

# ALGEBRAICALLY UNREALIZABLE COMPLEX ORIENTATIONS OF PLANE REAL PSEUDOHOLOMORPHIC CURVES

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ABSTRACT. We prove two inequalities for the complex orientations of a separating non-singular real algebraic curve in  $\mathbb{R}\mathbb{P}^2$  of any odd degree. We also construct a separating non-singular real (i.e., invariant under the complex conjugation) pseudoholomorphic curve in  $\mathbb{C}\mathbb{P}^2$  of any degree congruent to 9 mod 12 which does not satisfy one of these inequalities. Therefore the oriented isotopy type of the real locus of each of these curves is algebraically unrealizable.

## 1. INTRODUCTION

By a non-singular real algebraic curve in  $\mathbb{R}\mathbb{P}^2$  we mean a non-singular algebraic curve in  $\mathbb{C}\mathbb{P}^2$  invariant under the complex conjugation  $(x : y : z) \mapsto (\bar{x} : \bar{y} : \bar{z})$ . If such a curve is denoted by  $A$ , then we denote the set of its real points by  $\mathbb{R}A$ . A curve  $A$  is called *separating* (or *type I*) if  $A \setminus \mathbb{R}A$  is not connected. In this case  $A \setminus \mathbb{R}A$  has two connected components exchanged by the complex conjugation, and the boundary orientation induced by the complex orientation of any of these components is called a *complex orientation* of  $\mathbb{R}A$ . It is defined up to simultaneous reversing of the orientation of each connected component of  $\mathbb{R}A$ .

The main result of the paper (Theorem 1.1 below) is an inequality for the isotopy type of a plane nonsingular real algebraic curve endowed with a complex orientation (i.e., for the *complex scheme* of such curve according to Rokhlin's terminology [23]) which implies in particular that the oriented isotopy type shown in Figure 1, that is the complex scheme (in the notation of Viro [24])

$$J \sqcup 9_- \sqcup 1_- \langle 1_+ \langle 1_- \rangle \rangle \quad (1)$$

is unrealizable by a real algebraic curve of degree 9 in  $\mathbb{R}\mathbb{P}^2$ . Since this complex scheme is easily realizable by a real pseudoholomorphic curve (see Definition 1.3 below), it provides the first example of a complex scheme of a non-singular plane real projective curve which is algebraically unrealizable but pseudoholomorphically realizable. We also construct similar examples for any degree congruent to 9 modulo 12.

Let  $A$  be a non-singular separating real algebraic curve in  $\mathbb{R}\mathbb{P}^2$  of an odd degree  $m = 2k + 1$ . We fix a complex orientation on  $\mathbb{R}A$ . Let  $r$  be the number of connected components of  $\mathbb{R}A$ . Then  $l = r - 1$  is the number of *ovals* (components of  $\mathbb{R}A$  whose complement in  $\mathbb{R}\mathbb{P}^2$  is not connected). The component which is not an oval is called *pseudo-line* and we denote it by  $J$ . Following [23, 24], we say that an oval is even (resp. odd) if it is encircled by an even (resp. odd) number of other ovals. An

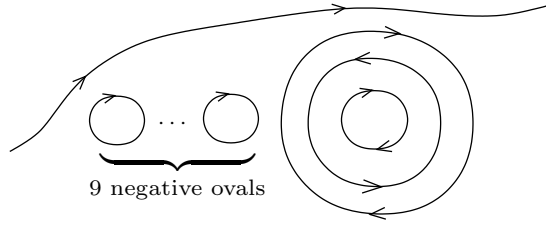


FIGURE 1. The complex scheme (1)

oval  $O$  is called *positive* if  $[O] = -2[J]$  in  $H_1(M)$  where  $M$  is the closure of the non-orientable component of  $\mathbb{RP}^2 \setminus O$ . Otherwise  $O$  is called *negative*. Traditionally, the number of even (resp. odd) ovals is denoted by  $p$  (resp. by  $n$ ), and the number of positive (resp. negative) ovals is denoted by  $\Lambda_+$  (resp.  $\Lambda_-$ ). Let

$$\begin{aligned} \Lambda_+^p &= \text{the number of positive even ovals,} \\ \Lambda_-^p &= \text{the number of negative even ovals,} \\ \Lambda_+^n &= \text{the number of positive odd ovals,} \\ \Lambda_-^n &= \text{the number of negative odd ovals.} \end{aligned}$$

**Theorem 1.1.** *If  $k > 0$ , then*

$$\Lambda_+^p + \Lambda_-^n + 1 \geq \frac{l - k^2 + 2k}{2} \quad \text{and} \quad \Lambda_+^n + \Lambda_-^p \geq \frac{l - k^2 + 2k}{2}. \quad (2)$$

Setting  $l = g - 2s$  one can equivalently rewrite (2) in the form

$$\Lambda_+^p + \Lambda_-^n + 1 \geq \frac{k^2 + k}{2} - s \quad \text{and} \quad \Lambda_+^n + \Lambda_-^p \geq \frac{k^2 + k}{2} - s. \quad (3)$$

This theorem is proven in §4 (see also Example 3.4). For the complex scheme (1) we have  $l = 12$  and  $\Lambda_+^p = \Lambda_-^n = 0$ , thus the left inequality in (2) is not satisfied for  $k = 4$ . So we obtain:

**Corollary 1.2.** *The complex scheme (1) is unrealizable by a real algebraic curve of degree 9.*

The main interest of Corollary 1.2 is that the complex scheme (1) admits a very simple realization by a real pseudo-holomorphic curve of degree 9 which we present just after the following definition and a brief discussion.

