \cdot A_i$. Hence $\inf \mathbf{c}(Y) > p$ by (a). Contradiction □

Lemma 4.3. *Let $Y = \delta^p \cdot A_1 \cdot \dots \cdot A_n$ be in left normal form, $n \ge 3$. Suppose that $\sup_s Y < p + n$.*

(a). If $\varphi(\mathbf{d}(Y)) = A_{n-1}$, then $\sup \mathbf{d}(Y) < p + n$.

(b). If $(\|A_1\|, \dots, \|A_{n-1}\|) \ne (2, \dots, 2)$, then $\sup \mathbf{d}(Y) < p + n$.

Proof. Apply Lemma 4.2 to Y^{-1}. □

Lemma 4.4.

(a). Let $\inf Y < \inf \mathbf{c}(Y)$ and $\sup \mathbf{c}(Y) = \sup Y$. Then $\ell_2(Y) \ge 2$.

(b). Let $\inf Y = \inf \mathbf{d}(Y)$ and $\sup \mathbf{d}(Y) < \sup Y$. Then $\ell_1(Y) \ge 2$.

Proof. (a). Let $A = \iota(Y)$, $Y = AY_1$, and $B = \iota^{\urcorner}(Y_1)$. The condition $\inf Y < \inf \mathbf{c}(Y) = \inf Y_1 A$ combined with Lemma 2.1a implies $\delta \preccurlyeq BA$. The condition $\sup \mathbf{c}(Y) = \sup Y$ implies $\delta \ne BA$. Hence $\|BA\| > \|\delta\| = 3$ whence $\|B\| = \|A\| = 2$.

(b). Apply (a) to Y^{-1}. □

Lemma 4.5. *Let $\ell(Y) \ge 3$* (note that this is so when $e(Y) \ge 2$ and $\inf Y < 0$).

(a). If $\inf Y < \inf_s Y$ and $\sup_s Y = \sup Y$, then $\inf Y < \inf \mathbf{c}(Y)$.

(b). If $\inf Y = \inf_s Y$ and $\sup_s Y < \sup Y$, then $\sup \mathbf{d}(Y) < \sup Y$.

(c). If $Y \notin \mathrm{SSS}(Y)$, then $\inf Y < \inf \mathbf{c}(Y)$ or $\sup \mathbf{d}(Y) < \sup Y$.

Proof. (a). If $\inf Y < \inf_s Y$, then $\inf Y = \inf X < \inf \mathbf{c}(X)$ where $X = \mathbf{c}^m(Y)$ for some $m \ge 0$ (see [5]). If, moreover, $\sup_s Y = \sup Y$, then $\ell_2(X) \ge 2$ by Lemma 4.4. We have $\ell_2(X) = \ell_2(Y)$ by Lemma 4.1, thus $\ell_2(Y) \ge 2$, and the result follows from Lemma 4.2b.

(b). Apply (a) to Y^{-1}.

(c). If $\inf Y = \inf_s Y$ or $\sup_s Y = \sup Y$, then the result follows from (a), (b). Otherwise it follows from Lemmas 4.2b, 4.3b because $\ell_2(Y) > 1$ or $\ell_1(Y) > 1$. □

Lemma 4.6. *Let $Y \in a^G b^G$ where $a, b \in \mathcal{A}$. Suppose that $\inf_s Y < 0$ and $\inf Y = \inf_s Y$ (i. e., Y is in its summit set). Then there exist $U, V \in G$ such that, up to exchange of a and b, we have $Y = UyV$ with $y \in a^G \cap \mathcal{A}$, $UV \sim b$ and the following conditions hold: $\ell(U) \ge 1$, $\ell(V) \ge 1$, the product $\varphi(U) \cdot y \cdot \iota(V)$ is left weighted, and hence $\ell(Y) = \ell(U) + 1 + \ell(V)$.*

Proof. Induction on $\sup Y - \sup_s Y$. If $\sup Y - \sup_s Y = 0$, then $Y \in \mathrm{SSS}(Y)$, and the result follows from Corollary 1.2. Indeed, if $Y = z^P y^Q$ with $\ell(Y) = 2 + 2\ell(P) + 2\ell(Q)$ and $\ell(Q) \ge 1$, then we set $U = z^P Q^{-1}$ and $V = Q$; if $Y = y^P z$ with $\ell(Y) = 2 + 2\ell(P)$, then we set $U = P^{-1}$ and $V = Pz$.

Suppose that $\sup Y - \sup_s Y > 0$. Then $\sup \mathbf{d}(Y) = \sup Y - 1$ by Lemma 4.5b. So, by the induction hypothesis, we assume that $\mathbf{d}(Y) = U'y'V'$ with the required properties. Without loss of generality we may assume also that $\inf V' = 0$.

