PETROVSKI-OLEINIK INEQUALITIES AND COMBINATORICS OF VIRO T-HYPERSURFACES

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Introduction

Let $X \subset \mathbf{RP}^{n-1}$ be a smooth real algebraic hypersurface defined by the equation $f(x_1,\dots,x_n)=0$ where f is a homogeneous polynomial of degree m with real coefficients. The Petrovski—Oleinik inequality (in the form given by Arnold [1]) states

$$|\tilde\chi(S^{n-1}_+)| \le \Pi_n(m), \tag{$*$}$$

where $\tilde\chi$ denotes the reduced (lowered by 1) Euler characteristic, $S^{n-1}_+ = \{x \in S^{n-1} \,|\, f(x) \ge 0\}$ (as usual, S^{n-1} denotes the $(n-1)$-dimensional sphere) and $\Pi_n(m)$ is the *Petrovski number*:

$$\Pi_n(m) = \#\{(k_1,\dots,k_n)\in \mathbf{Z}^n \,|\, 0<k_i<m;\ k_1+\dots+k_n = mn/2\}.$$

It is the number of integral interior points on the section of the n-dimensional cube with the side m by the hyperplane orthogonal to the diagonal and passing through the center of the cube. Petrovski showed that (*) is sharp for $n=3$; Viro [14] showed that (*) is sharp for $n=4$. This paper appeared as the result of an unseccessful attempt to prove the sharpness of (*) for all dimensions.

A real algebraic hypersurface is called *Viro T-hypersurface* if it can be constructed by the Viro method [15] starting with a triangulation and a polynomial which has non-zero monomials only at the vertices of the triangulation (see §2 for an exact definition). Viro T-hypersurfaces gave the first realizations of: counter-examples to Ragsdale's conjecture [7]; examples of M-hypersurfaces (and M-complete intersections) of any degree and any dimension [8]; examples of $\exp(Cm^{3/2})$ pairwise non-isotopic M-curves of degree m (see [12], the techniques from [6] were used there).

In this paper, we give a combinatorial interpretation of the Petrovski — Oleinik inequality for T-hypersurfaces in terms of the triangulations. Namely, we rewrite each side of (*) as a sum over all simplices of the triangulation (see (4.3), (6.2)) and show that each summand in the left hand side is less or equal than the corresponding summand in the right hand side (see (7.3). In other words, we decompose (*) into a sum of local inequalities.

First, this yeilds another proof of the Petrovski – Oleinik inequality for T-hypersurfaces. Second, for T-hypersurfaces, this provides a necessary and sufficient condition for the equality sign in (*): one has "=" in (*) iff one has "=" in all the local inequalities. The question of "=" in the local inequalities is discussed in §§7–9.

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The proof of the local inequalities is based on a relative version of the MacMullen inequalities for the numbers of k-dimensional faces of a simplicial polytope. The relative MacMullen inequalities are formulated and proven in the Appendix (joint with R. MacPherson).

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§1. DEFINITIONS AND NOTATION

(1.1). Throughout the paper n and m will denote respectively the dimension and the degree (see Introduction). Denote the set $\{1,2,\dots,n\}$ by $\bar{n}$. Let $\Delta\subset\mathbf{R}^n$ be the simplex $\Delta=\{x\in\mathbf{R}^n\,|\,x_i>0;\,x_1+\dots+x_n=m\}$.

We denote by $[p_1,\dots,p_k]$ the convex hull of points $p_1,\dots,p_k\in\mathbf{R}^n$.

For $x\in\mathbf{R}^n$, $a\in\mathbf{Z}^n$ we denote $x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$ by x^a.

For a finite set M we denote the number of elements in M by $|M|$ or by $\#M$.

For a polynomial $p(t)$ we denote by $\operatorname{coef}_\alpha(p)$ the coefficient of t^α.

The *affine span* of a set $A\subset\mathbf{R}^n$ is the minimal affine plane containing A. An affine plane $V\subset\mathbf{R}^n$ is called *integral* if it coincides with the affine span of $V\cap\mathbf{Z}^n$. Any k-dimensional integral affine plane is supposed to be endowed with the *lattice k-dimensional volume* normalized by the condition that the volume of a fundamental parallelepiped of $V\cap\mathbf{Z}^n$ is 1.

(1.2) Triangulations. *k-Simplex* in $\mathbf{R}^n$ $(k\le n)$ is the convex hull of $k+1$ points in general position. If τ is a face of a simplex σ then we write $\tau\le\sigma$. The empty simplex $\varnothing$ and σ itself are always considered as faces of σ. The *interiority* $\operatorname{Int}\sigma$ of a simplex σ is the interiority with respect to the affine span of σ (if $\dim\sigma=0$ then $\operatorname{Int}\sigma=\sigma$).

Simplicial complex in $\mathbf{R}^n$ is the set Σ of simplices satisfying the standard axioms: (1) if $\sigma\in\Sigma$ and $\tau\le\sigma$ then $\tau\in\Sigma$; (2) if $\tau=\sigma_1\cap\sigma_2$ then $\tau\le\sigma_1$ and $\tau\le\sigma_2$. (In particular, the empty simplex $\varnothing$ is always an element of Σ.)

For a simplicial complex Σ, we denote by $[\Sigma]$ its *support*: $[\Sigma]=\cup_{\sigma\in\Sigma}\sigma$ and we denote by $\operatorname{Som}\Sigma$ the set of the vertices. Σ is called a *triangulation* of a set $X\subset\mathbf{R}^n$ if $[\Sigma]=X$.

A simplex (or a triangulation) is called *integral* if all its vertices are integral points.

§2. VIRO T-HYPESURFACES

(2.1) Regular triangulations. Let $\Delta\in\mathbf{R}^n$ be as in (1.1). An integral triangulation Σ of Δ is called *regular* if there exists a convex function $\varphi:\Delta\to\mathbf{R}$ which is linear on any $\sigma\in\Sigma$ and is not linear on $\sigma_1\cup\sigma_2$ for any $\sigma_1,\sigma_2\in\Sigma$, $\sigma_1\ne\sigma_2$, $\dim\sigma_1=\dim\sigma_2=n-1$. Such a function φ is called *Σ-convex.* An example of a non-regular triangulation see [4; p. 119, Fig. 3].

(2.2) Induced triangulation of an octahedron. Let Σ be a regular triangulation of Δ (see (2.1)). Denote by g_i the reflection in the coordinate hyperplane $x_i=0$ and let $G=(\mathbf{Z}/2)^n$ be the group generated by $g_1,\dots,g_n$. Clearly, $G=\{g_I\,|\,I\subset\bar{n}\}$ where $g_I=\prod_{i\in I}g_i$. Set $\hat{\Delta}=G\Delta=\bigcup_{g\in G}g\Delta$ and $\hat{\Sigma}=\{g\sigma\,|\,\sigma\in\Sigma,\,g\in G\}$. Thus, $\hat{\Delta}$ is an n-dimensional octahedron and $\hat{\Sigma}$ is a triangulation of $\hat{\Delta}$.

Lemma. *$\hat\Sigma$ is combinatorially equivalent to the face complex of a convex polytope.*

Proof. Project Graph$(\varphi) \subset \mathbf{R}^n \times \mathbf{R}$ onto $\mathbf{R}^n \times 0$ from a point $(0, -y)$ for $y \gg 1$ and reflect the result with respect to all the coordinate hyperplanes. ☐

(2.3) Viro T-hypersurfaces. Let Σ be a regular triangulation of Δ (see (2.1)) and s a sign distribution on Σ. (*Sign distribution* is an arbitrary function $s : \operatorname{Som}\Sigma \to \{-1, +1\}$.) Let φ be a Σ-convex function (see (2.1)). Then *Viro T-hypersurface* associated with (Σ, s) is the hypersurface $X_{(\Sigma,s)} \subset \mathbf{RP}^{n-1}$ defined by $f_\varepsilon(x) = 0$, for ε sufficiently small, where

$$f_\varepsilon(x) = \sum_{a \in \operatorname{Som}\Sigma} s(a)\varepsilon^{\varphi(a)} x^a$$

If $0 < \varepsilon \ll 1$ then up to an ambient isotopy $X_{(\Sigma,s)}$ does not depend on the choice of φ and ε. The topological type of $X_{(\Sigma,s)}$ can be explicitly described as follows.