**Definition 1.3.** Let  $(X, \omega)$  be a symplectic 4-manifold and  $c : X \rightarrow X$  be a smooth involution such that  $c^*(\omega) = -\omega$ . A *real pseudoholomorphic curve* is a  $c$ -anti-invariant  $\mathcal{J}$ -holomorphic curve [8] for a smooth  $c$ -anti-invariant almost complex structure  $\mathcal{J}$  which is tamed by  $\omega$  (i.e.,  $\forall v \in TX, \omega(v, \mathcal{J}v) > 0$ ). In this case we denote the fix-point sets of  $X$  and  $A$  by  $\mathbb{R}X$  and  $\mathbb{R}A$ . Notice that  $\mathbb{R}X$  is a smooth 2-submanifold of  $X$  and  $\mathbb{R}A$  is a smooth 1-submanifold of  $\mathbb{R}X$  at smooth points of  $A$ . When  $X$  is  $\mathbb{CP}^2$ , we always consider the Fubini-Studi symplectic form and the standard complex conjugation. In this case we define the *degree* of  $A$  as its homological degree, that is  $\deg A = m$  if  $[A] = m[\mathbb{CP}^1]$  in  $H_2(\mathbb{CP}^2)$ .

In the setting of Definition 1.3, when a  $c$ -anti-invariant  $A$  is smooth, it is enough to demand that  $A$  is *symplectic* (i.e.,  $\omega|_A$  does not vanish) because in this case it

is necessarily  $\mathcal{J}$ -holomorphic for a suitable  $\omega$ -tame  $c$ -anti-invariant  $\mathcal{J}$ . Indeed, an  $\omega$ -tame  $\mathcal{J}$  is a section of a fibration over  $X$  by open balls [8, Lemma 2.3.C<sub>2</sub>], thus it can be extended from  $A$  to  $X$  (see [27, Prop. 1.1] for the real case). Similarly, if  $A$  is nodal (each singularity is an intersection of two smooth transverse local branches), then, in addition to the symplecticity, it is enough to demand that the intersections are positive (see [10, Lemma 1.4.2] whose proof can be easily adapted for the real case).

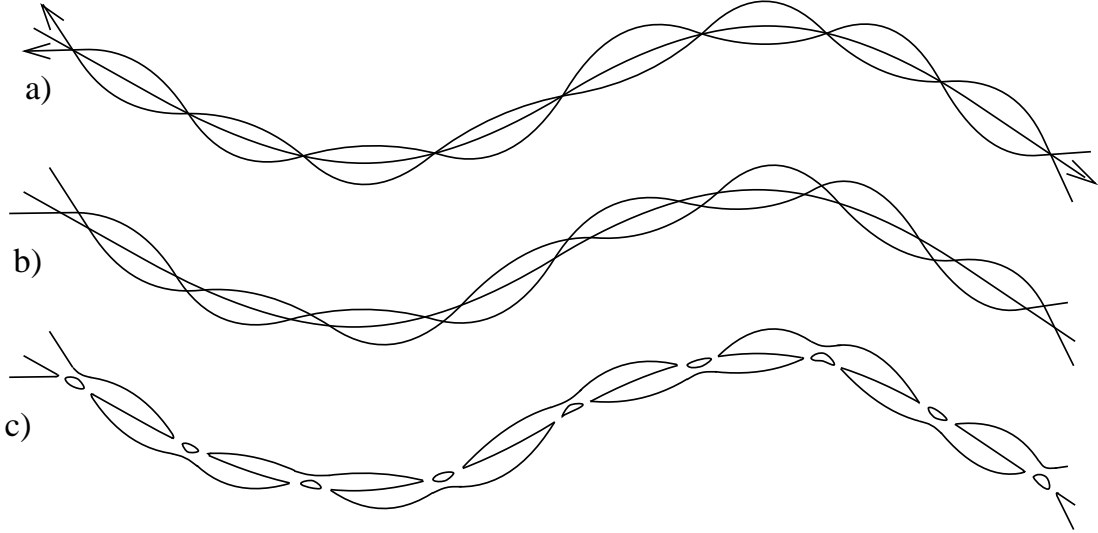


FIGURE 2. Pseudoholomorphic realization of (1)

Let us show that the complex scheme (1) is realizable by a real pseudoholomorphic curve in  $\mathbb{C}\mathbb{P}^2$  of degree 9 (see Remark 1.6 below for another realization). Let  $C = \{f = 0\}$  be a real cubic curve with an oval, and  $L = \{l = 0\}$  be the union of three lines, each cutting the pseudo-line of  $C$  at three distinct real points. Let  $A_{\text{sing}} = \{fg = 0\}$  with  $g = (f + \varepsilon l)(f - \varepsilon l)$  and  $0 < \varepsilon \ll 1$ . Then  $A_{\text{sing}}$  is a reducible algebraic curve of degree 9 with nine triple points. Its real locus consists of three nested ovals and a union of three pseudolines arranged as shown in Figure 2(a). In the class of real pseudoholomorphic curves, it can be perturbed as in Figure 2(b). If we consider  $f$  and  $l$  as holomorphic sections of the line bundle  $\mathcal{O}_{\mathbb{C}\mathbb{P}^2}(3)$  rather than homogeneous polynomials, then the perturbation can be realized by replacing  $f$  with  $f + h$  where  $h$  is a  $\mathcal{C}^1$ -small smooth (non-analytic)  $c$ -invariant section which is complex analytic in some neighbourhoods of the triple points. If  $h$  is small enough, the obtained curve is analytic near all double points. Finally, we perturb the double points by adding to  $(f + h)g$  a yet smaller  $c$ -invariant section of  $\mathcal{O}_{\mathbb{C}\mathbb{P}^2}(9)$  whose signs at the double points are chosen so that the real locus of the resulting curve  $A$  is the union of three nested ovals with the curve shown in Figure 2(c). If the complex orientations of the cubics are chosen as in Figure 2(a), the perturbation is coherent with them (see Figure 3), and hence (see [4], [12, p. II.4], or [23, §3.7]) the resulting curve of degree 9 is separating and its complex scheme is (1). The non-analytic part of  $A$  is close to  $A_{\text{sing}}$ , hence  $A$  is symplectic.

