POTENTIAL AT INFINITY OF A POLYNOMIAL IMAGE OF THE DISK

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Let D be a domain in $\mathbf{C}$ which is the image $q(\Delta)$ of the unit disk Δ under the mapping $t \mapsto q(t)$ where

$$q(t) = a_0 t + a_1 t^2 + ... + a_n t^{n+1}, \qquad a_0 \in \mathbf{R}, \quad |a_0| > 0 \tag{1}$$

is a polynomial, univalent in D.

Let $p(z) = \pi^{-1} \int_D (z-\zeta)^{-1} d\mu(\zeta)$ be the potential of D ($d\mu(x+iy) = dx\,dy$). The inverse problem of potential theory is the problem of reconstructing D given the germ of p at ∞ (see [1] and references therein). For $|z| \gg 1$ one has

$$p(z) = \sum_{k\ge 0} \frac{c_k}{z^{k+1}}, \qquad \text{where} \quad c_k = \frac{1}{\pi}\int_D \zeta^k d\mu(\zeta) = \frac{1}{\pi}\int_\Delta q(t)^k q'(t)\overline{q'(t)}\, d\mu(t). \tag{2}$$

(c_k are the moments of D). Using the right hand side of (2) one can define $p(z)$ for any polynomial $q(t)$ of the form (1), not necessarily univalent. Since

$$\frac{1}{\pi}\int_\Delta t^k \bar{t}^m \, d\mu(t) = \begin{cases} 1/(k+1), & \text{if } k = m \\ 0, & \text{if } k \ne m, \end{cases} \tag{3}$$

(2) allows us to express c_k as polynomials in $a_0, a_1, ..., a_n, \bar a_1, ..., \bar a_n$, with rational coefficients. It follows from (2), (3) that $c_k = 0$ for $k > n$. Hence,

$$p(z) = c_0 z^{-1} + c_1 z^{-2} + ... + c_n z^{-(n+1)}, \qquad c_0 \in \mathbf{R}, \quad |c_0| > 0 \tag{4}$$

Thus, we obtain a polynomial mapping $\eta : V^+ \to W^+$ where V (resp. W) is the vector space over $\mathbf{R}$, isomorphic to $\mathbf{R}\times\mathbf{C}^n$, with the coordinates $(a_0, ..., a_n)$ (resp. $(c_0, ..., c_n)$), V^+ (resp. W^+) is the half-spaces $a_0 > 0$, (resp. $c_0 > 0$) and η is defined by $\eta(q) = p$. We identify points of V^+ (resp. W^+) with polynomials q of the form (1) (resp. of the form (4)).

Denote by $j(\eta)$ the jacobian of η with respect to the volume forms $da_n \wedge ... \wedge da_1 \wedge da_0 \wedge d\bar a_1 \wedge ... \wedge d\bar a_n$ and $dc_n \wedge ... \wedge dc_1 \wedge dc_0 \wedge d\bar c_1 \wedge ... \wedge d\bar c_n$. Given $q \in V$, let us define $t_1, ..., t_n$ by $q'(t) = \prod_{i=1}^n (1 - t_i t)$.

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Theorem. $j(\eta) = 2a_0^{n^2+n+1}\operatorname{Res}_t(t^n\bar{q}'(t^{-1}),\, q'(t)) = 2a_0^{n^2+3n+1}\prod_{i=1}^n\prod_{j=1}^n(1-t_i\bar{t}_j)$.

Remark. (P. Etingof) A similar formula appears in the Conformal Field Theory as a formula for the cocycle defining the central extension of the complexification of the group $\operatorname{Diff}(S^1)$.

Let $\mathcal{I}_n = \{q \in V^+ \mid q' \text{ has no root in } \Delta\}$.

Corollary 1. *(*see [1]*) The restriction of* η *onto* $\operatorname{Int}\mathcal{I}_n$ *is an immersion.*

Corollary 2. *The closure of the fold of* η *is the (real) hypersurface* $F = \{q \in V^+ \mid \exists t_i,\, q'(t_i) = 0 \text{ and } |t_i| = 1\}$.