Let g_i and g_I be as in (2.2). Extend the sign distribution s onto $\operatorname{Som}\hat\Sigma$: if $a = (a_1, ..., a_n) \in \operatorname{Som}\hat\Sigma$ and $s(a)$ is already defined then put $s(g_i(a)) = (-1)^{a_i} s(a)$. Thus, for $a \in \operatorname{Som}\Sigma$ one has $s(g_I(a)) = s(a) \cdot \prod_{i \in I} (-1)^{a_i}$. Denote: $\hat\Sigma_+ = \{\sigma \mid s(v) = +1$ for any vertex v of $\sigma\}$. Then $\operatorname{Som}\hat\Sigma_+ = \{a \in \operatorname{Som}\hat\Sigma \mid s(a) = +1\}$.

Let $\hat\Delta$ and $\hat\Sigma$ be as in (2.2) and let $\hat\Sigma'$ be the barycentric subdivision of $\hat\Sigma$. Denote: $S^{n-1}_+ = S^{n-1} \cap \{f_\varepsilon \geq 0\}$ (like in (1)) and $\hat\Delta_+ = \bigcup_{a \in \operatorname{Som}\hat\Sigma_+} \operatorname{Star}_{\hat\Sigma'}(a)$.

Theorem. (Viro [15]) *For $\varepsilon > 0$ sufficiently small there is a homeomorphism $(S^{n-1}, S^{n-1}_+) \approx (\hat\Delta, \hat\Delta_+)$.*

§3. Combinatorial polynomials

(3.1) Relative H-polynomial of a convex polytope. Let $P \in \mathbf{R}^n$ be a convex simplicial polytope such that $\dim P = n$. Let f_k be the number of its faces of dimension k. Define the H-polynomial[1] of P as

$$H_P(t) = \sum_{i=0}^{n} h_i t^i = (t-1)^n + \sum_{k=1}^{n} f_{k-1} \cdot (t-1)^{n-k} = \sum_{\tau < P} (t-1)^{n-d(\tau)}$$

where $d(\tau) = 1 + \dim\tau$ (Recall, that $\tau < P$ means that τ is a face of P; by convention, $\varnothing < P$ and $d(\varnothing) = 0$.)

If $\alpha = \{\alpha_1, \ldots, \alpha_k\}$, $k \leq n$ is a set of hyperplanes in general position which agrees with P, then we call $H^{rel}_{P,\alpha}$ *the relative H-polynomial of P with respect to α* (see Appendix).

Examples. (a) If P is a simplex then $H_P(t) = 1 + t + \cdots + t^n$. (b) If P is an octahedron then $H_P(t) = (1+t)^n$. (c) If S is the k-suspension over P then $H_S(t) = (t+1)^k H_P(t)$.

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[1] In the Appendix, the H-polynomial of a polytope is called the *Poincaré polynomial.* However, in the main part of the paper we use the term *H-polynomial* because following Arnold [1], we introduce in §5 the Poincaré polynomial of a face.

(3.2) Combinatorial polynomial of a face of a triangulation of Δ. Let Δ be as in (1.1) and Σ a regular triangulation of Δ (see (2.1)). Let τ be any simplex from Σ (possibly, $\tau=\varnothing$). Following [1], define the *combinatorial polynomial of* τ as

$$R_\tau(t)=\sum_{\sigma\geq\tau}(-1)^{n-k(\sigma)}(t-1)^{k(\sigma)-d(\sigma)},$$

where $d(\sigma)=1+\dim\sigma$ is the dimension of the cone over σ, and $k(\sigma)$ is the dimension of the minimal coordinate hyperplane which contains σ.

(3.3) Slice polytope of a face. Let τ be a face of a convex simplicial polytope $P\subset\mathbf{R}^n$, such that $0\in\operatorname{Int}P$. Let L be a linear functional which defines a hyperplane of support of τ, i.e. $L|_P\leq 1$ and $L(x)=1$ iff $x\in\tau$. Let β_τ be the intersection of the hyperplane $\{L=1-\varepsilon\}$, $0<\varepsilon\ll 1$ with a plane of dimension $n-\dim\tau$ which is transversal to τ and intersects $\operatorname{Int}\tau$. Define the *slice polytope of* τ as $\tau^*=P\cap\beta_\tau$. The following Lemma A is a standard fact about convex polytopes and Lemma B below can be proven in a similar way.

Lemma A. *The mapping $\sigma\mapsto\sigma\cap\beta_\tau$ defines a monotonic (i.e. respecting the order "$\leq$") bijection of $\{\sigma\,|\,\tau\leq\sigma<P\}$ onto the face complex of τ^*.* $\square$

Let $\alpha=\{\alpha_i\}$ be a set of hyperplanes which agrees with P (see Appendix). Set $\alpha_\tau=\{\alpha_i\cap\beta_\tau\,|\,\alpha_i\in\alpha\ \ \&\ \ \tau\subset\alpha_i\}$

Lemma B. *α_τ agrees with τ^*.* $\square$

(3.4) Notation. Let $\hat\Delta$, $\hat\Sigma$ be as in (2.2). Denote by

$$\hat{\sum_{cond(\sigma)}}\mathit{expr}(\sigma);\qquad\text{respectively:}\quad\sum_{cond(\sigma)}\mathit{expr}(\sigma)$$

the sum of the expression $\mathit{expr}(\sigma)$ over all simplices $\sigma\in\hat\Sigma$ (respectively: $\sigma\in\Sigma$; the empty simplex included in the both cases!) satisfying a condition $\mathit{cond}(\sigma)$.

Let $k(\sigma)$ be as in (3.2). The following lemma is evident.

Lemma. *If $\tau\in\Sigma$ then*

$$\hat{\sum_{\sigma\geq\tau;\,cond(\sigma)}}\mathit{expr}(\sigma)=\sum_{\sigma\geq\tau;\,cond(\sigma)}2^{k(\sigma)-k(\tau)}\mathit{expr}(\sigma)$$

(3.5) Comparing H^{rel} and R_τ. Let Δ be as in (1.1) and Σ a regular triangulation of Δ. Let $\hat\Delta$ and $\hat\Sigma$ be as in (2.2). Denote by $\alpha=\{\alpha_i\}_{i=1,\dots,n}$ the set of the coordinate hyperplanes $\alpha_i=\{x_i=0\}$. Let τ be any face of $\hat\Delta$. Define τ^* and α_τ as in (3.3) assuming that P is a convex realization of $\hat\Delta$ (see Lemma (2.2)).

Proposition. *If $\tau\in\Sigma$ then $H^{rel}_{\tau^*,\alpha_\tau}(t)=2^{n-k(\tau)}R_\tau(t)$.*

Proof. For $I \subset \bar{n}$ denote: $\alpha_I = \bigcap_{i\in I} \alpha_i$ and $k(\alpha_I) = \dim \alpha_I = n - |I|$. Then

$$\begin{aligned}
H^{rel}_{\tau^*,\alpha_\tau}(t) &= \sum_{\alpha_I \geq \tau} (-1)^{|I|}(t+1)^{|I|} H_{\tau^*\cap\alpha_I}(t) \\
&= \sum_{\alpha_I\geq\tau} (-1)^{|I|}(t+1)^{|I|} \hat{\sum_{\tau\leq\sigma\leq\alpha_I}} (t-1)^{k(\alpha_I)-d(\sigma)} && \text{by Lemma (3.3.A)} \\
&= \sum_{\alpha_I\geq\tau} (-1)^{|I|}(t+1)^{|I|} \sum_{\tau\leq\sigma\leq\alpha_I} 2^{k(\sigma)-k(\tau)}(t-1)^{k(\alpha_I)-d(\sigma)} && \text{by Lemma (3.4)} \\
&= \sum_{\sigma\geq\tau} (-1)^{n-k(\sigma)}(t-1)^{k(\sigma)-d(\sigma)} 2^{k(\sigma)-k(\tau)} \sum_{\alpha_I\geq\sigma} (t+1)^{n-k(\alpha_I)}(1-t)^{k(\alpha_I)-k(\sigma)} \\
&= \sum_{\sigma\geq\tau} (-1)^{n-k(\sigma)}(t-1)^{k(\sigma)-d(\sigma)} 2^{k(\sigma)-k(\tau)} \cdot 2^{n-k(\sigma)} = 2^{n-k(\tau)} R_\tau(t). \qquad \square
\end{aligned}$$

Together with Theorem 1 of Appendix and (2.2), (3.3.B) this yeilds

(3.6) Corollary. *R_τ is symmetric and unimodal.* $\square$

§4. LEFT HAND SIDE OF THE PETROVSKI–OLEINIK INEQUALITY FOR T-HYPERSURFACES

(4.1) Notation. Let $\tau \subset \mathbf{R}^n$ be an integral simplex such that its vertices $v_1, ..., v_d$ are linerly independent. Set

$$e(\tau) = \begin{cases} 1 & \text{if } v_1 + ... + v_d \in 2\mathbf{Z}^n \text{ or if } \tau = \varnothing \\ 0 & \text{otherwise.} \end{cases}$$

If $e(\tau) = 1$ we say that τ is *even*, otherwise τ is *odd.*

Let G, $\hat{\Delta}$, $\hat{\Sigma}$ be as in (2.2) and $\tau \in \hat{\Sigma}$. Then we denote: $s(\tau) = \prod_{i=1}^d s(v_i)$ where $v_1, ..., v_d$ are the vertices of τ.