Let $\delta^p \cdot A_1 \cdot \ldots \cdot A_n$ be the left normal form of Y. Then the left normal form $B_1 \cdot \ldots \cdot B_{n-1}$ of $\delta^{-p}\mathbf{d}(Y)$ is obtained from $\tau^p(A_n) \cdot A_1 \cdot \ldots \cdot A_{n-1}$ by the procedure described in Lemma 2.3. It follows that for some $i \geq 1$, we have $(\|A_n\|, \|A_1\|, \ldots, \|A_{i-1}\|, \|A_i\|) = (1, 2, \ldots, 2, 1)$, $(\|B_1\|, \ldots, \|B_i\|) = (2, \ldots, 2)$, and $A_\nu = B_\nu$ for $\nu > i$; see Figure 1 (left). Hence we have $U' = \delta^p B_1 \ldots B_{j-1}$, $y' = B_j$, $V' = B_{j+1} \ldots B_{n-1}$ for some j in the range $i < j < n-1$ and we obtain the desired decomposition $Y = UyV$ by setting $U = A_n^{-1}U' = \delta^p A_1 \ldots A_{j-1}$, $y = y' = A_j$, $V = V'A_n = A_{j+1} \ldots A_n$. □

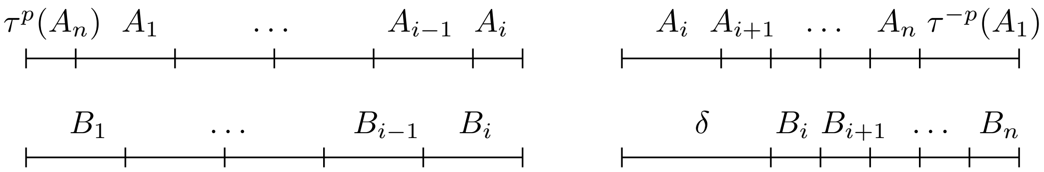

FIGURE 1. Illustration to the proof of Lemma 4.6 (on the left) and Lemma 4.7 (on the right)

Lemma 4.7. *Let $Y \in a^G b^G$ where $a, b \in \mathcal{A}$. Suppose that $\sup_s Y > 1$, $\sup Y = \sup_s Y$ (i. e., Y^{-1} is in its summit set), and $\|\varphi(Y)\| = 1$. Then there exist $U, V \in G$ such that, up to exchange of a and b, we have $Y = UyV$ with $y \in a^G \cap \mathcal{A}$, $UV \sim b$ and the following conditions hold:*

(i) *$\ell(V) \geq 1$;*
(ii) *$\ell(yV) = 1 + \ell(V)$;*
(iii) *if $\ell(U) > 0$, then the product $\varphi(U) \cdot \iota(yV)$ is left weighted;*
(iv) *if $\ell(U) > 0$, then $\ell_2\big(\varphi(U)yV\big) \geq 1$.*

Note that (ii) and (iii) imply $\ell(Y) = \ell(U) + 1 + \ell(V)$.

Proof. Induction on $\inf_s Y - \inf Y$. If $\inf_s Y - \inf Y = 0$, then $Y \in \mathrm{SSS}(Y)$, and the result follows from Corollary 1.2. Indeed, if $Y = y^P z^Q$ with $\ell(Y) = 2 + 2\ell(P) + 2\ell(Q)$, then we set $U = P^{-1}$ and $V = Pz^Q$.

Suppose that $\inf_s Y - \inf Y > 0$. Then $\inf \mathbf{c}(Y) = \inf Y + 1$ by Lemma 4.5a. Let $\delta^p \cdot A_1 \cdot \ldots \cdot A_n$ be the left normal form of Y. We set $\tilde{A}_1 = \tau^{-1}(A_1)$. Then the left normal form $\delta \cdot B_2 \cdot \ldots \cdot B_n$ of $\delta^{-p}\mathbf{c}(Y)$ is obtained from $(A_2 \cdot \ldots \cdot A_n)\tilde{A}_1$ by the procedure described in Lemma 2.5:

$$\begin{aligned}(A_2 \cdot \ldots \cdot A_n)\tilde{A}_1 &= (A_2 \cdot \ldots \cdot A_{n-1})(C_n \cdot B_n) = \ldots \\ &= (A_2 \cdot \ldots \cdot A_{i-1} \cdot A_i)(C_{i+1} \cdot B_{i+1} \cdot \ldots \cdot B_n) \\ &= (A_2 \cdot \ldots \cdot A_{i-1})(\ \delta \ \cdot \ B_i \ \cdot \ B_{i+1} \cdot \ldots \cdot B_n) = \ldots \\ &= (\delta \cdot B_2 \cdot \ldots \cdot B_{i-1} \cdot B_i \cdot \ldots \cdot B_n)\end{aligned}$$

where $2 \leq i \leq n$, all the products in the parentheses are left weighted, and $B_\nu = \tau(A_\nu)$ for $\nu = 2, \ldots, i-1$. It follows that $(\|A_i\|, \|A_{i+1}\|, \ldots, \|A_n\|, \|A_1\|) = (2, 1, \ldots, 1, 2)$ and $(\|B_i\|, \ldots, \|B_n\|) = (1, \ldots, 1)$; see Figure 1 (right).Note that the condition $\|\varphi(Y)\| = 1$ reads as $\|A_n\| = 1$. Since $\|A_i\| = 2$, this yields $i < n$.

Since $\varphi(\mathbf{c}(Y)) = B_n$ and $\|B_n\| = 1$, we may assume that the induction hypothesis holds, so, we have a decomposition $\mathbf{c}(Y) = U'y'V'$ with the required properties. Without loss of generality we may assume also that $\inf V' = 0$. We shall refer to Conditions (i)–(iv) applied to the decomposition $\mathbf{c}(Y) = U'y'V'$ by writing (i)′–(iv)′. Condition (iii)′ means that $U' = \delta^{p+1}B_2 \dots B_{j-1}$ and $y'V' = B_j \dots B_n$ for some $j \geq 2$. Condition (iv)′ combined with $\|B_i\| = \dots = \|B_n\| = 1$ implies $j \leq i$.