Corollary 3. *The closure of the branching set of* η *is the codimension 2 subvariety* $B = \{q \in V^+ \mid \exists t_i \neq t_j,\, t_i\bar{t}_j = 1\}$

Proof of Theorem. Denote $\partial/\partial a_m$ by ∂_m and $\partial/\partial\bar{a}_m$ by $\bar\partial_m$, $m = 0, ..., n$. ($\partial_0 = \bar\partial_0$). Put

$$J = \begin{pmatrix} \bar\partial_n c_n & \dots & \bar\partial_1 c_n & \partial_0 c_n & \partial_1 c_n & \dots & \partial_n c_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \bar\partial_n c_1 & \dots & \bar\partial_1 c_1 & \partial_0 c_1 & \partial_1 c_1 & \dots & \partial_n c_1 \\ \bar\partial_n c_0 & \dots & \bar\partial_1 c_0 & \partial_0 c_0 & \partial_1 c_0 & \dots & \partial_n c_0 \\ \bar\partial_n \bar c_1 & \dots & \bar\partial_1 \bar c_1 & \partial_0 \bar c_1 & \partial_1 \bar c_1 & \dots & \partial_n \bar c_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \bar\partial_n \bar c_n & \dots & \bar\partial_1 \bar c_n & \partial_0 \bar c_n & \partial_1 \bar c_n & \dots & \partial_n \bar c_n \end{pmatrix},$$

$$A = \begin{pmatrix} A_{n,n} & & \mathbf{0} & 0 & & & \\ \vdots & \ddots & & \vdots & & \mathbf{0} & \\ A_{1,n} & \dots & A_{1,1} & 0 & & & \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ & & & 0 & \bar A_{1,1} & \dots & \bar A_{1,n} \\ & \mathbf{0} & & \vdots & & \ddots & \vdots \\ & & & 0 & \mathbf{0} & & \bar A_{n,n} \end{pmatrix}, \quad M = \frac{1}{2a_0}\begin{pmatrix} & & & -(n+1)\bar a_n & & & \\ & 2a_0E & & \vdots & & \mathbf{0} & \\ & & & -2\bar a_1 & & & \\ 0 & \dots & 0 & a_0 & 0 & \dots & 0 \\ & & & 2a_1 & & & \\ & \mathbf{0} & & \vdots & & 2a_0E & \\ & & & (n+1)a_n & & & \end{pmatrix}$$

where $A_{k,j}$, $j \geq k > 1$ are polynomials in $a_0^{-1}, a_n, ..., a_1, a_0, \bar a_1, ..., \bar a_n$, such that $\sum_{j\geq k} A_{k,j}q^j = t^k$. Let us calculate AJ. Put $A_{k,j} = 0$ for $j < k$, $A_{0,0} = 1$, $A_{0,j} = 0$ for $j > 0$, and $a_m = 0$ unless $0 \leq m \leq n$. Denote $\frac{1}{\pi}\int_\Delta f(t)\overline{g(t)}\,d\mu(t)$ by $\langle f, g\rangle$. Then (3) means $\langle t^k, t^m\rangle = \delta_{km}/(k+1)$. Note that $\partial_m q = t^{m+1}$, $\partial_m q' = (m+1)t^m$ and

$$(\partial_m q^j)\cdot q' = \frac{dq^j}{dq}\cdot\partial_m q\cdot q' = \frac{dq^j}{dq}\cdot\frac{dq}{dt}\cdot\partial_m q = \frac{dq^j}{dt}\cdot\partial_m q.$$