Lemma. *For $\tau \in \Sigma$ one has $\sum_{\tau'\in G\tau} s(\tau') = 2^{k(\tau)} s(\tau) e(\tau)$.*

Proof. Clearly that $|G\tau| = 2^{k(\tau)}$. Let $v_1, ..., v_d$ be the vertices of τ and let $v = (x_1, ..., x_n) = v_1 + ... + v_n$. Then $s(g_I\tau) = (-1)^{x_I} s(\tau)$ where $x_I = \sum_{i\in I} x_i$. Hence, if $e(\tau) = 1$ then all x_I are even, and $\sum_{\tau'\in G\tau} s(\tau') = |G\tau| s(\tau) = 2^{k(\tau)} s(\tau)$. If $e(\tau) = 0$ then x_j is odd for some j. Put $G_j = \{g_I \mid j \notin I \subset \bar{n}\}$. Then $\sum_{\tau'\in G\tau} s(\tau) = \sum_{\tau'\in G_j\tau} \big(s(\tau') + s(g_j\tau')\big) = 0$. $\square$

Corollary. (see (3.4)) *For any expression $expr(\tau)$ one has*

$$\hat{\sum_\tau} s(\tau)\, expr(\tau) = \sum_\tau s(\tau) e(\tau) 2^{k(\tau)}\, expr(\tau)$$

(4.2) Lemma. *Let the notation be as in (2.3). Then $[\hat{\Sigma}_+]$ is a deformation retract of $\hat{\Delta}_+$ (see Fig. 1).*

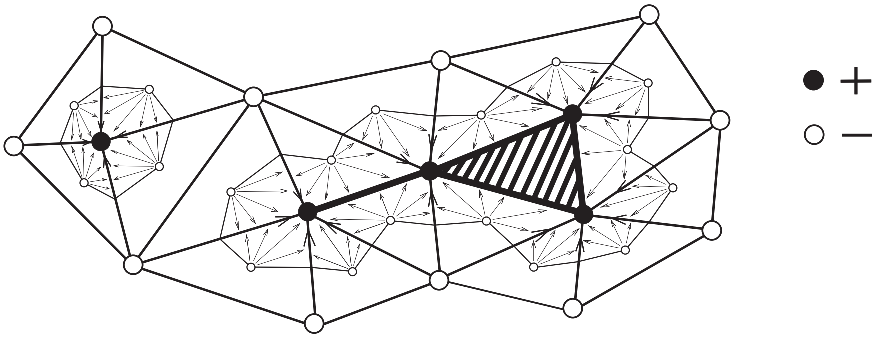

FIG. 1.

Proof. Consider a sequence of sets $[\hat{\Sigma}_+] = X_0 \subset X_1 \subset ... \subset X_n = \operatorname{Int}\hat{\Delta}_+$ where

$$X_i = [\hat{\Sigma}_+] \cup \big([\operatorname{Skel}^i \hat{\Sigma}] \cap \operatorname{Int}\hat{\Delta}_+).$$

Construct a sequence of deformation retractions $X_n \to X_{n-1} \to ... \to X_0$ as follows.

If $\sigma \in \hat{\Sigma} - \hat{\Sigma}_+$ is an i-dimensional simplex and b is the barycenter of σ then $b \notin X_i$ and hence, $\sigma \cap X_i$ can be blown from b onto $\partial\sigma \cap X_{i-1}$. Performing this procedure for all i-simplices $\sigma \in \hat{\Sigma} - \hat{\Sigma}_+$, we obtain the required retraction $X_i \to X_{i-1}$. □

(4.3) Proposition. *Let $X = X_{(\Sigma,s)}$ be a Viro T-hypersurface (see (2.3)) defined by $f = 0$. Let $S^{n-1}_+ = S^{n-1} \cap \{f \geq 0\}$ (as in the left hand side of (*)). Then*

$$\tilde{\chi}(S^{n-1}_+) = (-1)^{n-1} \sum_{\tau \in \Sigma} e(\tau) s(\tau) R_\tau(-1)$$

where $e(\tau)$ and $s(\tau)$ are defined in (4.1) and $R_\tau(t)$ is the combinatorial polynomial of τ (see (3.2)).

Proof. It follows from (2.3) and (4.2) that $\tilde{\chi}(S^{n-1}_+) = \tilde{\chi}(\hat{\Delta}_+) = \tilde{\chi}([\hat{\Sigma}_+])$. Let $\mathbf{1}_{\hat{\Sigma}_+} : \hat{\Sigma} \to \{0,1\}$ and $\mathbf{1}_{\operatorname{Som}\hat{\Sigma}_+} : \operatorname{Som}\hat{\Sigma} \to \{0,1\}$ be the characteristic functions of $\hat{\Sigma}_+$ and $\operatorname{Som}\hat{\Sigma}_+$ i.e. $\mathbf{1}_{\hat{\Sigma}_+}(\sigma) = 1$ iff $\sigma \in \hat{\Sigma}_+$ and $\mathbf{1}_{\operatorname{Som}\hat{\Sigma}_+}(v) = 1$ iff $v \in \operatorname{Som}\hat{\Sigma}_+$. Clearly, that $\mathbf{1}_{\operatorname{Som}\hat{\Sigma}_+}(v) = (s(v)+1)/2$. Let $d(\sigma)$, $k(\sigma)$ be as in (3.2). Then,

$$\mathbf{1}_{\hat{\Sigma}_+}(\sigma) = \prod_{i=1}^{d(\sigma)} \mathbf{1}_{\operatorname{Som}\hat{\Sigma}_+}(v_i) = \prod_{i=1}^{d(\sigma)} \frac{s(v_i)+1}{2} = \Big(\frac{1}{2}\Big)^{d(\sigma)} \sum_{\tau \leq \sigma} s(\tau)$$

where $v_1, ..., v_{d(\sigma)}$ are the vertices of σ (recall that $\varnothing \leq \sigma$). Let $\hat{\sum}$ and $\sum$ mean the same as in (3.4). Then we have

$$
\begin{aligned}
-\tilde{\chi}(S^{n-1}_{+}) &= \widehat{\sum_{\sigma}}(-1)^{d(\sigma)}\mathbf{1}_{\hat{\Sigma}_{+}}(\sigma) = \widehat{\sum_{\sigma}}(-2)^{-d(\sigma)}\widehat{\sum_{\tau\le\sigma}}s(\tau) = \widehat{\sum_{\tau}}s(\tau)\widehat{\sum_{\sigma\ge\tau}}(-2)^{-d(\sigma)} \\
&= \sum_{\tau} s(\tau)e(\tau)2^{k(\tau)}\widehat{\sum_{\sigma\ge\tau}}(-2)^{-d(\sigma)} && \text{by Corollary (4.1)} \\
&= \sum_{\tau} s(\tau)e(\tau)2^{k(\tau)}\sum_{\sigma\ge\tau}2^{k(\sigma)-k(\tau)}(-2)^{-d(\sigma)} && \text{by Lemma (3.4)} \\
&= (-1)^{n}\sum_{\tau} s(\tau)e(\tau)\sum_{\sigma\ge\tau}(-1)^{n-k(\sigma)}(-2)^{k(\sigma)-d(\sigma)} \\
&= (-1)^{n}\sum_{\tau} s(\tau)e(\tau)R_{\tau}(-1). \qquad \square
\end{aligned}
$$

§5. Poincaré polynomial of a simplex

(5.1) Definition. Given a set $S\subset\mathbf{R}^n$ and a linear functional $L:\mathbf{R}^n\to\mathbf{R}$, define the *Poincaré series of S with respect to L* as $[S]^L=\sum_{a\in S\cap\mathbf{Z}^n}t^{L(a)}=\sum_{\alpha}c_\alpha t^\alpha$ where c_α is the number of integral points on the hyperplane section $S\cap\{L=\alpha\}$.