Let $U = \delta^p A_1 \dots A_{j-1}$, $y = \tau^{-1}(y')$, and $V = y^{-1}A_j \dots A_n$. First, let us show that $y \preccurlyeq A_j$. Indeed, if $j < i$, then $y = \tau^{-1}(y') \preccurlyeq \tau^{-1}(B_j) = A_j$. If $j = i$, then $yC_{i+1} \preccurlyeq y\delta = \delta y' = \delta B_i = A_i C_{i+1}$ whence $y \preccurlyeq A_i = A_j$. Thus

$$A_j = ys, \qquad V = s \cdot (A_{j+1} \cdot \ldots \cdot A_n), \qquad s \in [1, \delta[. \tag{4.2}$$

We have $y \sim y' \sim a$ and $UV = \tilde{A}_1 U'V'\tilde{A}_1^{-1} \sim U'V' \sim b$. Let us show that the decomposition $Y = UyV$ satisfies (i)–(iv). Indeed, $i < n$ implies (i), $\|A_i\| = 2$ implies (iv), and the fact that $A_1 \cdot \ldots \cdot A_n$ is left weighted implies (iii). So, it remains to check that (ii) holds. By (4.2) we have $\ell(V) \leq \ell(yV) \leq \ell(V) + 1$, thus it is enough to exclude the case $\ell(V) = \ell(yV)$, that is $\ell(V) = n - j + 1$.

Suppose that $\ell(V) = n - j + 1$. The product of $n - j$ factors in the parentheses in (4.2) is left weighted, hence $A_n \succcurlyeq \varphi(V)$ by Lemma 2.3. Since $A_n = \varphi(Y)$, we have $\|A_n\| = 1$ by the hypothesis of the lemma. Thus the condition $A_n \succcurlyeq \varphi(V)$ implies $A_n = \varphi(V)$. We have

$$\begin{aligned} \sup V\tilde{A}_1 &= \sup V' + 1 && \text{because } V\tilde{A}_1 = \delta V' \\ &= \sup y'V' && \text{because } \ell(y'V') = \ell(V') + 1 \text{ by (ii)}' \\ &= n - j + 1 && \text{because } y'V' = B_j \dots B_n \end{aligned}$$

hence $\sup V\tilde{A}_1 = \sup V$ which implies $A_n\tilde{A}_1 = \varphi(V)\tilde{A}_1 \preccurlyeq \delta$ by Lemma 2.1b. Hence $\sup \mathbf{c}(Y) < \sup Y$ which is impossible because $\sup Y = \sup_s Y$. □

Lemma 4.8. *Let $V \in G$ and $x, y \in \mathcal{A}$ be such that:*

(i) $\ell(yV) = 1 + \ell(V) \geq 2$;
(ii) $\inf yVx = \inf yV$;
(iii) $\sup yVx = \sup yV$.

Let $t = \varphi(yVx)$ and $yVx = Wt$. Then $y \preccurlyeq \varphi^{\urcorner}(W)$.

Proof. Without loss of generality we may assume that $\inf yVx = \inf V = 0$. Then we have $\ell(U) = \sup U$ for elements U of G considered in this proof. Let $r = \ell(V)$. The fact that $t = \varphi(Wt)$ implies $\ell(W) = \ell(Wt) - 1$, hence

$$\ell(W) = \ell(yVx) - 1 = \ell(yV) - 1 = \ell(V) = r. \tag{4.3}$$

Let $A_1 \cdot \ldots \cdot A_r$ and $B_0 \cdot \ldots \cdot B_r$, $r \geq 1$, be the left normal form of V and of yV respectively. By (ii) and (iii) we have $\delta \succ B_r x$. Since $B_r x \succcurlyeq t$, we may write $B_r x = st$ with $s \in [1, \delta[$. It follows from Lemma 2.5 that $B_{r-1}s \cdot t$ is the left normal form of $B_{r-1} \cdot B_r x$, in particular,

$$B_{r-1}s \preccurlyeq \delta \tag{4.4}$$

Let i be the minimal non-negative integer such that $A_j = B_j$ for all $j > i$.

Case 1. $i = r$. Then we have $\|B_0\| = \dots = \|B_{r-1}\| = 2$ by Lemma 2.3. Hence the left normal form of yVx is $B_0 \cdot \ldots \cdot B_{r-1} \cdot B_r x$. Therefore the right normal form of W is $B_0 \cdot \ldots \cdot B_{r-1}$, and we obtain $y \preccurlyeq B_0 = \varphi^{\urcorner}(W)$.

Case 2. $i = r-1$ and $s = 1$. Then $t = A_r x = B_r x$ and $W = y \cdot A_1 \cdot \ldots \cdot A_{r-1}$, hence $y = \varphi^{\urcorner}(W)$ by (4.3).

Case 3. $i = r-1$ and $s \neq 1$. Then we have $\|B_0\| = \dots = \|B_{r-2}\| = 2$ by Lemma 2.3. Hence the left normal form of yVx is $B_0 \cdot \ldots \cdot B_{r-2} \cdot B_{r-1}s \cdot t$ and the left normal form of W is $B_0 \cdot \ldots \cdot B_{r-2} \cdot B_{r-1}s$. The right normal form of W coincides with the left normal form because the letter length of each canonical factor is 2. Hence $y \preccurlyeq B_0 = \varphi^{\urcorner}(W)$.