**Remark 1.4.** Since (1) is algebraically unrealizable, so is the intermediate nodal curve. This fact however is much easier: it immediately follows from Abel's theorem applied to the divisors cut by any two of the cubic curves on the third one.

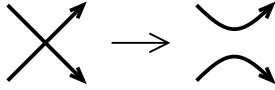


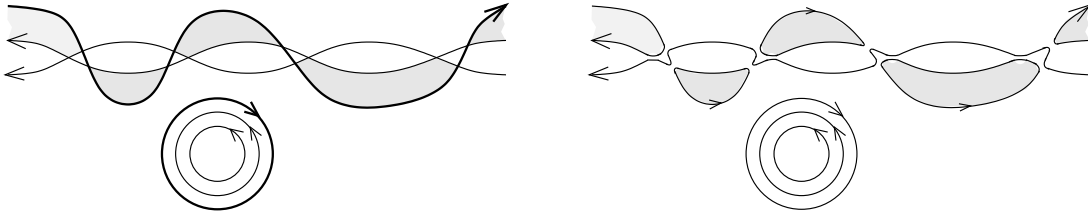
FIGURE 3. A perturbation according to complex orientations.

The same tripling construction applied to suitable Hilbert's  $M$ -curves yields similar examples of higher degrees. The following proposition is proven in §5.

**Proposition 1.5.** *For any positive integer  $p$  there exists a real pseudoholomorphic curve of degree  $m = 12p - 3$  with  $l = 40p^2 - 38p + 10$  ovals such that  $\Lambda_+^p = 0$  and  $\Lambda_-^n = 2p^2 - p - 1$ . The complex scheme of this curve is unrealizable by an algebraic curve of the same degree.*

If a real pseudoholomorphic separating curve has two nested ovals of opposite signs bounding an annulus free of other ovals, then the complex scheme obtained by reversing the orientations of these two ovals is also realizable by a separating real pseudoholomorphic curve of the same degree. I will call this operation *swapping of parallel ovals*. This is a conjugation-equivariant version of Auroux–Donaldson–Katzarkov's braiding construction [1] (see Proposition 6.1 in §6 for more details).

**Remark 1.6.** The complex schemes  $J \sqcup 9_- \sqcup 1_+ \langle 1_- \langle 1_- \rangle \rangle$  and  $J \sqcup 9_- \sqcup 1_- \langle 1_- \langle 1_+ \rangle \rangle$  (which are obtained from (1) by swapping the orientations of any two consecutive nested ovals) are realizable by real algebraic curves of degree 9. Indeed, choose the suitable complex orientations on the cubics in Figure 2(a), then perturb the cubics generically, and smooth the double points according to the chosen orientations, see Figure 4. Thus the swapping of parallel ovals on any of these two algebraic curves (see Proposition 6.1) provides another pseudoholomorphic realization of the complex scheme (1).

FIGURE 4. Algebraic realization of  $J \sqcup 9_- \sqcup 1_{\pm} \langle 1_- \langle 1_{\mp} \rangle \rangle$ 

**Question 1.7.** Is it true that if (2) does not hold for a separating real pseudoholomorphic curve  $A$ , then  $A$  can be transformed to a curve satisfying (2) by swappings of parallel ovals?

The answer to Question 1.7 is affirmative in all cases considered in this paper (see Remark 1.6 and the proof of Proposition 1.5).

**Remark 1.8.** Let  $p$ ,  $n$ ,  $\Lambda_+$ , and  $\Lambda_-$  be the number of even, odd, positive, and negative ovals respectively. The arguments in [23, §3.3] (adapted for an odd degree) show that  $p - n \equiv k^2 + k + \Lambda_+ - \Lambda_- \pmod{4}$ , whence

$$\Lambda_+^p + \Lambda_-^n \equiv l + \frac{k^2 + k}{2} \pmod{2} \quad \text{and} \quad \Lambda_-^p + \Lambda_+^n \equiv \frac{k^2 + k}{2} \pmod{2}. \quad (4)$$

This observation allows us to increase the r.h.s. of the left (resp. right) inequality (3) by 1 when  $l \equiv s \pmod{2}$  (resp. when  $s$  is odd). However, the congruences (4) are satisfied by all separating real pseudoholomorphic curves, thus this improvement is useless for distinguishing between algebraic and pseudoholomorphic realizability.

**Remark 1.9.** (On the sharpness of (3).) The Rokhlin-Mishachev formula [23, §2.3] implies that the right inequality (3) is sharp for Harnack curves ( $M$ -curves without any nest). One can easily prove by induction that the left inequality (3) improved according to Remark 1.8 is sharp for Hilbert's  $M$ -curves (cf. §5 below). I checked that both improved inequalities are sharp for  $k \leq 3$  and for any  $s$  as soon as this fact does not contradict the non-negativity of  $\Lambda_{\pm}^p + \Lambda_{\mp}^n$ .

**Remark 1.10.** If  $1 \leq k \leq 3$ , then all separating real pseudoholomorphic curves of degree  $2k + 1$  satisfy the inequalities (2). For  $k \leq 2$ , this is a consequence of the Rokhlin-Mishachev formula. For  $M$ -curves of degree 7, this is proven in [13, Thm. 2.1], and the same arguments work for other separating curves of degree 7.