Let $\sigma\in\mathbf{R}^n$ be an integral simplex whose vertices $v_1,\dots,v_d$ are linearly independent. Let $C_\sigma=\mathbf{R}_{+}\sigma=\{x_1v_1+\dots+x_dv_d\,|\,x_i\ge 0\}$ be the closed cone generated by σ and $\Pi_\sigma=\{x_1v_1+\dots+x_dv_d\,|\,0\le x_i<1\}$ be the "half-closed" parallelepiped.

Let L be a linear functional such that $L|_\sigma=1$. Following Arnold [1],[2] define the *Poincaré series p_σ* (resp.: q_σ) and the *Poincaré polynomial P_σ* (resp.: Q_σ) *of the face σ* (resp.: *of the interiority of the face σ*) as follows:

$$
\begin{aligned}
p_\sigma(t) &= [C_\sigma]^L, & q_\sigma(t) &= [\operatorname{Int} C_\sigma]^L, \\
P_\sigma(t) &= [\Pi_\sigma]^L, & Q_\sigma(t) &= [\operatorname{Int}\Pi_\sigma]^L
\end{aligned}
$$

(for $\sigma=\varnothing$, set by definition $p_\varnothing=q_\varnothing=P_\varnothing=Q_\varnothing=1$).

(5.2) Examples. (see [1]) (a). For Δ as in (1.1) one has

$$
\begin{aligned}
p_\Delta(t) &= (1-t^{1/m})^{-n} & q_\Delta(t) &= t^{n/m}(1-t^{1/m})^{-n} \\
P_\Delta(t) &= \left(\frac{1-t}{1-t^{1/m}}\right)^{n} & Q_\Delta(t) &= \left(\frac{t^{1/m}-t}{1-t^{1/m}}\right)^{n}
\end{aligned}
$$

(b). The Petrovski number (see Introduction) is $\Pi_n(m)=\operatorname{coef}_{n/2}Q_\Delta(t)$.

(5.3) Lemma. (see [1]).

$$
\begin{aligned}
&(a)\quad p_\sigma(t)=\sum_{\tau\le\sigma}q_\tau(t), && (b)\quad q_\sigma(t)=\sum_{\tau\le\sigma}(-1)^{d(\sigma)-d(\tau)}p_\tau(t), \\
&(c)\quad P_\sigma(t)=\sum_{\tau\le\sigma}Q_\tau(t), && (d)\quad Q_\sigma(t)=\sum_{\tau\le\sigma}(-1)^{d(\sigma)-d(\tau)}P_\tau(t),
\end{aligned}
$$

Proof. (a), (c) are evident; (b), (d) follow from the inclusion-exclusion formula.

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[2] Our notation for Poincaré series and polynomials differs from that in [1].

(5.4) Lemma. (see [1]). $P_\sigma(t) = p_\sigma(t)\cdot(1-t)^{d(\sigma)}$.

Proof. Let M be the semigroup generated by the vertices $v_1,...,v_d$ of σ. Clearly that C_σ is the disjoint union of the sets $m+\Pi_\sigma$ over all $m\in M$. Note also that for any $m = m_1v_1+...+m_dv_d\in M$ and for any subset $S\subset\mathbf{R}^n$ one has $[m+S]^L = t^{m_1+\dots+m_d}[S]^L$. Hence,

$$p_\sigma = [C_\sigma]^L = \sum_{m\in M}[m+\Pi_\sigma]^L = P_\sigma\sum_{m\in M}t^{m_1+\dots+m_d} = P_\sigma\cdot(1+t+t^2+...)^d$$

(5.5) Lemma. *Let τ be a face of a simplex σ and a, b elements of any commutative ring. Then $\sum_{\tau\le\lambda\le\sigma}a^{d(\sigma)-d(\lambda)}b^{d(\lambda)-d(\tau)} = (a+b)^{d(\sigma)-d(\tau)}$.* □

(5.6) Lemma.

$$Q_\sigma(t) = \sum_{\tau\le\sigma}(-t)^{d(\sigma)-d(\tau)}q_\tau(t)\,(1-t)^{d(\tau)};\quad q_\sigma(t)\,(1-t)^{d(\sigma)} = \sum_{\tau\le\sigma}t^{d(\sigma)-d(\tau)}Q_\tau(t).$$

Proof.

$$\begin{aligned}Q_\sigma(t) &\overset{(5.3d)}{=} \sum_{\lambda\le\sigma}(-1)^{d(\sigma)-d(\lambda)}P_\lambda(t) \overset{(5.4)}{=} \sum_{\lambda\le\sigma}(-1)^{d(\sigma)-d(\lambda)}p_\lambda(t)\,(1-t)^{d(\lambda)}\\ &\overset{(5.3a)}{=} \sum_{\lambda\le\sigma}(-1)^{d(\sigma)-d(\lambda)}(1-t)^{d(\lambda)}\sum_{\tau\le\lambda}q_\tau(t)\\ &= \sum_{\tau\le\sigma}q_\tau(t)\,(1-t)^{d(\tau)}\sum_{\tau\le\lambda\le\sigma}(-1)^{d(\sigma)-d(\lambda)}(1-t)^{d(\lambda)-d(\tau)}\\ &\overset{(5.5)}{=} \sum_{\tau\le\sigma}q_\tau(t)\,(1-t)^{d(\tau)}\cdot(-t)^{d(\sigma)-d(\tau)};\end{aligned}$$

$$\begin{aligned}q_\sigma(t)(1-t)^{d(\sigma)} &\overset{(5.3b)}{=} (1-t)^{d(\sigma)}\sum_{\lambda\le\sigma}(-1)^{d(\sigma)-d(\lambda)}p_\lambda(t)\\ &\overset{(5.4)}{=} \sum_{\lambda\le\sigma}(t-1)^{d(\sigma)-d(\lambda)}P_\lambda(t) \overset{(5.3c)}{=} \sum_{\lambda\le\sigma}(t-1)^{d(\sigma)-d(\lambda)}\sum_{\tau\le\lambda}Q_\tau(t)\\ &= \sum_{\tau\le\sigma}Q_\tau(t)\sum_{\tau\le\lambda\le\sigma}(t-1)^{d(\sigma)-d(\lambda)} \overset{(5.5)}{=} \sum_{\tau\le\sigma}Q_\tau(t)\,t^{d(\sigma)-d(\tau)}.\end{aligned}$$

§6. RIGHT HAND SIDE OF THE PETROVSKI – OLEINIK INEQUALITY FOR T-HYPERSURFACES

(6.1) Proposition. *Let Σ be a regular triangulation of Δ (see (1.1), (2.1)). Then $Q_\Delta(t) = \sum_{\tau\in\Sigma}Q_\tau(t)R_\tau(t)$.*

Proof. Note that if $\sigma \in \Sigma$ and $\operatorname{Int}\sigma \subset \operatorname{Int}\Delta'$ for some face Δ' of Δ then $d(\Delta') = k(\sigma)$. Thus,

$$\begin{aligned}
Q_\Delta(t) &= \sum_{\Delta' \le \Delta} (-t)^{n-d(\Delta')} q_{\Delta'}(t)\,(1-t)^{d(\Delta')} && \text{by (5.6; left)}\\
&= \sum_{\sigma\in\Sigma} (-t)^{n-k(\sigma)} q_\sigma(t)(1-t)^{k(\sigma)} && \text{since } q_{\Delta'} = \sum_{\operatorname{Int}\sigma\subset\operatorname{Int}\Delta'} q_\sigma\\
&= \sum_{\sigma} (-t)^{n-k(\sigma)} (1-t)^{k(\sigma)-d(\sigma)} \sum_{\tau\le\sigma} t^{d(\sigma)-d(\tau)} Q_\tau(t) && \text{by (5.6; right)}\\
&= \sum_{\tau} Q_\tau(t)\, t^{n-d(\tau)} \sum_{\sigma\ge\tau} (-1)^{n-k(\sigma)} (t^{-1}-1)^{k(\sigma)-d(\sigma)} &&\\
&= \sum_{\tau} Q_\tau(t)\, t^{n-d(\tau)} R_\tau(t^{-1}) = \sum_{\tau} Q_\tau(t) R_\tau(t) && \text{by symmetricity of } R_\tau.
\end{aligned}$$