Case 4. $i \le r-2$. Then $B_r = A_r$, $B_{r-1} = A_{r-1}$, and $W = y \cdot A_1 \cdot \ldots \cdot A_{r-2} \cdot B_{r-1}s$. By (4.4), this is a decomposition of W into a product of r simple elements. Hence $y = \varphi^{\urcorner}(W)$ by (4.3). □

5. Proof of Theorem 1.4

Let the hypothesis of Theorem 1.4 hold. For a permutation (λ, μ, ν) of $(1, 2, 3)$ and an integer n, we set

$$\mathcal{Q}^{(\lambda)}_{n,p} = \{(x, Y) \mid xY \sim X,\ x \in a_\lambda^G \cap \mathcal{A},\ Y \in a_\mu^G a_\nu^G,\ \ell(Y) \le n,\ \inf Y \ge p\},$$

$$\mathcal{Q}_{n,p} = \mathcal{Q}^{(1)}_{n,p} \cup \mathcal{Q}^{(2)}_{n,p} \cup \mathcal{Q}^{(3)}_{n,p}, \qquad \mathcal{Q}_n = \bigcup_p \mathcal{Q}_{n,p}. \qquad \text{and} \qquad \mathcal{Q} = \bigcup_n \mathcal{Q}_n.$$

Till the end of the section (x, y, z) will always denote some permutation of (b_1, b_2, b_3) with $b_i \in a_i^G \cap \mathcal{A}$, and $x_1, x_2, \dots$ (resp. $y_1, y_2, \dots$ or $z_1, z_2, \dots$) will stand for some atoms which are conjugate to x (resp. to y or to z). All these new atoms will be obtained from x, y, z by applying Lemma 3.1.

Lemma 5.1. *Let $(x, Y) \in \mathcal{Q}_{n,p}$ and $p < 0$. Suppose that $\inf xY > p$ or $\inf Yx > p$. Then $\mathcal{Q}_{n-1} \neq \varnothing$.*

Proof. By symmetry, it is enough to consider the case when $\inf xY > \inf Y$. Let $A = \iota(Y)$. Then $\delta \preccurlyeq xA$ by Lemma 2.1a. Since $\|x\| = 1$ and $\|A\| \le 2$, this means

$$xA = Ax_1 = \delta. \tag{5.1}$$

Case 1. $Y \in \mathrm{SSS}(Y)$. Then, by Corollary 1.2, we have $Y = AUyV$ with $\ell(Y) = \ell(U) + \ell(V) + 2$ and $AUV \sim z$. Hence, for $Z = VxAU = V\delta U$, we have $yZ = yVxAU \sim xAUyV = xY \sim X$ and $Z = VxAU \sim xAUV \in x(z^G)$. Since $\ell(Z) = \ell(V\delta U) \le \ell(V) + \ell(U) = \ell(Y) - 2 \le n - 2$, we obtain $(y, Z) \in \mathcal{Q}_{n-2}$.

Case 2. $Y \notin \mathrm{SSS}(Y)$. By Lemma 4.5c, $\inf Y < \inf \mathbf{c}(Y)$ or $\sup \mathbf{d}(Y) < \sup Y$. If $\inf Y < \inf \mathbf{c}(Y)$, then $(xY)^A = x_1 \mathbf{c}(Y)$ by (5.1), whence $(x_1, \mathbf{c}(Y)) \in \mathcal{Q}_{n-1}$.

Suppose that $\sup \mathbf{d}(Y) < \sup Y$. Let $B = \varphi(Y)$, $Y = Y_1 B$. Then $\mathbf{d}(Y) = BY_1$ and $\ell(Y_1) = \ell(Y) - 1$. Let $C = \varphi^{\urcorner}(Y_1)$, $Y_1 = CY_2$. Then $\ell(Y_2) = \ell(Y_1) - 1 = \ell(Y) - 2$. Since

$$\sup(BY_1) = \sup \mathbf{d}(Y) < \sup Y = \sup B + \sup Y_1,$$

we obtain $BC \preccurlyeq \delta$ by Lemma 2.1b. We have $C = \varphi^{\urcorner}(Y_1) \preccurlyeq \iota(Y_1) = \iota(Y) = A$ whence $xC \preccurlyeq xA = \delta$ by (5.1). Hence $xC = Cx_2$ and we obtain $(xY)^C = x_2 Y^C$ with

$$\ell(Y^C) = \ell(Y_2 BC) \le \ell(Y_2) + \ell(BC) = \ell(Y_2) + 1 = \ell(Y) - 1,$$

thus $(x_2, Y^C) \in \mathcal{Q}_{n-1}$. □

Lemma 5.2. *Let* $(x, Y) \in \mathcal{Q}_{n,p}$ *and* $p < 0$. *Suppose that* $\sup xY \le \sup Y$ *or* $\sup Yx \le \sup Y$. *Then either* $xY \in \mathrm{SSS}(X)$, *or* $Yx \in \mathrm{SSS}(X)$, *or* $\mathcal{Q}_{n-1} \ne \varnothing$.

Proof. By symmetry, it is enough to consider only the case $\sup Yx \le \sup Y$. Then we have $Ax = x_1 A \preccurlyeq \delta$ with $x_1 \in x^G \cap \mathcal{A}$ and $A = \varphi(Y)$, $Y = Y_1 A$. By Lemma 5.1 we may assume that

$$\inf xY = \inf Yx = \inf Y. \tag{5.2}$$

Let $B = \iota^{\urcorner}(Yx)$. Since the simple element Ax divides $\delta^{-p}Yx$ from the right but δ does not due to (5.2), we conclude that $B \succcurlyeq Ax$. Since $\|Ax\| = 2$, this means that $B = Ax$. Then $\mathbf{c}^{\urcorner}(Yx) = BY_1$. If $B \cdot \iota(Y_1)$ is not left weighted, then $\inf BY_1 > \inf Y_1 = p$ and the result follows from Lemma 5.1 applied to $(x_1, \mathbf{d}(Y))$ because $x_1\mathbf{d}(Y) = x_1 A Y_1 = BY_1$. So, we assume that $B \cdot \iota(Y_1)$ is left weighted whence $\inf BY_1 = \inf Y_1$ which means that $\inf \mathbf{c}^{\urcorner}(Yx) = \inf Yx$. By Lemma 4.2b this implies that either

$$\inf Yx = \inf\nolimits_s Yx \tag{5.3}$$

or $\ell_2(Y) = 0$.