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## 2. REAL ALGEBRAIC AND REAL PSEUDOHOLOMORPHIC CURVES

This section is not used in the rest of the paper and it can be considered as an extension of the introduction. A reader interested in the proofs of the results formulated above can skip it.

It is still unknown if there exists an isotopy type of configurations of disjoint embedded circles in  $\mathbb{R}\mathbb{P}^2$  (a real scheme according to Rokhlin's terminology [23]) which is realizable by a smooth pseudoholomorphic curve but algebraically unrealizable with the same degree. It seems very plausible that the 6 open cases for  $M$ -curves of degree 8 (see [14]) as well as the most of the pseudoholomorphic curves constructed in [18] are such examples but the existing methods are insufficient to prove it (see the discussion in [5, §1]). It is also unknown if there exists a real or complex scheme (again in Rokhlin's sense) in  $\mathbb{R}\mathbb{P}^2$  which is realizable by a flexible curve (in the sense of Viro [24]) but pseudoholomorphically unrealizable.

There are known examples of singular (in particular, reducible) real pseudoholomorphic curves in  $\mathbb{P}^2$  whose real loci are algebraically unrealizable with the same degree (same degrees of irreducible components). Some simplest examples are given in [5, §1]. Notice that the algebraic unrealizability of the Pappus-Ringel arrangement discussed there, can be deduced from Abel's theorem as in Remark 1.4, if one perturbs one triple of lines into a cubic curve and considers the divisors which are cut on it by two other triples of lines (of course, a school geometry proof is also possible).

The following is a brief account of methods used in different settings to prove the algebraic unrealizability of real pseudoholomorphic curves.

**Hilbert-Rohn-Gudkov method.** (See [6, 20, 21, 22].) Assuming the existence of an algebraic curve  $A$  with a given isotopy type of  $\mathbb{R}A$ , one can consider its one-parameter equisingular deformations chosen in such a way that some quantity (for example the length of a line segment or the area of some component of the complement to  $\mathbb{R}A$ ) monotonically grows or monotonically decreases. The monotonicity

ensures that the curve must degenerate. Then one chooses another equisingular one-parameter family and so on. This gives a tree of a priori possible degenerations whose leaves are excluded one-by-one by various topological or algebraic arguments. Notice that in collaboration with Eugenii Shustin, by this method we found a very long and complicated proof of Corollary 1.2. However the proof was never written, so we cannot to be sure that it was complete.

**Bezout's theorem for the intersection with unstable curves.** (See [26].) Isotopy types of some real pseudoholomorphic curves constructed in [26] on real Hirzebruch surfaces are unrealizable algebraically because their algebraicity contradicts Bezout's theorem for the number of intersection points with the exceptional curve (a curve  $E$  with  $E^2 \leq -2$ ) which exists for the integrable complex structure but does not exist for a generic almost complex structure.

**Trigonal curves.** (See [2, 6].) The construction from [16] provides an algorithm to decide if a given fiberwise isotopy type is realizable or not by a real algebraic trigonal curve.

**Cubic resolvent of a quadrilateral curve.** (See [6, 15, 17, 21].) Using the cubic resolvent, the algebraic unrealizability of a fiberwise arrangement of a quadrilateral curve can be reduced to that of a mutual arrangement of a trigonal curve and a line. Then one can try to prove that the trigonal curve itself is algebraically unrealizable (as in [6, 15]) or that its mutual arrangement with the line is topologically unrealizable (as in [17, 21]).

This method can be also applied to curves of higher gonality as follows. Let us consider a plane real curve as the graph of an  $n$ -valued function on the upper half-plane. Then any four univalent branches of this function can be continued to a 4-valued function on a suitable Riemann surface, and we can study its cubic resolvent. In this way I proved the algebraic unrealizability in all cases marked by " $\mathcal{A}^*$  alg" in the lists in [17] (unpublished).

**Auxiliary pencils of cubics.** (See [5].) A promising idea was to exploit the fact that a pencil of algebraic cubics through 8 base points on  $\mathbb{P}^2$  always has one more base point, whereas in a family of pseudoholomorphic curves through 8 fixed points, the 9th crossing point of its members may float. However, the implementation of this idea in [5] appeared to be erroneous (see [6]) and so far there are no examples where this method allows to prove the algebraic unrealizability of pseudoholomorphically realizable isotopy types.

### 3. SOME PROPERTIES OF SEPARATING MORPHISMS

Let  $A$  be an abstract real algebraic curve (a Riemann surface endowed with an antiholomorphic involution) and  $f : A \rightarrow \mathbb{P}^1$  a real (i.e. equivariant under the complex conjugation) morphism. Following [3, 11], we say that  $f$  is *separating* if  $f^{-1}(p) \subset \mathbb{R}A$  for any  $p \in \mathbb{R}\mathbb{P}^1$ . It is clear that if there exists a separating morphism  $A \rightarrow \mathbb{P}^1$ , then  $A$  is separating. The converse is also true and, moreover, the following estimate takes place (which plays a crucial rôle in our proof of Theorem 1.1).

**Theorem 3.1.** (Alexandre Gabard [7, Thm. 7.1]). *Let  $A$  be a smooth connected real algebraic separating curve of genus  $g$ . Let  $r$  be the number of connected components of  $\mathbb{R}A$ . Then there exists a separating morphism  $A \rightarrow \mathbb{P}^1$  of degree at most  $(g + r + 1)/2$ .*

As shown in [3], the bound  $(g+r+1)/2$  is sharp for any fixed  $g$  and  $r$ . Notice that the restriction to  $\mathbb{R}A$  of a separating morphism  $A \rightarrow \mathbb{P}^1$  of degree  $n$  is a covering over  $\mathbb{R}\mathbb{P}^1$  of degree  $n$ . The next theorem is a combination of the adjunction formula (in terms of Poincaré residues) with the Abel-Jacobi theorem.