(6.2) Corollary. *For any regular triangulation Σ of Δ one has $\sum_{\tau\in\Sigma} \operatorname{coef}_{n/2}\big(Q_\tau(t) R_\tau(t)\big) = \Pi_n(m)$ where $\Pi_n(m)$ is the Petrovski number (see Introduction). Thus, for a Viro T-hypesurface $X_{(\Sigma,s)}$ (see §2) (*) is equivalent to*

$$\left|\sum_{\tau\in\Sigma} e(\tau)s(\tau)R_\tau(-1)\right| \le \sum_{\tau\in\Sigma} \operatorname{coef}_{n/2}\big(Q_\tau(t)R_\tau(t)\big)$$

where $e(\tau)$, $s(\tau)$ are defined in (4.1), R_τ is the combinatorial polynomial of τ (see (3.2)) and Q_τ the Poincaré polynomial of $\operatorname{Int}\tau$ *(see (5.1)).*

Proof. Combine (*), (4.3), (5.2b), and (6.1). □

§7. THE LOCAL INEQUALITIES

(7.1) Symmetric and unimodal polynomials. Let $H(t) = \sum h_i t^i$ be a polynomial and $d \in \mathbf{Z}$. Say that H is *symmetric with center $t^{d/2}$* if $h_i = h_{d-i}$; H is *unimodal with center $t^{d/2}$* if all its coefficients are non-negative, $h_{i-1} \le h_i$ for $i \le d/2$ and $h_i \ge h_{i+1}$ for $i \ge d/2$.

If a polynomial $H(t)$ is symmetric with center $t^{d/2}$ then we shall denote the coefficient of $t^{d/2}$ by $\operatorname{mcoef} H$.

We shall use the convention: if we say that a polynomial written in the form $\sum_{i=0}^{d} h_i t^i$ is symmetric and/or unimodal then the center is supposed to be at $t^{d/2}$, even if $h_d = 0$.

Lemma. *Let $H(t) = \sum_{i=0}^{d} h_i t^i$ be symmetric and unimodal. Then:*

(a) $|H(-1)| \le h_{d/2}$;

(b) Let $d = 2k$. Then $H(-1) = h_k$ iff $h_{2i} = h_{2i+1}$, $i = 0, \dots, [(k-1)/2]$;

(c) Let $d = 2k$. Then $H(-1) = -h_k$ iff $h_0 = 0$ and $h_{2i-1} = h_{2i}$, $i = 1, \dots, [k/2]$;

Proof. If d is odd then the both sides in (a) are zero. If $d = 2k$ then $h_k - H(-1) = 2(h_1 - h_0) + 2(h_3 - h_1) + \dots$ and $h_k + H(-1) = 2h_0 + 2(h_2 - h_1) + 2(h_4 - h_3) + \dots$ □

(7.2) Corollary. *Let H_P be the H-polynomial of a convex simplicial polytope of dimension $d = 2k$. (see (3.1)). Then the following statements are equivalent:*
(a). $|H_P(-1)| = h_k$; (b). $H_P(-1) = h_k$; (c). P is a simplex.

Proof. H_P is symmetric and unimodal (see [13]). Hence we can apply Lemma (7.1):
(a) $\Longrightarrow$ (b). Otherwise (7.1c) would imply $h_d = 0$.
(b) $\Longrightarrow$ (c). By (7.1b) we have $1 = h_{d-1}$, hence, $f_0 = d + 1$ (see (3.1)).
(c) $\Longrightarrow$ (b) $\Longrightarrow$ (a). See Example (3.1a). $\square$

(7.3) Corollary. *Let Σ be a regular triangulation of Δ and $\tau \in \Sigma$. Then* $e(\tau)|R_\tau(-1)| \le \operatorname{coef}_{n/2}\big(Q_\tau(t)R_\tau(t)\big)$.

Proof. Put $q = \operatorname{mcoef} Q_\tau$ and $r = \operatorname{mcoef} R_\tau$. Evidently that $\operatorname{mcoef}(Q_\tau R_\tau) \ge qr$, $q \ge e(\tau)$, and it follows from (3.6) and (7.1a) that $r \ge |R_\tau(-1)|$. $\square$

Together with (6.2) this gives a combinatorial proof of (*) for T-hypersurfaces.

(7.4) Definition. A triangulation of Δ (see (1.1)) is called *locally extremal* if it is regular and for each simplex τ (including $\tau = \varnothing$) one has

$$e(\tau)|R_\tau(-1)| = \operatorname{coef}_{n/2}\big(Q_\tau(t)R_\tau(t)\big). \qquad (**)$$

Corollary. *Let $X = X_{(\Sigma,s)}$ be a Viro T-hypersurfaces. If one has "=" in (*) then Σ is locally extremal.*

Proof. Compare (6.2) and (7.3). $\square$

(7.5) Reduced Poincaré polynomial. Given $Q(t) = \sum_{\alpha\in A} q_\alpha t^\alpha$, $A \subset \mathbf{Q}$ and $\beta \in \mathbf{Q}$, we define the β-reduction of $Q(t)$ as $\operatorname{red}_\beta Q(t) = \sum_{\alpha\in A\cap(\beta+\mathbf{Z})} q_\alpha t^\alpha$

For Σ as in (2.1) and $\tau \in \Sigma$ we define the *reduced Poincaré polynomial of* $\operatorname{Int}\tau$ as $\tilde Q_\tau = \operatorname{red}_{n/2} Q_\tau$ (see §5). It easily follows from (6.1) (see also (5.2b)) that

$$\Pi_n(m) = \sum_{\tau\in\Sigma} \operatorname{mcoef}\big(\tilde Q_\tau(t)R_\tau(t)\big).$$

§8. THE CASE OF A PRIMITIVE TRIANGULATION

(8.1) Definition. An integral i-dimensional simplex $\tau \in \mathbf{R}^n$ is called *minimal* if $\tau \cap \mathbf{Z}^n = \operatorname{Som}\tau$. It is called *primitive* if its i-dimensional volume is $1/i!$. A triangulation is called *primitive* (resp. *minimal*) if each simplex is primitive (resp. minimal).

Clearly, each primitive simplex is minimal; if $\dim\tau \le 2$ then minimality is equivalent to primitivity; if τ is minimal and $\dim\tau \ge 3$ then its volume can be arbitrary big.

Lemma. *Let $\sigma \ne \varnothing$ be an integral primitive simplex. Then:*
(a) If σ is even (see (4.1)) then $d(\sigma)$ is odd (i.e. $\dim\sigma$ is even).
(b) If the vertices of σ are linearly independent then σ has not more than one even non-empty face.
(c) If $\sigma \subset \Delta$ (see (1.1)) and m is even then σ has exactly one even non-empty face.

Proof. Let V be the linear span of σ. Since σ is primitive, there exist $a \in V$ and a base $e_1, ..., e_d$ of $M = \mathbf{Z}^n \cap V$, such that the vertices of σ are $a + e_1, ..., a + e_d$.

Let $a = \sum a_i e_i$ and let $I = \{i \mid a_i \text{ is odd}\}$. Let τ be a face of σ spanned on $\{a + e_j \mid j \in J\}$. Suppose that τ is even. We shall show (and this will prove (b)) that then $J = I$. Indeed, let v be the sum of the vertices of τ. Then $v = |J|a + \sum_{j\in J} e_j \in 2M$. If $|J|$ were even then $|J|a$ would be an even vector and each x_j, $j \in J$ would be odd where $v = \sum x_i e_i$ is the expansion of v in the base $\{e_i\}$. Thus, $|J|$ is odd (this proves (a)). Note that $\sum_{i\in I} e_i \equiv a \pmod 2$, hence, $\sum_{i\in I} e_i + \sum_{j\in J} e_j \equiv a + \sum_{j\in J} e_j \equiv v \equiv 0 \pmod 2$. But $\{e_i\}_{i\in\bar n}$ is the base of $M \otimes \mathbf{Z}_2$, thus, $J = I$. To prove (c), note that $J = I = \varnothing$ implies $a \in 2M$ which contradicts $m \in 2\mathbf{Z}$. □

(8.2) Proposition. *Let $\tau \in \mathbf{R}^n$ be a primitive simplex with linearly independent vertices. Then $\tilde{Q}_\tau(t) = e(\tau)t^{d(\tau)/2}$. In particular,* $\operatorname{mcoef}(Q_\tau R_\tau) = \operatorname{mcoef}(\tilde{Q}_\tau R_\tau) = e(\tau)\operatorname{mcoef} R_\tau(t)$.