Case 1. $\ell_2(Y) = 0$. Let $C = \iota(Y)$, $Y = CY_2A$. If $A \cdot C$ is left weighted, then Y is rigid, hence $Y \in \mathrm{SSS}(Y)$ which contradicts [20; Corollary 3]. Hence $AC \preccurlyeq \delta$ and we obtain $(x_1, \mathbf{d}(Y)) \in \mathcal{Q}_{n-1}$ because

$$x_1\mathbf{d}(Y) = x_1\mathbf{d}(Y_1A) = x_1AY_1 = AxY_1 \sim xY_1A = xY \sim X$$

and

$$\ell(\mathbf{d}(Y)) = \ell(\mathbf{d}(CY_2A)) = \ell(ACY_2) \le \ell(AC) + \ell(Y_2) = 1 + \ell(Y_2) = \ell(Y) - 1.$$

Case 2. $\ell_2(Y) > 0$, thus (5.3) holds. If $\sup_s Yx = \sup Yx$, then $Yx \in \mathrm{SSS}(X)$ and we are done. So, we assume that $\sup_s Yx < \sup Yx$ which implies by Lemma 4.5b

$$\sup \mathbf{d}(Yx) < \sup Yx. \tag{5.4}$$

Case 2.1. $\sup_s Y = \sup Y$. Suppose that $\sup_s Y \le 1$. Then $\inf_s Y = 0$ and $\sup_s Y = 1$ by [20; Theorem 1b] (or by Corollary 1.2). By Lemma 4.1, this yields

$$p = \inf Y = e(Y) - 2\sup Y + \ell_1(Y) = 2 - 2 \times 1 + \ell_1(Y) \ge 0$$

which contradicts the hypothesis $p < 0$. Thus $\sup_s Y > 1$. Recall also that $\varphi(Y) = A$ and $Ax \prec \delta$ whence $\|A\| = 1$.

So, we may use Lemma 4.7. Hence $Y = UyV$ where $UV \sim z$ and Conditions (i)–(iv) of Lemma 4.7 hold. Condition (iii) implies $\varphi(yV) = \varphi(Y) = A$. Condition (iv) implies that the left normal form of Vx coincides with the tail of the left normal form of Yx, in particular, $\varphi(Yx) = \varphi(yVx)$; we denote this element by t and we set $yVx = Wt$ as in Lemma 4.8. Then we have $y \preccurlyeq \varphi^{\urcorner}(W)$ by Lemma 4.8 and we set $\varphi^{\urcorner}(W) = ys = sy_1$, $W = ysW_1$with $s \in [1, \delta[$.

We are going to prove that $(y_1, Z) \in \mathcal{Q}_{n-1}$ for $Z = W_1tUs$. We evidently have:

$$y_1Z = y_1W_1tUs \sim sy_1W_1tU = ysW_1tU = WtU = yVxU \sim xUyV = xY \sim X,$$
$$Z = W_1tUs \sim sW_1tU = VxU \sim xUV \in x^Gz^G.$$

So, it remains to show that $\ell(Z) < n$. We have $\mathbf{d}(Yx) = \mathbf{d}(UWt) = tUW$ and $\sup(\mathbf{d}(Yx)) < \sup(Yx) = \sup(Y)$ by (5.4), thus

$$\sup tUW < \sup Y. \tag{5.5}$$

If $\sup tU + \sup W < \sup Y$, then

$$\ell(Z) = \ell(W_1 tUs) \leq \ell(W_1) + \ell(tU) + 1 = \ell(W) + \ell(tU) < \ell(Y) = n$$

and we are done. So, we assume that $\sup tU + \sup W \geq \sup Y$. Since

$$\begin{aligned}\sup tU + \sup W &\leq 1 + \sup U + \sup W = \sup U + \sup Wt \\ &= \sup U + \sup yVx = \sup U + \sup yV = \sup Y,\end{aligned}$$

it follows that $\sup tU + \sup W = \sup Y$. Then (5.5) combined with Lemma 2.1b yields $\varphi(tU)\varphi^{\urcorner}(W) \preccurlyeq \delta$ whence $Bs \preccurlyeq Bsy_1 = B\varphi^{\urcorner}(W) \preccurlyeq \delta$ where $B = \varphi(tU)$. Thus, by setting $tU = U_1 B$, we obtain

$$\begin{aligned}\ell(Z) &= \ell(W_1 tUs) \leq \ell(W_1 U_1 Bs) \leq \ell(W_1 U_1) + \ell(Bs) = \ell(W_1 U_1) + 1 \\ &\leq \ell(W_1) + \ell(U_1) + 1 = \ell(W) + \ell(U_1) = \ell(Wt) + \ell(U) - 1 \\ &= \ell(yVx) + \ell(U) - 1 = \ell(U) + \ell(yV) - 1 = \ell(Y) - 1 = n - 1.\end{aligned}$$