**Theorem 3.2.** *Let  $S$  be a smooth real algebraic surface,  $A$  be a smooth irreducible real separating curve on  $S$ , and  $D$  be a real divisor on  $S$  belonging to the linear system  $|A + K_S|$ . Assume that  $D$  does not have  $A$  as a component. We may always write  $D = 2D_0 + D_1$  with a reduced curve  $D_1$  and an effective divisor  $D_0$ . Let us fix a complex orientation on  $\mathbb{R}A$  and an orientation on  $\mathbb{R}S \setminus (\mathbb{R}A \cup \mathbb{R}D_1)$  which changes each time when we cross  $\mathbb{R}A \cup \mathbb{R}D_1$  at its smooth point (this is possible because  $D \in |A + K_S|$ ). The latter orientation induces a boundary orientation on  $\mathbb{R}A \setminus (\mathbb{R}A \cap D_1)$ . Let  $f : A \rightarrow \mathbb{C}\mathbb{P}^1$  be a separating morphism. Then it is impossible that, for some  $p_0 \in \mathbb{R}\mathbb{P}^1$ , the set  $f^{-1}(p_0) \setminus \text{supp}(D)$  is non-empty and the two orientations coincide at each point of this set.*

*Proof.* We have  $D - A \sim K_S$ . So, let  $\omega_S$  be a real meromorphic 2-form realizing this divisor. Let  $\omega$  be the Poincaré residue of  $\omega_S$  on  $A$  (if we have  $A = \{F(x, y) = 0\}$  and  $\omega_S = g(x, y) dx \wedge dy$  in some local holomorphic coordinates  $(x, y)$  on  $S$ , then  $\omega$  is the restriction to  $A$  of the 1-form  $g dx / F'_y$ ). This is a holomorphic 1-form on  $A$ . By construction, the form  $\omega_S$  (after a change of sign if necessary) defines the orientation on  $\mathbb{R}S \setminus (\mathbb{R}A \cup D_1)$  described in the statement of the theorem, and  $\omega$  defines the induced boundary orientation on  $\mathbb{R}A \setminus (\mathbb{R}A \cap D_1)$ .

Let us fix a point  $p_0 \in \mathbb{R}\mathbb{P}^1$ , and let  $t$  be the coordinate on an affine chart of  $\mathbb{R}\mathbb{P}^1$  centered at  $p_0$ . Since  $f|_{\mathbb{R}A}$  is a covering, the parameter  $t$  lifts to a local parameter near each points of  $f^{-1}(p_0)$ . This means that, for some interval  $I$  containing 0, there are smooth functions  $p_k : I \rightarrow \mathbb{R}A$ ,  $k = 1, \dots, n := \deg(f)$ , such that  $f^{-1}(t) = \{p_1(t), \dots, p_n(t)\}$  for any  $t \in I$ . Let  $v_0 = d/dt \in T_{p_0}(\mathbb{R}\mathbb{P}^1)$  and let  $v_k = p'_k(0) \in T_{p_k(0)}(\mathbb{R}A)$ . Then  $v_0 = f_*(v_k)$  for each  $k = 1, \dots, n$ . The coordinate  $t$  defines a complex orientation on  $\mathbb{R}\mathbb{P}^1$  which is lifted by  $f$  to a complex orientation of  $\mathbb{R}A$ . Without loss of generality we may assume that it is the one chosen in the statement of the theorem.

Since all the divisors  $f^{-1}(t)$  are equivalent to each other, the Abel-Jacobi theorem implies that, for any  $t \in I$ ,

$$\sum_{k=0}^n \int_{p_k([0, t])} \omega = 0.$$

Differentiating this identity at  $t = 0$ , we get  $\omega(v_1) + \dots + \omega(v_n) = 0$ . It remains to notice that  $\omega(v_k) = 0$  iff  $p_k \in \mathbb{R}A \cap \text{supp}(D)$ , and otherwise the sign of  $\omega(p_k)$  is positive (resp. negative) iff the orientation induced by  $\omega$  coincides with (resp. is opposite to) the chosen complex orientation of  $\mathbb{R}A$ .  $\square$

Let us illustrate on simple examples how Theorem 3.2 works.

**Example 3.3.** (Cf. [11, Examples 2.8 and 3.8].) In [19, Thm. 1] I proved that a hyperbolic quartic curve  $A$  in  $\mathbb{R}\mathbb{P}^2$  does not admit any separating morphism to  $\mathbb{R}\mathbb{P}^1$  whose restriction to the exterior oval has degree 1 (the real locus of a hyperbolic quartic consists of two nested ovals). This result immediately follows from Theorem 3.2. Indeed, suppose that such a morphism  $f$  exists. Fix  $p_0 \in \mathbb{R}\mathbb{P}^2$  and let  $p_1$  be the only point of  $f^{-1}(p_0)$  lying on the exterior oval. Then the elements of  $|A + K|$  are lines. Let  $D = D_1$  be a line which separates  $p_1$  from the interior

oval (see Figure 5). The complex orientation of  $\mathbb{R}A$  is shown in Figure 5 by double arrows. The orientation defined by  $\omega$  (from Theorem 3.2) is shown by ordinary arrows. This contradicts Theorem 3.2.