Proof. If $d(\tau)$ is even then $\tilde{Q}(t) = 0$ and the claim is trivial. Suppose that $d = d(\tau)$ is odd. Let V be the linear span of τ, L the linear functional on V such that $L|_\tau = 1$, and $M = \{m \in \mathbf{Z}^n \mid 2L(m) \in \mathbf{Z}\}$. Denote by $v_1, \dots, v_d$ the vertices of τ and let Π_τ be as in (5.1). We have to show that $m \in M \cap \operatorname{Int}\Pi_\tau \implies 2m = \sum v_i$. Indeed, the fact that τ is primitive means that there exist $a \in M$ with $L(a) = 1/2$ and a base $e_1, \dots, e_d$ of M such that $v_i = a + e_i$. Then $m = \sum m_i e_i$ with integer m_i's. On the other hand, if $m \in \operatorname{Int}\Pi_\tau$ then $m = \sum x_i v_i$ where $0 < x_i < 1$. Hence, $a \cdot \sum m_i = \sum (m_i - x_i)v_i$. But $2a$ lies in the affine span of τ and τ is primitive, this implies that the coefficients of a in the base $\{v_i\}$ are half-integer. Therefore, $m_i - x_i$ is half-integer for any i, hence $x_i = 1/2$. □

Thus, for a primitive simplex τ the local extremality condition (**) is equivalent to

$$e(\tau) = 1 \quad \implies \quad |R_\tau(-1)| = \operatorname{mcoef} R_\tau\,,$$

and if τ is primitive, $d(\tau) \equiv n \bmod 2$, and τ is not contained in the union of coordinate hyperplanes then (**) is equivalent to

$$e(\tau) = 1 \quad \implies \quad \tau^* \text{ is a simplex}.$$

Recall that τ^* is the slice polytope of τ (see (3.3))

(8.3). Even dimension. Let n be even and Σ be a primitive triangulation of Δ (see (1.1)). Let S_+^{n-1} and $\Pi_n(m)$ be as in (*) (see Introduction) for the Viro T-hypersurface $X = X_{\Sigma,s}$ (s is an arbitrary sign distribution).

Proposition.

$$-\tilde{\chi}(S_+^{n-1}) = R_\varnothing(-1); \qquad \Pi_n(m) = \operatorname{coef}_{n/2} R_\varnothing.$$

In particular, for $n = 4$ one has $R_\varnothing = c_1t^3 + c_2t^2 + c_1t$ where $c_1 = \binom{m-1}{3}$ and $c_2 = \Pi_4(m) = \frac{2}{3}m^3 - 2m^2 + \frac{7}{3}m - 1$, hence, $-\tilde{\chi}(S_+^{n-1}) = c_2 - 2c_1 = \frac{1}{3}m^3 - \frac{4}{3}m + 1$ does not depend on Σ (nor on s). Thus, one has "$=$" in (*) for $m \le 3$ and "$<$" for $m \ge 4$.

Proof. If $\tau \ne \varnothing$ then either $R_\tau(-1) = \operatorname{mcoef} R_\tau = 0$ (when $d(\tau)$ is odd) or $e(\tau) = 0$ (when $d(\tau)$ is even). Thus, the contribution of τ in the both sides of (*) iz zero.

To compute $R_\varnothing$ for $n = 4$, note that the number of vertices and 3-faces is known for a primitive triangulation, and the number of edges and triangles can be found from Dehn – Sommerville equations (see Appendix). □

(8.4). Odd dimension. Suppose that n is odd and one has "=" in (*) for a Viro hypersurface $X_{(\Sigma,s)}$ where Σ is a primitive triangulation of Δ. Let $\tau\in\Sigma$. If $d(\tau)$ is even (in particular, if $\tau=\varnothing$) then the contribution of τ to the both sides of (*) is zero. Thus, a necessary condition on a primitive triangulation Σ for "=" in (*) is the condition:

The slice polytope τ^* is a simplex for each simplex τ such that $d(\tau)$ is odd and $k(\tau)=n$.

§9. THE CASE OF LOW DIMENSIONS

Racall (see 1.1) that all integral planes are endowed with the lattice volume, in particular, the length of a segment $[a,b]$, $a,b\in\mathbf{Z}^n$ is $\#\mathbf{Z}^n\cap[a,b)$.

Given a k-simplex σ in an affine integral k-plane V and a point $p\in\mathbf{Z}^n\setminus V$, define the *height* h_p of the simplex $[p\sigma]$ as the length of the segment $\varphi([p\sigma])$ where $\varphi:\mathbf{R}^n\to\mathbf{R}^{n-k}$ is the projection along V, such that $\varphi(\mathbf{Z}^n)=\mathbf{Z}^{n-k}$. Thus, we have $\operatorname{vol}_{k+1}[p\sigma]=h_p\operatorname{vol}_k\sigma/(k+1)$.

$\underline{n=3}$.

(9.1) Local condition. Let us interpret the local condition (**) for each values of $(d(\tau),k(\tau))$. We suppose that m (see (1.1)) is even (for m odd (*) is just $0=0$).

$d(\tau)=0$ (i.e. $\tau=\varnothing$): $Q_\tau=1$, $R_\tau(-1)=\operatorname{coef}_{3/2}R_\tau=0$, hence (**) always holds.

$d(\tau)=1$: $\tilde{Q}_\tau=e(\tau)t^{1/2}$. Denote the number of edges of $\hat{\Sigma}$ incident to τ by $\hat{\nu}$.
$k(\tau)=1,2$: $2^{3-k(\tau)}R_\tau=(\hat{\nu}-4)t$, hence (**) holds automatically;
$k(\tau)=3$: $R_\tau=1+(\hat{\nu}-2)t+t^2$, hence (**) holds iff $e(\tau)=0$ or $\hat{\nu}=3$.

$d(\tau)=2$:
$k(\tau)=2$: $R_\tau=0$, hence (**) always holds.
$k(\tau)=3$: $R_\tau=t+1$, hence $\operatorname{coef}_{3/2}(Q_\tau R_\tau)=2\operatorname{coef}_{1/2}Q_\tau$. Thus, (**) holds iff $(\operatorname{Int}\tau)\cap2\mathbf{Z}^3=\varnothing$.

$d(\tau)=3,k(\tau)=3$: $R_\tau=1$, hence (**) is equivalent to $\operatorname{coef}_{3/2}Q_\tau=e(\tau)$. This is so if and only if one of the following conditions hold:

(i) τ is primitive;
(ii) $\tau=[abc]$ where the line ac contains an even point and the height h_a equals 1.
(iii) the barycenter b of τ is even and $\tau\cap\mathbf{Z}^3=\operatorname{Som}\tau\cup\{b\}$.

Analyzing these conditions, one easily obtains

(9.2) Proposition. *($n=3$, m is even). (a). Any locally extremal (see (7.4)) triangulation of Δ can be subdivided up to a primitive locally extremal triangulation.*

(b). Let Σ be locally extremal and

$$s(a)=\begin{cases}-1, & \text{if } k(a)=2 \text{ and } a\notin2\mathbf{Z}^3\\ 1, & \text{otherwise.}\end{cases}$$

Then one has "=" in (*) for the Viro T-hypersurface $X=X_{(\Sigma,s)}$.

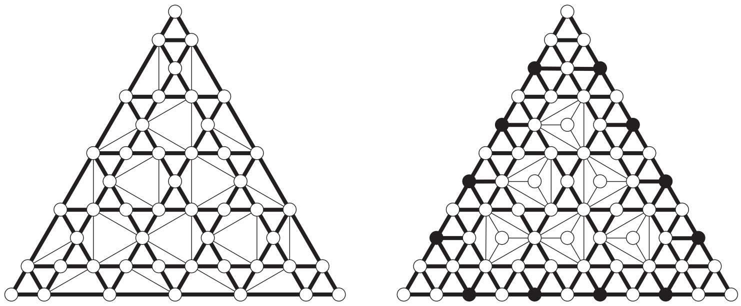

FIG. 2.

Examples of (Σ, s), providing "=" in (*) are given in Fig. 2, ("+" is white, "−" is black). The regularity follows from (10.1), (10.2) using the hexagonal subdivision shown by thick lines.

$\underline{n = 4}$.

(9.3) Local condition. Like in (9.1), we study the local extremality condition (**) for each pair (d, k).