Case 2.2. $\sup \mathbf{d}(Y) < \sup Y$. Recall that $Yx = Y_1 Ax = Y_1 B$ where $A = \varphi(Y)$ and $B = Ax = \iota^{\urcorner}(Yx)$. So, we have $\mathbf{d}(Y) = AY_1$. Thus the condition $\sup \mathbf{d}(Y) < \sup Y$ reads as $\sup AY_1 < \sup Y_1 A = \sup A + \sup Y_1$, hence, by Lemma 2.1b, we have $AC \preccurlyeq \delta$ where we set $C = \varphi^{\urcorner}(Y_1)$, $Y_1 = CY_2$. Since $B = \iota^{\urcorner}(Yx)$ and $Yx = Y_1 B$, we have $\varphi^{\urcorner}(Yx) = \varphi^{\urcorner}(Y_1 B) = \varphi^{\urcorner}(Y_1) = C$. Thus $(x_1, \mathbf{d}(Y)) \in \mathcal{Q}_{n-1}$ because

$$x_1 \mathbf{d}(Y) = x_1 \mathbf{d}(Y_1 A) = x_1 AY_1 = AxY_1 \sim Y_1 Ax = Yx \sim X$$

and $\mathbf{d}(Y) = \mathbf{d}(Y_1 A) = AY_1 = ACY_2$ whence

$$\ell(\mathbf{d}(Y)) \leq \ell(AC) + \ell(Y_2) = 1 + \ell(Y_2) = \ell(Y_1) = \ell(Y) - 1.$$

Case 2.3. $\sup_s Y < \sup \mathbf{d}(Y) = \sup Y$. Let us show that this case is impossible. Indeed, the condition $\sup \mathbf{d}(Y) = \sup Y$ combined with Lemma 4.3b yields $\ell_1(Y_1) = 0$. Since, moreover, $Yx = Y_1 B$, $B = Ax = \iota^{\urcorner}(Yx)$ and $\|B\| = 2$, we obtain $\ell_1(Yx) = 0$. By (5.3) this implies that Yx is rigid which contradicts (5.4). □

Lemma 5.3. *Let $(x, Y) \in \mathcal{Q}_{n,p}$, $p < 0$. Suppose that $\ell(xY) = \ell(Yx) = 1 + \ell(Y)$. Then either $xY \in \mathrm{SSS}(X)$, or $Yx \in \mathrm{SSS}(X)$, or $\mathcal{Q}_{n-1} \neq \varnothing$, or $\mathcal{Q}_{n,p+1} \neq \varnothing$*

Proof. The condition $\ell(Yx) = \ell(Y) + 1$ implies $\varphi(Yx) = x$ and hence $\mathbf{d}(Yx) = xY$.

Case 1. $\sup Yx > \sup_s Yx$. By [5] we then have

$$\sup \mathbf{d}(xY) = \sup \mathbf{d}^2(Yx) < \sup Yx. \tag{5.6}$$

Since $\ell(xY) = \ell(Yx)$, we have $\sup \mathbf{d}(Yx) = \sup xY = \sup Yx$. Hence $\ell_1(Y) = 0$ by Lemma 4.3b. Let $A = \varphi(Y)$, $B = \varphi(xY)$, $C = \iota(xY)$, and let $xY = CUB$.

We have $A \neq B$ (otherwise we would obtain $\sup \mathbf{d}(Yx) < \sup Yx$ by Lemma 4.3a) and we have $A \not\succcurlyeq B$ by Lemma 2.3. Hence $\|B\| = 1$. By combining this fact with $\ell_1(CUB) = \ell_1(xY) = \ell_1(Yx) = 1$, we obtain $\ell_1(CU) = 0$. It follows that the left normal form of $\delta^{-p}CU$ coincides with its right normal form, in particular, $\varphi^{\urcorner}(CU) = \iota(CU) = C$. By (5.6), we have

$$\sup BCU = \sup \mathbf{d}(xY) < \sup xY = \sup CUB = \sup B + \sup CU.$$

Hence, by Lemma 2.1b, we have $B\varphi^{\urcorner}(CU) \preccurlyeq \delta$, that is $BC \preccurlyeq \delta$. This implies $BC = \delta$ because $\|C\| = 2$ (recall that $\ell_1(CU) = 0$) and $\|B\| = 1$. We have $x \preccurlyeq \iota(xY) = C$, hence $Bx \preccurlyeq BC = \delta$ which yields $Bx = x_1 B$ with $x_1 \in x^G \cap \mathcal{A}$. Since $x \preccurlyeq C$, we may write $C = xC'$, $C' \in [1, \delta]$. So, for $Z = BC'U$, we obtain

$$x_1 Z = x_1 BC'U = BxC'U = BCU = \mathbf{d}(CUB) = \mathbf{d}(xY) \sim X$$

and $Z = BC'U \sim C'UB = x^{-1}CUB = Y$. We have

$$\ell(Z) \leq \ell(BC') + \ell(U) = 1 + \ell(U) = \ell(xY) - 1 = \ell(Y) = n,$$

hence $(x_1, Z) \in \mathcal{Q}_{n,p}$. Since $x_1 Z = x_1 BC'U = BxC'U = BCU = \delta U$, we have $\inf x_1 Z > \inf U = p$, thus the result follows from Lemma 5.1.