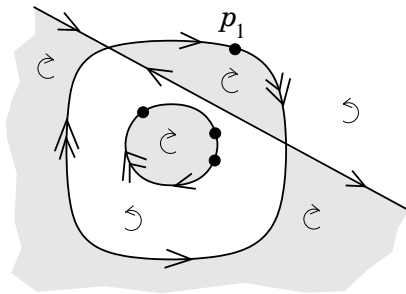


FIGURE 5. Hyperbolic quartic in Example 3.3

**Example 3.4.** (A specialization of the proof of Theorem 1.1 for the case of the 9th degree complex scheme (1), i.e., a direct proof of Corollary 1.2). We argue by contradiction. Suppose (1) is algebraically realizable. We have  $g = 28$ ,  $r = 13$ ,  $l = 12$ ,  $s = 8$ . By Theorem 3.1 there exists a separating morphism  $f$  of degree  $\leq (28+13+1)/2 = 21$ . Elements of the linear system  $|A+K|$  are sextic curves. We choose a double cubic for  $D$ . Then  $D = 2D_0$  and  $D_1$  is empty, thus the orientations are as in Figure 6 (presented in the same style as in the previous example). We see that the two orientations coincide on all ovals of  $\mathbb{R}A$  and they do not coincide on the pseudoline, which we denote by  $J$ . Since  $l = 12$ , among the points of  $f^{-1}(p_0)$ , at least 12 are on the ovals, hence at most 9 of them are on  $J$ . Thus the cubic curve  $D_0$  can be chosen so that  $J \cap f^{-1}(p_0) \subset D_0$ . Then we obtain a contradiction with Theorem 3.2 unless  $f^{-1}(p_0) \subset D_0$ . However, the latter case is impossible. Indeed, even if there are less than 9 points in  $J \cap f^{-1}(p_0)$ , we can choose any nine-point subset of  $J$  containing  $J \cap f^{-1}(p_0)$  and trace the cubic  $D_0$  through it. Then, if  $f^{-1}(p_0) \subset D_0$ , then  $D_0$  cuts each oval because each oval has a point of  $f^{-1}(p_0)$ . But  $D_0$  must cut each oval at an even number of points (counting the multiplicities), hence it cuts the union of all ovals at least at 24 points in addition to the 9 points where it cuts  $J$ . This contradicts the Bezout theorem.

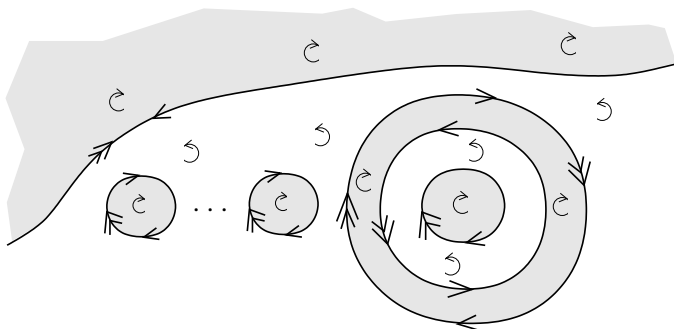


FIGURE 6. The orientations for (1) in Example 3.4

**Example 3.5.** (Cf. [11, Example 2.14].) Let  $S$  be the complexification of the standard sphere in  $\mathbb{R}^3 \subset \mathbb{R}P^3$ , and  $A$  a curve of degree 6 on it (a complete intersection of  $S$  with a real cubic surface). Suppose that  $\mathbb{R}A$  consists of three nested



embedded circles (see Figure 7), and let  $A_0$  be the middle one. We are going to show that there is no separating morphism  $f : A \rightarrow \mathbb{P}^1$  whose restriction to  $A_0$  has covering degree 1. The proof is the same as in Example 3.3. In this case, the elements of  $|A + K|$  are plane sections. So, if such a morphism exists, we chose  $D = D_1$  to be a circle (cut on  $S$  by a plane) which separates the single point of  $A_0 \cap f^{-1}(p_0)$  from the other points of  $f^{-1}(p_0)$ , and we obtain a contradiction with Theorem 3.2 (see Figure 7).

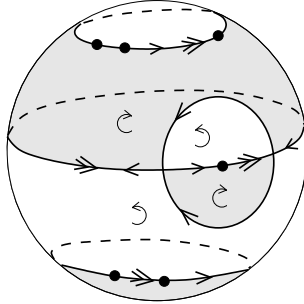


FIGURE 7. The orientations in Example 3.5

**Remark 3.6.** For an abstract real algebraic curve  $A$  with  $r$  components  $A_1, \dots, A_r$  of the real locus, Kummer and Shaw [11] defined the *separating semigroup*  $\text{Sep}(A)$  of  $A$  as the set of  $r$ -tuples  $(\deg f|_{A_1}, \dots, \deg f|_{A_r})$  for all separating morphisms  $f : A \rightarrow \mathbb{P}^1$ . They proved some properties of it, in particular, they computed  $\text{Sep}(A)$  for all curves  $A$  of genus  $\leq 2$ . In [19], I computed the separating semigroup for all hyperelliptic curves and for all curves of genus 3. Since each non-hyperelliptic curve of genus 4 embeds to a quadric surface  $S$  in  $\mathbb{P}^3$ , Theorem 3.2 allows to compute  $\text{Sep}(A)$  for any curve  $A$  of genus 4 (cf. Example 3.5). In particular, it appears (though it is not evident a priori) that  $\text{Sep}(A)$  for  $A$  of genus 4 depends only on the equivariant deformation class (called also the rigid isotopy type) of the pair  $(S, A)$ . These results will be written in a forthcoming paper.

#### 4. PROOF OF THEOREM 1.1

Let the notation be as in Introduction. So,  $A$  is a plane real algebraic non-singular separating curve of degree  $m = 2k + 1$ , and hence of genus  $g = (m - 1)(m - 2)/2 = k(2k - 1)$ . Let  $f : A \rightarrow \mathbb{P}^1$  be a separating morphism of degree  $n \leq (g + r + 1)/2$  which exists by Theorem 3.1.