$d(\tau) = 0$ (i.e. $\tau = \varnothing$): by the definition (see (3.2)),

$$R_\varnothing(t) = \sum_{0 \le d \le k} (-1)^{4-k}(t-1)^{k-d} f_{k;d}$$

where $f_{k;d} := \#\{\sigma \in \Sigma \mid k(\sigma) = k, d(\sigma) = d\}$. Consider separately the two cases:

$$R_\varnothing(-1) = \operatorname{mcoef} R_\varnothing; \tag{9.3.1}$$

$$R_\varnothing(-1) = -\operatorname{mcoef} R_\varnothing. \tag{9.3.2}$$

It is clear that $\operatorname{coef}_4 R_\varnothing = 0$ and $\operatorname{coef}_3 R_\varnothing = f_{4;1} = \#\big(\operatorname{Som}(\Sigma) \cap \operatorname{Int}\Delta\big)$. Thus, we see from (7.1b) that (9.3.1) holds iff $f_{4;1} = 0$. (This means that all the vertices of Σ lie on $\partial\Delta$.)

Analogously, (9.3.2) is equivalent to $f_{4;2} = 4f_{4;1} + f_{3;1}$.

$d(\tau) = 1$: $\tilde{Q}_\tau = 0$ and $R_\tau(-1) = 0$. Hence, (**) holds automatically;

$d(\tau) = 2$: $\tilde{Q}_\tau = qt$ where $q = \#(\mathbf{Z}^4 \cap \operatorname{Int}\tau)$. Put $\hat{\nu} = \#\{\tau < \sigma \in \hat{\Sigma} \mid d(\sigma) = 4\}$.

$k(\tau) = 2, 3$: $2^{4-k(\tau)} R_\tau = (\hat{\nu}-4)t$, hence (**) is equivalent to $(\hat{\nu}-4)(q-e(\tau)) = 0$. (Note that $q = e(\tau)$ if and only if $q \le 1$).

$k(\tau) = 4$: $R_\tau = 1 + (\hat{\nu} - 2)t + t^2$, hence (**) takes form $e(\tau)|4 - \hat{\nu}| = q(\hat{\nu} - 2)$. This holds if and only if either (i) $q = 0$ or (ii) $\hat{\nu} = 3$ and $q = 1$.

$d(\tau) = 3$:

$k(\tau) = 3$: $R_\tau = 0$, hence (**) holds automatically.

$k(\tau) = 4$: $R_\tau = 1 + t$, $\tilde{Q}_\tau = q + qt$ where $q = \#(\mathbf{Z}^4 \cap \operatorname{Int}\tau)$. Hence, (**) is equivalent to $q = 0$.

$d(\tau) = k(\tau) = 4$: $R_\tau = 1$, hence (**) is equivalent to the condition

$$\operatorname{coef}_2 Q_\tau = e(\tau). \tag{9.3.3}$$

It is possible to list more or less explicitly all the 3-simplices satisfying (9.3.3) as we did it for the other values of (k, d). However the answer is rather complicated and we restrict ourselves by deriving some consequences of (9.3.3).

(9.4) Poincaré polynomial of the interiority of a 3-simplex. Let $\tau \subset \mathbf{R}^4$ be an integral 3-simplex. Denote by V, S, and l, respectively its lattice volume, the sum of the lattice areas of the faces, and the sum of the lattice lengths of the edges. Put $i = \#(\mathbf{Z}^4 \cap \operatorname{Int}\tau)$. Let $\tilde{Q}_\tau(t) = c_1 t + c_2 t^2 + c_1 t^3$ be as in (7.5).

(9.4.1) Proposition. *(a)* $c_1 = i$; *(b)* $c_2 = 6V - 2S + l - 2i - 3$.

Proof. (a). Evident. (b). Replacing if necessary $\mathbf{Z}^4$ with the lattice generated by the integral points of the affine span of τ we may suppose that $\operatorname{coef}_\alpha p_\tau = 0$ for $\alpha \notin \mathbf{Z}$ (in particular, $\tilde{Q}_\tau = Q_\tau$). By Ehrart formula [5] we have

$$\operatorname{coef}_k p_\tau = Vk^3 + (S/2)k^2 + \Delta k + 1, \quad k \geq 0, \qquad \text{where } \Delta = i - V + (S/2) + 1.$$

The summation of $t^k \operatorname{coef}_k p_\tau$ over $k = 0, 1, \ldots$ yeilds

$$p_\tau = V \cdot \frac{t^3 + 4t^2 + t}{(1-t)^4} + \frac{S}{2} \cdot \frac{t^2 + t}{(1-t)^3} + \frac{\Delta t}{(1-t)^2} + \frac{1}{1-t}.$$

Similarly, we find $p_{\tau,d} := \sum_{\sigma \leq \tau, d(\sigma) = d} p_\sigma$ by the summation of

$$\operatorname{coef}_k p_{\tau,3} = Sk^2 + lk + 4, \qquad \operatorname{coef}_k p_{\tau,2} = lk + 6, \qquad \operatorname{coef}_k p_{\tau,1} = 4$$

and apply $Q_\tau = \sum_{d=0}^{4} (t-1)^d p_{\tau,d}$ (see (5.3d), (5.4)). □

Lemma. *There exists a triangulation of* τ *with vertices at* $\operatorname{Som}(\tau) \cup (\mathbf{Z}^4 \cap \operatorname{Int}\tau)$ *and with* $\geq 3i + 1$ *tetrahedra.*

Proof. Denote the points of $\mathbf{Z}^4 \cap \operatorname{Int}\tau$ by $p_1, \ldots, p_i$. Let $\Sigma_0 = \{\tau\}$ and let Σ_j be obtained from Σ_{j-1} by adding the point p_j and subdividing the simplices containing it. Clearly, each time we add ≥ 3 tetrahedra. □

(9.4.2) Corollary. *(a). If* $i > 0$ *then* $6V \geq 2S + 3(i-1)$; *(b). If* $i > 0$ *then* $c_2 \geq i + l - 6$; *(c).* $c_2 \geq c_1$.

Proof. (a). In the triangulation of the Lemma, the volume of the 4 tetrahedra having a common face with τ, is $\geq S/3$. The volume of the others is $\geq (\#\text{tetrahedra} - 4)/6 \geq (3i + 1 - 4)/6$
(b). Put (a) into (9.4.1b). (c). Put $c_1 = i$ and $l \geq 6$ into (b). □

Conjecture. Q_τ is unimodal for any polyhedron τ with vertices at integral points.

Remark. By the arguments as above one can prove this conjecture when $d(\tau) = 4$.

(9.4.3) Corollary. *If* τ *is minimal (see (8.1)) then* $c_2 = 6V - 1$.

Proof. Put $i = 0$, $l = 6$, $S = 2$ into (9.4.1b). □

(9.4.4) Proposition. *If* τ *is minimal then the following conditions are equivalent:*
(a) τ *satisfies (9.3.3);* *(b)* $V = (1 + e(\tau))/6$; *(c)* V *is* $1/6$ *or* $1/3$.

Proof. (a) $\iff$ (b) by (9.4.3); (b) $\implies$ (c) is evident.
(c) $\implies$ (b). For $V = 1/6$ this follows from Lemma (8.1a). Suppose that $V = 1/3$ and let us prove that $e(\tau) = 1$. Let $v_0, \ldots, v_3$ be the vertices of τ. Set $e_j = v_j - v_0$, $j = 1, 2, 3$. Denote by M the lattice generated by e_1, e_2, e_3. Let $M' = \mathbf{Z}^4 \cap (M \otimes \mathbf{R})$. We have $M' : M = 2$. Hence, M' is generated by e_1, e_2, e_3' and $e_3 = a_1 e_1 + a_2 e_2 + 2e_3'$. Since $v_0 + \cdots + v_4 = 4v_0 + (a_1 + 1)e_1 + (a_2 + 1)e_2 + 2e_3'$, it suffices to show that the both a_1 and a_2 are odd. Indeed, if $a_1 \equiv a_2 \equiv 0 \bmod 2$ then the segment $[v_0 v_3]$ would not be minimal; if $a_1 + 1 \equiv a_2 \equiv 0 \bmod 2$ then $[v_2 v_3]$ would not be minimal. □

§10. REGULARITY CRITERIA

(10.1) Regular polyhedral decomposition. Given a convex polytope $\Delta \in \mathbf{R}^n$, define its *(regular) polyhedral decomposition* replacing everywere in (1.2) and (2.1):

"simplex"	$\longrightarrow$	"convex polyhedron" (omit $k \le n$ in the definition)
"simplicial complex"	$\longrightarrow$	"polyhedral complex"
"triangulation"	$\longrightarrow$	"polyhedral decomposition"

Proposition. *Let Σ be a polyhedral decomposition of a convex n-dimensional polytope $\Delta \subset \mathbf{R}^n$. Suppose that (possibly, after an affine change of coordinates) each face $\sigma \in \Sigma$ can be inscribed into a sphere whose center lies either in* $\operatorname{Int}\sigma$ *or in* $\operatorname{Int}(\sigma \cap \Delta')$ *for some face Δ' of Δ. Then Σ is regular.*

Proof. Put $\varphi(x) = \sum x_i^2$ for $x \in \operatorname{Som}\Sigma$ and extend φ linearly onto each face. □

(10.2) Polyhedral subdivisions. Let Σ, Σ' be polyhedral decompositions of a convex polytope Δ. For $\sigma \in \Sigma$ put $\Sigma'_\sigma = \{\sigma' \in \Sigma' \,|\, \sigma' \subset \sigma\}$. Say that Σ' is a *polyhedral subdivision of* Σ if $\forall \sigma \in \Sigma$ one has $[\Sigma'_\sigma] = \sigma$.

