REAL QUINTIC SURFACE WITH 23 COMPONENTS

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Abstract. We construct a real surface of degree 5 in $\mathbf{RP}^3$ which has 23 connected components

Theorem. *There exists a real algebraic surface in* $\mathbf{RP}^3$ *of degree* 5 *which has* $b_0 = 23$ *connected components:* 22 *spheres and* $\mathbf{RP}^2$ *with two handles.*

A quintic surface in $\mathbf{RP}^3$ with 21 components (and a surface with $b_1 = 45$) was constructed by Kharlamov [5]. He used Horikawa's result [4] which allows to deform the double covering of the quadratic cone branched along a certain curve into a quintic surface in $\mathbf{RP}^3$. Following the same scheme, Kharlamov and Itenberg [6] constructed a quintic surface in $\mathbf{RP}^3$ with 22 components. They constructed the branching curve by Viro's method using Itenberg's counter-example [3] to Ragsdale's conjecture. Bihan [1] (also using Horikawa's results and elements of Itenberg's example) constructed a real algebraic surface with $b_0 = 23$ (and a surface with $b_1 = 47$) which can be deformed as complex algebraic surface into a smooth quintic surface in $\mathbf{CP}^3$ but which is not embeddable into $\mathbf{RP}^3$. The question of existence of quintic surfaces in $\mathbf{RP}^3$ with $b_0 = 24, 25$ and those with $b_1 = 47$ is still open (it is known that $b_0 \le 25$ and $b_1 \le 47$; see [5,6]). More information about real surfaces can be found in a recent survey [2].

Our construction is just a slight modification of that in [6].

Lemma. *There exists a curve* C_6 *of degree* 6 *on* $\mathbf{RP}^2$ *arranged with respect to two lines* L_1 *and* L_2 *as in Figure 4 and which has two non-real (conjugated) ordinary triple points lying on the complexification of* L_2.

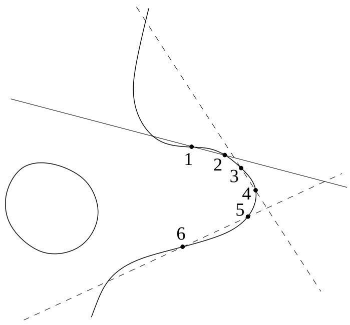

Figure 1

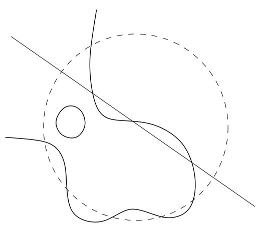

Figure 2

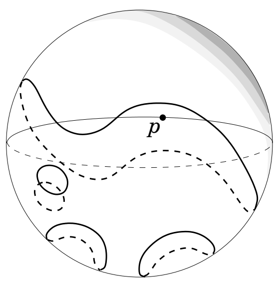

Figure 3

Proof. Let C_3 be a plane real algebraic cubic curve with two connected components. Let $1, \dots, 6$ be points on C_3 such that 1 and 6 are inflection points and $2, \dots, 5$ lye (in this order) on the convex arc 1-6. Then the lines 1-2, 3-4, and 5-6 are arranged with respect to C_3 as in Figure 1. Perturbing the union of the (dashed) lines 3-4 and 5-6, we obtain the arrangement of a line C_1, a conic C_2 and the cubic C_3 depicted in Figure 2. Let us choose affine coordinates where C_2 is a circle. Let S^2 be the sphere which has C_2 as an equator. The cylinders over C_1 and C_3 cut on S^2 curves $\hat{C}_1$ and $\hat{C}_3$ of degrees 2 and 6 arranged as in Figure 3. The complexification of S^2 is $\mathbf{CP}^1 \times \mathbf{CP}^1$. Let us denote the projections on the factors by π_1 and π_2. Let us choose a generic point p on $\hat{C}_1$ as it is shown in Figure 3, blow it up, and then let us blow down the proper transforms of the lines $\pi_j^{-1}(\pi_j(p))$, $j = 1, 2$ (these lines are mapped to each other by the complex conjugation and p is a single real point on each of them). We obtain $\mathbf{RP}^2$. The transforms of $\hat{C}_3$, $\hat{C}_1$, and the exceptional divisor of the blowup give us C_6, L_1, and L_2 respectively.