Suppose that one of the inequalities (2) does not hold. Let  $C_0$  be the union of the components of  $\mathbb{R}A$  which are counted in the left hand side of that inequality (here we assume that the “1” in the left hand side of the left inequality counts the pseudoline of  $A$ ), and let  $C_1 = \mathbb{R}A \setminus C_0$ . Let  $G$  be a homogeneous polynomial of degree  $k - 1$ , and  $D = 2D_0$  be the divisor of  $G^2$ . Then  $\deg D = 2k - 2 = m - 3$ , i.e.,  $D \in |A + K_{\mathbb{P}^2}|$ . So, let us introduce an orientation on  $\mathbb{R}\mathbb{P}^2 \setminus \mathbb{R}A$  as in Theorem 3.2 (notice that  $D_1$  is empty in our case). Then, up to reversing the chosen orientation, we may assume that the boundary orientation induced from  $\mathbb{R}\mathbb{P}^2 \setminus \mathbb{R}A$  and the complex orientation are coherent on  $C_1$  and not coherent on  $C_0$ .

For  $j = 0, 1$ , let  $n_j$  be the covering degree of  $f|_{C_j}$  and  $r_j$  be the number of components of  $C_j$ . Then  $r_0$  is the left hand side of the inequality in (3) which fails,

i. e.,

$$r_0 \leq \frac{k^2 + k}{2} - s - 1. \quad (5)$$

We also have  $r_1 \leq n_1 = n - n_0$ , hence

$$r_0 = r - r_1 \geq r - n + n_0 \geq r + n_0 - \frac{g + r + 1}{2} = n_0 + \frac{r - g - 1}{2} = n_0 - s.$$

By combining the two inequalities, we obtain  $n_0 \leq k(k+1)/2 - 1$ , hence we can choose  $G$  so that the support of  $D$  passes through  $f^{-1}(p_0) \cap C_0$ .

Thus, to obtain a contradiction with Theorem 3.2, it remains to check that the above choice of  $G$  can be done so that  $\text{supp } D$  does not pass through all the  $n$  points of  $f^{-1}(p_0)$ . Indeed, we may choose  $D$  so that it passes through at least  $k(k+1)/2 - 1$  points of  $C_0$ . Suppose that  $f^{-1}(p_0) \subset \text{supp}(D)$ . Each component of  $C_1$  has at least one point of  $f^{-1}(p_0)$ , and at least  $r_1 - 1$  of them are ovals. Since  $D_0$  intersects each oval at least twice, by the Bezout theorem we obtain

$$(k-1)(2k+1) \geq D_0 \cdot C_0 + D_0 \cdot C_1 \geq ((k^2 + k)/2 - 1) + (2r_1 - 1). \quad (6)$$

Since  $r_0 \geq 0$ , the inequality (5) implies

$$s \leq (k^2 + k)/2 - 1, \quad (7)$$

hence, denoting  $(k^2 + k)/2$  by  $a$ , we obtain

$$r_1 = r - r_0 \stackrel{(5)}{\geq} (g + 1 - 2s) - (a - s - 1) = g - a - s + 2 \stackrel{(7)}{\geq} g - 2a + 3.$$

By plugging this bound for  $r_1$  into (6), we obtain

$$2k^2 - k - 1 \geq (5k^2 - 7k + 8)/2,$$

that is  $-k^2 + 5k - 10 \geq 0$  which is a contradiction. Theorem 1.1 is proven.

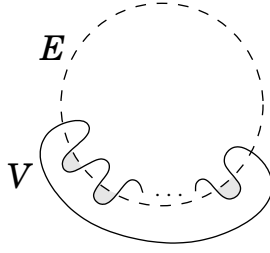
## 5. CONSTRUCTION OF CURVES OF DEGREE $12p - 3$ (PROOF OF PROPOSITION 1.5)

The recursive construction in the proof of the following lemma is nothing else than a particular case of Hilbert's construction of  $M$ -curves in [9] (see also [24, §1.10]), and we compute the quantities  $\Lambda_+^p$  and  $\Lambda_-^n$  for the resulting curves.

Let us fix a smooth real conic  $E$  on  $\mathbb{P}^2$  with  $\mathbb{R}E \neq \emptyset$ . Let  $\Delta$  be the disk bounded by  $\mathbb{R}E$  on  $\mathbb{R}\mathbb{P}^2$ . For a separating curve  $C$ , let  $\Lambda_-(C, \Delta)$  be the number of its negative ovals contained in  $\Delta$ .

**Lemma 5.1.** *For any  $p \geq 2$  there exists an  $M$ -curve  $C_d$  of degree  $d = 4p - 1$  in  $\mathbb{R}\mathbb{P}^2$  such that:*

- (i)  $C_d$  is transverse to  $E$  and all the intersection points are real and belong to an oval  $V$  of  $C_d$  which is arranged with respect to  $\mathbb{R}E$  as shown in Figure 8;
- (ii) the oval  $V$  and all ovals of  $C_d \cap \Delta$  do not encircle other ovals, and the grey digons in Figure 8 do not contain ovals of  $C_d$ ;
- (iii)  $E$  is encircled by  $2p - 3$  ovals of  $C_d$  and  $V$  is positive;
- (iv)  $\Lambda_-(C_d, \Delta) = \Lambda_-^n(C_d) = 2p^2 - p - 1$  and  $\Lambda_+^p(C_d) = 0$ .


 FIGURE 8. Mutual arrangement of  $V$  and  $\mathbb{R}E$ 

*Proof.* Induction on  $p$ . For the base case  $p = 2$  (i.e.,  $d = 4p - 1 = 7$ ), see Figure 9. The inductive step is shown in Figure 10. We see there that when passing from  $C_{4p-1}$  to the intermediate curve  $C_{4p+1}$ , the newly appearing ovals are:

- $4p - 1$  positive ovals in  $\Delta$ ,
- $4p - 3$  negative ovals outside  $\Delta$ ,
- a negative oval crossing  $E$  at  $2(4p + 1)$  points,
- a positive oval which encircles  $E$  and all the other new ovals.