Proposition. *Let Σ be a regular polyhedral decomposition of a convex polytope Δ and Σ' a polyhedral subdivision of Σ. Suppose that there exists a continuous function $\psi : \Delta \to \mathbf{R}$ such that $\forall \sigma \in \Sigma$ the restriction $\psi|_\sigma$ is (Σ'_σ)-convex (i.e. the decompositions Σ'_σ are "coherently regular"). Then Σ' is regular.*

Proof. If φ is Σ-convex and $0 < \varepsilon \ll 1$ then $\varphi + \varepsilon\psi$ is Σ'-convex. □

APPENDIX: RELATIVE MACMULLEN INEQUALITIES

by R. MacPherson and S. Orevkov

Let P be a convex simplicial polytope in $\mathbf{R}^n$. Define its *Poincaré polynomial* H_P as

$$H_P(t) = (t-1)^n + \sum_{i=1}^{n} f_{i-1}(t-1)^{n-i},$$

where f_i is the number of i-dimensional simplices of P.

Necessary and sufficient conditions on a polynomial

$$h_n t^n + h_{n-1}t^{n-1} + ... + h_1 t + h_0 \tag{1}$$

with $h_n = 1$ for it to be a Poincaré polynomial of a convex simplicial polytope, are

$$h_i = h_{n-i}, \qquad i = 0, ..., [n/2] \quad \text{(Dehn-Sommerville equations);} \tag{2}$$

$$h_i \le h_{i-1}, \qquad i = 1, ..., [n/2]; \tag{3}$$

$$(h_{i+1} - h_i) \le (h_i - h_{i-1})^{<i>}, \qquad i = 1, ..., [n/2] - 1; \tag{4}$$

where $m^{<k>}$ is some explicitly defined function of the integers m and k.

These conditions were conjectured by MacMullen [11] and proved by Stanley [13] (necessity) and Billera and Lee [3] (sufficiency). The proof of the necessity uses toric varieties and the hard Lefschetz theorem.

A polynomial (1) is said to be *symmetric and unimodal* if $h_n \geq 0$ and the conditions (2), (3) are satisfied.

Here we give a relative version of the inequality (3) (Theorem 1 below) for coefficients of Poincaré polynomials of a polytope and its intersections with hyperplanes in general position. The proof is based on the the relative hard Lefschetz theorem of Beilinson, Bernstein, Deligne, and Gabber.

Let P be a convex simplicial polytope in $\mathbf{R}^n$ and let $\alpha = \{\alpha_1, ..., \alpha_k\}, k \leq n$ be a set of hyperplanes in general position. Denote $\{1, ..., k\}$ by $\bar{k}$. For $I \subset \bar{k}$, let $\alpha_I = \cap_{i \in I} \alpha_i$, $P_I = P \cap \alpha_I$ (by convention, $\alpha_\varnothing = \mathbf{R}^n$, $P_\varnothing = P$). Say that P *argees* with α if any α_I intersects $\operatorname{Int} P$ and each face of P_I is a face of P. If P agrees with α, we define the *relative Poincaré polynomial of P with respect to α* as

$$H^{rel}_{P,\alpha}(t) = \sum_{I \subset \bar{k}} (-1)^{|I|} (t+1)^{|I|} H_{P_I}(t)$$

Theorem 1. *The polynomial $H^{rel}_{P,\alpha}(t)$ is symmetric and unimodal.*

Proof. Since the hyperplanes $\alpha_1, ..., \alpha_k$ are in general position, we can chose coordinates $(x_1, ..., x_n)$ in $\mathbf{R}^n$ so that α_i is defined by $x_i = 0$. The condition that P agrees with α implies that the origin can be chosen inside P. Since P is simplicial, we may perturb it so that all its vertices are rational. The perturbation can be chosen so that all the incidence relations are preserved.

For any face σ of P consider the cone obtained as the union of all rays with vertex at the origin, which intersect σ. All such cones define a fan Σ in $\mathbf{R}^n$, and let X be the toric variety over $\mathbf{C}$ associated to Σ (see [4]). Let Y be $(\mathbf{CP}^1)^k$, which we shall consider as the toric variety associated to the fan Σ_Y consisting of all coordinate octants in $\mathbf{R}^k$.

The mapping $\mathbf{R}^n \to \mathbf{R}^k$ defined by $y_i = x_i$, (where $(y_1, ..., y_k)$ are coordinates in $\mathbf{R}^k$) is simplicial (sends any cone of Σ to a cone of Σ_Y). Hence, it defines a toric morphism $f : X \to Y$ (see [4]).

The structure of toric variety defines the following stratification of Y. Let $Y_0 = \mathbf{C} - \{0\}$ be the 1–dimensional and $Y_1 = \{0\}$, $Y_2 = \{\infty\}$ the 0–dimensional strata of $\mathbf{CP}^1$. Denote by M the set of all k-tuplets $(m_1, ..., m_k)$ where $m_i = 0, 1, 2$. For $m \in M$ let us define

$$Y_m = \{(y_1, ..., y_k\} \in Y \mid y_j = Y_{m_j} \text{ if } m_j > 0\}.$$

We apply the Decomposition theorem [2; Section 5.4.5] (see also [9; Section 12]) to the map f. It expresses the pushforward of the intersection complex of X as a direct sum of intersection complexes of subvarieties of Y. Since P is simplicial, X is rationally smooth, the intersection complex of X is the constant sheaf. By directly examining the map f, one can see that only subvarieties Y_m of Y occur, and that all the intersection complexes involved have un-twisted coefficients. Taking Poincare polynomials, we get the following statement (where the unimodality comes from the relative hard Lefschetz theorem, [2; Section 5.4.10])

Lemma. *There exist symmetric unimodal polynomials φ_m with integral coefficients such that for any open $V \subset Y$,*

$$H(f^{-1}(V)) = \sum_m \varphi_m H(V \cap Y_m)$$

See [10] for a fuller exposition of the Decomposition theorem from this point of view.

Let $U \subset Y_0$ be an open disk. For $I \subset \bar{k}$ put

$$U_I = \{(y_1, ..., y_k) \in Y \mid y_i \in U \text{ if } i \in I\}$$

Define $J(m)$ as $\{j \mid m_j = 0\}$.

Then

$$U_I \cap Y_m = \begin{cases} (\mathbf{CP}^1)^{|J(m)-I|} \times U^{|I|}, & I \subset J(m) \\ \varnothing & \text{otherwise.} \end{cases}$$

The lemma applied to U_I gives us

$$H_{P_I} = H(f^{-1}(U_I)) = \sum_{m\in M} \varphi_m(t) H(U_I \cap Y_m) = \sum_{m \in M, I\subset J(m)} \varphi_m(t)(t+1)^{|J(m)-I|}.$$

For $J \in \bar{k}$ put $\varphi_J(t) = \sum_{m\in M, J(m)=J} \varphi_m(t)$. Then $H_{P_I} = \sum_{I\subset J} \varphi_J(t)(t+1)^{|J|-|I|}$, and

$$H^{rel}_{P,\alpha} = \sum_{I\subset \bar{k}} (-1)^{|I|} \sum_{I\subset J\subset \bar{k}} \varphi_J(t)(t+1)^{|J|} = \sum_{J\subset \bar{k}} \varphi_J(t)(t+1)^{|J|} \sum_{I\subset J} (-1)^{|I|} = \varphi_\varnothing(t).$$

□

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