Case 2. $\sup Yx = \sup_s Yx$. If $\inf Yx = \inf_s Yx$, then $Yx \in \mathrm{SSS}(X)$ and we are done. So, we suppose that $\inf Yx < \inf_s Yx$. Then, by Lemma 4.5a, we have

$$\inf Yx < \inf \mathbf{c}(Yx). \tag{5.7}$$

Let $A = \iota(Y)$, $Y = AY_1$. The condition $\ell(Yx) = \ell(Y) + 1$ implies that $\varphi(Y) \cdot x$ is left weighted whence $\iota(Yx) = \iota(Y) = A$. Thus $\mathbf{c}(Y) = Y_1 A$, $\mathbf{c}(Yx) = Y_1 x A$, and

$$\varphi(Yx) = \varphi(Y_1 x) = x. \tag{5.8}$$

Case 2.1. $\inf Y = \inf_s Y$. Let $t = \mathfrak{p}(Yx)$, $A = tA'$, thus $\mathfrak{s}(Yx) = A'Y_1 xt$. Then $xt \preccurlyeq \delta$, hence $xt = tx_2$, $x_2 \in x^G \cap \mathcal{A}$, and we obtain $\mathfrak{s}(Yx) = Y^t x_2$. By (5.7) combined with [13; Lemma 4] we have

$$\inf Yx < \inf \mathfrak{s}(Yx). \tag{5.9}$$

Since $t \preccurlyeq A = \iota(Y)$, we have $\ell(Y^t) \leq \ell(Y) + 1$. If $\ell(Y^t) \leq \ell(Y)$, then the result follows from Lemma 5.1 applied to (x_2, Y^t), because $x_2 Y^t \sim Y^t x_2 = \mathfrak{s}(Yx) \sim X$ and $\inf Y^t x_2 > p$ by (5.9). So, we assume that

$$\ell(Y^t) = \ell(Y) + 1. \tag{5.10}$$

The condition $t \preccurlyeq A = \iota(Y)$ implies $\inf Y^t \geq \inf Y$. Since $\inf Y = \inf_s Y$, it follows that $\inf Y^t = \inf_s Y$. Hence, by the 'right-to-left' version of Lemma 4.6, we have $Y^t = UyV$ with $UV \sim z$, $\ell(U) + \ell(V) + 1 = \ell(Y^t)$, $\ell(V) \geq 1$, and $\iota^{\urcorner}(Uy) \cdot \varphi^{\urcorner}(V)$ right weighted. The last two conditions imply $\iota^{\urcorner}(Y^t) = \iota^{\urcorner}(V)$; we denote this element by B and we set $V = V_1 B$. By (5.9) and (5.10) we have

$$\inf Y^t + \inf x_2 = \inf Y^t = \inf Y = \inf Yx < \inf \mathfrak{s}(Yx) = \inf Y^t x_2,$$

hence $\delta \preccurlyeq \iota^{\Lsh}(Y^t)x_2 = Bx_2$ by Lemma 2.1a. Since $\|Bx_2\| \le \|\delta\|$, this means that $Bx_2 = \delta$, and we obtain

$$\mathfrak{s}(Yx) = UyVx_2 = UyV_1Bx_2 = UyV_1\delta \sim yZ$$

where $Z = V_1\delta U$. Since $UV \sim z$, we have $Z \sim UV_1\delta = UV_1Bx_2 = UVx_2 \in z^G x^G$. Since, moreover,

$$\ell(Z) \le \ell(U) + \ell(V_1) = \ell(U) + \ell(V) - 1 = \ell(Y^t) - 2 = \ell(Y) - 1,$$

we conclude that $(y, Z) \in \mathcal{Q}_{n-1}$.

Case 2.2. $\inf Y < \inf \mathbf{c}(Y)$. Recall that $Y = AY_1$ and $A = \iota(Y) = \iota(Yx)$. Let $B = \iota^{\Lsh}(Y_1)$, $Y_1 = Y_2B$. Since

$$\inf Y_1 + \inf A = \inf Y < \inf \mathbf{c}(Y) = \inf Y_1A,$$

we have $\delta \preccurlyeq BA$ by Lemma 2.1a. Hence $B = CD$ and $DA = \delta$ for some simple elements C and D. By Theorem 3.2, the left normal form of DxA is $D' \cdot x_1 \cdot A'$ with $A', D' \in \mathcal{P}$, $x_1 \in x^G \cap \mathcal{A}$, and $D'A' = \delta$.

Since $\iota(Yx) = \iota(Y) = A$, we have $\mathbf{c}(Yx) = Y_1xA = (Y_2C)(DxA)$. Hence, $\delta \preccurlyeq Y_2C\,\iota(DxA) = Y_2CD'$ by (5.7) combined with Lemma 2.1a. Hence, for $Z = A'Y_2CD'$, we have $\inf Z > \inf Y$ and

$$\ell(Z) \le \ell(A') + \ell(Y_2) + \ell(C) + \ell(D') - 1 \le \ell(Y).$$

Since $Z \sim Y_2CD'A' = Y_2C\delta = Y_2CDA = Y_1A \sim Y$ and

$$x_1Z \sim Y_2CD'x_1A' = Y_2CDxA = Y_1xA \sim Yx \sim X,$$

we conclude that $(x_1, Z) \in \mathcal{Q}_{n,p+1}$.