Hence, for any $m \geq 0$ we have

$$\begin{aligned}\sum_j A_{k,j}\langle\partial_m(q^j q'), q'\rangle &= \Big\langle(\partial_m q)\cdot\frac{d}{dt}\sum A_{k,j}q^j, q'\Big\rangle + \Big\langle(\partial_m q')\cdot\sum A_{k,j}q^j, q'\Big\rangle \\ &= \langle t^{m+1}\cdot kt^{k-1}, q'\rangle + \langle(m+1)t^m\cdot t^k, q'\rangle \\ &= (m+k+1)\langle t^{m+k}, q'\rangle = (m+k+1)\bar a_{m+k},\end{aligned}$$

$$\sum_j A_{k,j}\langle q^j q', \partial_m q'\rangle = \langle q't^k, (m+1)t^m\rangle = (m-k+1)a_{m-k}$$

Therefore, the entries of the right upper quadrant of AJ (resp. the left upper quadrant, the upper part of the central column) are equal resp. to

$$\begin{aligned}
\sum A_{k,j}\partial_m c_j &= \sum A_{k,j}\langle \partial_m(q^j q'), q'\rangle = (m+k+1)\bar a_{m+k}, \qquad m>0,\\
\sum A_{k,j}\bar\partial_m c_j &= \sum A_{k,j}\langle q^j q', \partial_m q'\rangle = (m-k+1)a_{m-k}, \qquad m>0,\\
\sum A_{k,j}\partial_0 c_j &= \sum A_{k,j}(\langle \partial_0(q^j q'), q'\rangle + \langle q^j q', \partial_0 q'\rangle) = (k+1)\bar a_k - (k-1)a_{-k}.
\end{aligned}$$

Each entry of the lower part of AJ is conjugated to the centrally symmetric entry of the upper one. Thus,

$$AJ = \begin{pmatrix}
a_0 & & \mathbf{0} & & \text{(n+1)}\bar a_n & & & \mathbf{0} & \\
2a_1 & a_0 & & & n\bar a_{n-1} & \text{(n+1)}\bar a_n & & & \\
\vdots & 2a_1 & \ddots & & \vdots & n\bar a_{n-1} & \ddots & & \\
na_{n-1} & \vdots & \ddots & a_0 & 2\bar a_1 & \vdots & \ddots & \text{(n+1)}\bar a_n & \\
\text{(n+1)}a_n & na_{n-1} & \dots & 2\bar a_1 & 2a_0 & 2\bar a_1 & \dots & n\bar a_{n-1} & \text{(n+1)}\bar a_n \\
 & \text{(n+1)}a_n & \ddots & \vdots & 2a_1 & a_0 & \ddots & \vdots & n\bar a_{n-1} \\
 & & \ddots & n\bar a_{n-1} & \vdots & & \ddots & 2\bar a_1 & \vdots \\
 & & & \text{(n+1)}\bar a_n & na_{n-1} & & & a_0 & 2a_1 \\
 & \mathbf{0} & & & \text{(n+1)}a_n & & \mathbf{0} & & a_0
\end{pmatrix}$$

Multiplying this matrix by M from the right, we replace the central column with $(0,...,0,a_0,2a_1,...,(n{+}1)a_n)^t$ and obtain the matrix whose upper row is $(a_0,0,...,0)$ and the complementary minor of the a_0 is the transposed Sylvester matrix for the resultant of $a_0+2a_1t+...+(n+1)a_nt^n$ and $(n+1)\bar a_n + n\bar a_{n-1}t+...+\bar a_0 t^n$. Thus,

$$\det A \det J \det M = a_0 \operatorname{Res}_t(q', t^n \bar q'(t^{-1})).$$

Clearly that $\det M = 1/2$ and $A_{k,k} = a_0^{-k}$ which implies $\det A = \prod A_{k,k}^2 = a_0^{-n(n+1)}$. It remains to note that the reversing the order of ∂a_m's in J changes the sign the same way as the swapping the arguments of the resultant. □

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References

1. A.I. Prilepko, *Potential theory, inverse problems in*, in: Mathematical Encyclopedia, vol. 4, pp. 266–269.

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