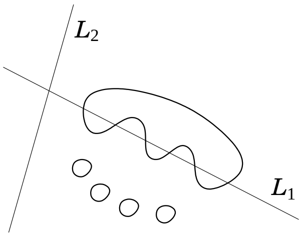

FIGURE 4

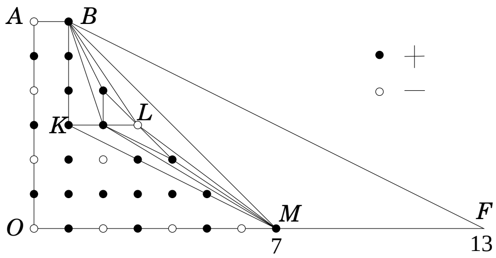

FIGURE 5

Proof of Theorem. Let Σ_2 be Hirzebruch surface of degree 2 (the standard quadratic cone in $\mathbf{P}^3$ blown up at the vertex). It is smoothly fibered over $\mathbf{P}^1$. Let L be a fiber and E the exeptional section ($L.L = 0$, $E.E = -2$). Let C be a curve such that $C.L = 6$, $C.E = 1$. Suppose that C has only two singular points, these points are ordinary triple points and they lye on L. Let us blow up the triple points and consider the double covering branched along the proper transforms L' and C' of L and C. Then we blow down the preimage of L'. According to Horikawa's theorem [4], the obtained surface can be deformed into a quintic in $\mathbf{CP}^3$. If Σ_2 and C are real then this construction gives us two real surfaces (because the double covering branched along $f(x, y) = 0$ is $z^2 = \pm f(x, y)$). The both of them can be deformed into quintic surfaces in $\mathbf{RP}^3$ (see [6; Proposition 1]).

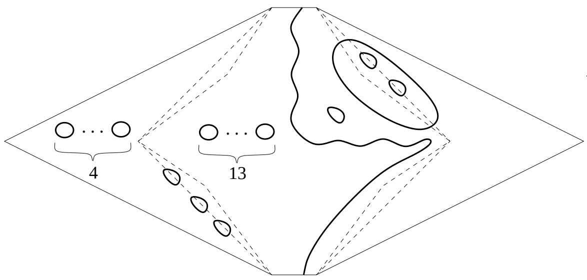

FIGURE 6

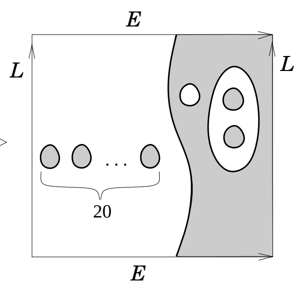

FIGURE 7

We shall construct C glueing it by Viro according to the triangulation of the quadrangle $OABF$ depicted in Figure 5 (the pentagon $OABLM$ should be arbitrarily further subdivided into triangles of area 1/2). The charts corresponding to the triangles in $OABLM$ are determined by the signs which are shown as $\bullet$ and $\circ$. The part $OABM$ is reproduced without changes from [6] (preserving even the notation of the vertices).

Choosing coordinates in Figure 4 so that L_1 and L_2 are coordinate lines, we may consider this picture as a chart of a polynomial whose Newton polygon is BFM (see Figures 5 and 6). The sides BM and BF correspond to L_1 and L_2 respectively. The obtained curve C on $\mathbf{R}\Sigma_2$ is depicted in Figure 7 where $\mathbf{R}\Sigma_2$ is cut along L (vertical sides) and E (horizontal sides). The triple points of C_6 can be saved on C by the results of Shustin [7]. The covering corresponding to the gray half provides the required surface.

Remark 1. The covering correponding to the white half of Figure 7 gives a surface with $b_1 = 45$. Surfaces with $b_1 = 45$ are known since [5].

Remark 2. Replacing only the chart $OABM$ in our construction, one can not obtain more than 23 components (nor $b_1 > 45$). Indeed, otherwise, the union of a curve C_7 with this chart (viewed as a plane 7th degree curve) and the line corresponding to the edge BM could be smoothed out into a plane curve of degree 8 with $p > 19$, i.e. into a counter-example to Ragsdale's Conjecture of degree 8 (as usual, p and n denote the numbers of even and odd ovals). This is impossible by Petrovski inequality ($p - n \le 19$), Arnold-Gudkov-Rohlin congruence ($p - n \equiv 0 \mod 8$ if $p + n = 22$), and Gudkov-Krahnov-Kharlamov congruence ($p - n \equiv \pm 1 \mod 8$ if $p + n = 21$).

I am gratefull to I. Itenberg who informed me that nobody tried non-real triple points in the construction [6].

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