Case 2.3. $\inf Y = \inf \mathbf{c}(Y) < \inf_s Y$. Then $\ell_2(Y_1) = 0$ by Lemma 4.2b. By (5.8), this implies $\ell_2(Y_1x) = 0$ whence $\iota^{\Lsh}(Y_1x) = \varphi(Y_1x) = x$. By (5.7), we have

$$\inf Y_1xA = \inf \mathbf{c}(Yx) > \inf Yx = \inf Y_1x + \inf A.$$

Hence $\delta \preccurlyeq \iota^{\Lsh}(Y_1x)A = xA$ by Lemma 2.1a. Since $\|xA\| \le 3$, this means that $xA = \delta$. Hence $xA = Ax_1$, $x_1 \in \mathcal{A}$, and we obtain $\mathbf{c}(Yx) = Zx_1$ where $Z = Y_1A = \mathbf{c}(Y) \sim Y$ and $\delta \preccurlyeq Zx_1$, so, the result follows from Lemma 5.1 applied to (x_1, Z). $\square$

Lemma 5.4. *Let $(x, Y) \in \mathcal{Q}$ and $Yx \in \mathrm{SSS}(X)$. Then there exists $(x_1, Y_1) \in \mathcal{Q}$ such that $x_1Y_1 \in \mathrm{SSS}(X)$.*

Proof. Let $A = \iota^{\Lsh}(Yx)$ and $Yx = UA$. Then $A \succcurlyeq x$ whence $A = sx = x_1s$ and $Y = Us$ for a simple element s. Let $X_1 = \mathbf{c}^{\Lsh}(Yx)$ and $Y_1 = sU$. Then we have $X_1 = AU = x_1sU = x_1Y_1$, hence $(x_1, Y_1) \in \mathcal{Q}$ and $x_1Y_1 \in \mathrm{SSS}(X)$. $\square$

Theorem 1.4 immediately follows from Lemmas 5.1 – 5.4.

6. Structure of SSS(X) when $\|\Delta\| = 3$ (after S.-J. Lee)

Here we give a summary of those results from [14; Chapter 4] which extend to any homogeneous Garside group with Garside element of letter length 3.

Let $(G, \mathcal{P}, \Delta)$ be a homogeneous Garside structure with set of atoms $\mathcal{A}$ such that $\|\Delta\| = 3$.

We say that $X \in G$ is **rigid** if $\varphi(X) \cdot \iota(X)$ is left weighted. Following [14], we say that X is **strictly rigid** if it is rigid and $\ell_1(X) = 0$ or $\ell_2(X) = 0$ (see (4.1)). If $X \in \mathrm{USS}(X)$, then we define the cycling orbit of X as $O_X = \{\mathbf{c}^m \tau^k(X) \mid k, m \geq 0\}$.

Proposition 6.1. *Let* $X \in \mathrm{USS}(X)$, $\ell(X) \geq 2$. *Then:*

(a). $\mathrm{SC}(X) = \mathrm{USS}(X)$.

(b). $\mathrm{SSS}(X) = \bigcup_{m \geq 0} \mathbf{c}^{\urcorner m}(\mathrm{USS}(X))$.

(c). One and only one of the following alternatives holds:

(i) *each element of* $\mathrm{USS}(X)$ *is strictly rigid and* $\mathrm{SSS}(X) = \mathrm{USS}(X)$;

(ii) *each element of* $\mathrm{USS}(X)$ *is rigid but not strictly rigid, and* $\mathrm{USS}(X) = O_X$;

(iii) *no element of* $\mathrm{SSS}(X)$ *is rigid and* $\mathrm{SSS}(X) = \mathrm{USS}(X) = O_X$.

Lemma 6.2. *If X is not rigid and* $X \in \mathrm{SSS}(X)$, *then* $\mathbf{c}^{\urcorner}(\mathbf{c}(X)) = \mathbf{d}^{\urcorner}(\mathbf{d}(X)) = X$.

Proof. If $\ell(X) = 1$, the statement is evident. Assume that $\ell(X) > 1$. Since X is not rigid, the product $\varphi(X) \cdot \iota(X)$ is not left weighted. Since $X \in \mathrm{SSS}(X)$, this implies $\|\varphi(X)\| = 1$ and $\|\iota(X)\| = 2$. Let $X = \iota(X)U$. Then $\mathbf{c}(X) = U\iota(X)$, hence $\iota^{\urcorner}(\mathbf{c}(X)) \succcurlyeq \iota(X)$. This fact combined with $\|\iota(X)\| = 2$ implies $\iota^{\urcorner}(\mathbf{c}(X)) = \iota(X)$ whence $\mathbf{c}^{\urcorner}(\mathbf{c}(X)) = X$. Similarly $\mathbf{d}^{\urcorner}(\mathbf{d}(X)) = X$. □

Lemma 6.3. *Let* $X \in G$, $\ell(X) > 1$. *Suppose that X^G does not contain any rigid element. Then* $\mathrm{SC}(X) = \mathrm{SC}^{\urcorner}(X) = \mathrm{SSS}(X)$.

Proof. Lemma 6.2 implies that $\mathbf{c}$ and $\mathbf{d}$ are bijective mappings from $\mathrm{SSS}(X)$ to itself and that $\mathbf{c}^{\urcorner}$ and $\mathbf{d}^{\urcorner}$ are their inverse mappings. Hence $\mathfrak{s}$ and $\mathfrak{s}^{\urcorner}$ also are bijective mappings from $\mathrm{SSS}(X)$ to itself. □

Proof of Proposition 6.1. (a). If X^G does not contain a rigid element, then the result follows from Lemma 6.3. Otherwise it follows from [3; Theorem 3.15] which states that if X^G contains a rigid element, then all elements of $\mathrm{USS}(X)$ are rigid.

(b). Let $X \in \mathrm{SSS}(X)$ and let $m \geq 0$ be the minimal number such that $Y = \mathbf{c}^m(X) \in \mathrm{USS}(X)$. Then $\mathbf{c}^{\urcorner m}(Y) = X$ by Lemma 6.2.

(c). The fact that $\mathrm{USS}(X) = O_X$ when X is not strictly rigid is proven in [14; Theorem 4.4.1]. All the other statements follow from (a) and [3; Theorem 3.15]. □

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