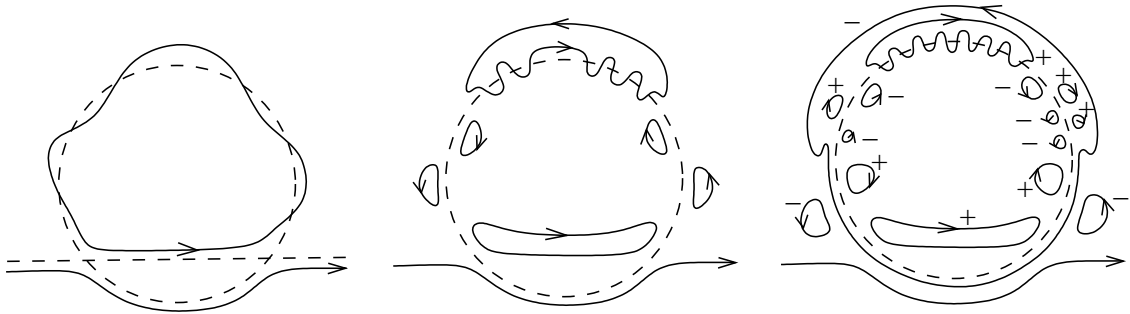
Similarly, when passing from  $C_{4p+1}$  to  $C_{4p+3}$ , the newly appearing ovals are:

- $4p + 1$  negative ovals in  $\Delta$ ,
- $4p - 1$  positive ovals outside  $\Delta$ ,
- a positive oval crossing  $E$  at  $2(4p + 3)$  points,
- a negative oval which encircles  $E$  and all the other new ovals.

Thus, by induction,

$$\Lambda_-(C_{4p+3}, \Delta) = (2p^2 - p - 1) + (4p + 1) = 2(p + 1)^2 - (p + 1) - 1,$$

and we see that  $4p + 1$  new ovals contribute to  $\Lambda_-^n$ , no new oval contributes to  $\Lambda_+^p$ , and the contributions of the old ovals are not changed.  $\square$


 FIGURE 9. Construction of  $C_7$  ( $p = 2$ )

*Proof of Proposition 1.5.* We apply the same construction as was explained in Introduction in the case  $p = 1$  (see Figure 2). Namely, for any  $p \geq 2$  and  $d = 4p - 1$ , let us consider the curve  $C_d$  from Lemma 5.1 and let  $C'_d$  and  $C''_d$  be two its smooth  $\mathcal{C}^1$ -small perturbations invariant under the complex conjugation (by a  $\mathcal{C}^1$ -small perturbation of a subset  $Y$  of a manifold  $X$  we mean the image of  $Y$  under a diffeomorphism of  $X$  which is  $\mathcal{C}^1$ -close to identity). If the perturbations are small enough,  $C'_d$  and  $C''_d$  are smooth symplectic surfaces and we endow them with the

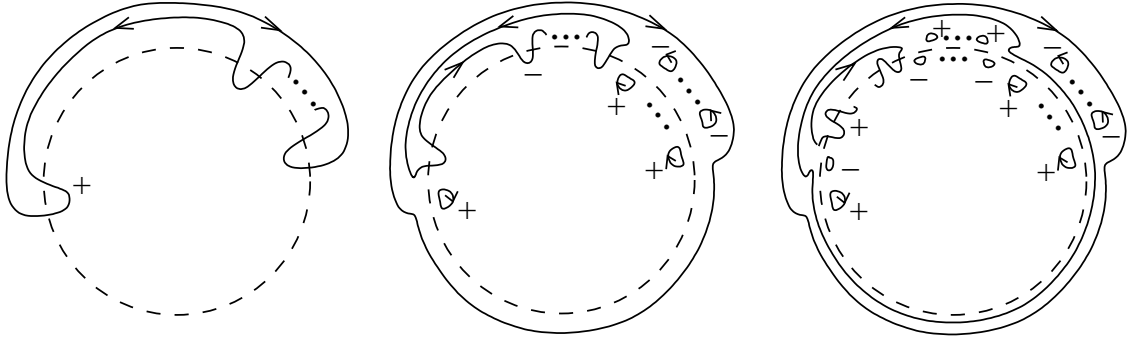


FIGURE 10. The inductive step in the proof of Lemma 5.1

orientation given by the standard symplectic form on  $\mathbb{C}\mathbb{P}^2$ . Let  $A_{\text{sing}} = C_d \cup C'_d \cup C''_d$ . The perturbations can be chosen so that the union of the pseudoline components of  $C_d$ ,  $C'_d$ , and  $C''_d$  looks as in Figure 2(a) (with  $d^2$  triple points), the curves are analytic near the triple points, do not have other intersections, and for each oval of  $\mathbb{R}C_d$  there are ovals of  $\mathbb{R}C'_d$  and  $\mathbb{R}C''_d$  which are  $\mathcal{C}^1$ -close to it. To achieve these properties, we first construct  $C'_d$  as a small perturbation of  $C_d$ , and then  $C''_d$  as a yet smaller perturbation of  $C_d$ . When constructing each of  $C'_d$ ,  $C''_d$ , we start by perturbing a neighborhood of  $\mathbb{R}C_d$  with the required properties, and then we extend the perturbation to the whole  $C_d$ . If crossing points appear in the complement of  $\mathbb{R}\mathbb{P}^2$ , the signed number of them is zero, hence they can be removed by pairs by modifying the perturbation along paths in  $C_d \setminus \mathbb{R}C_d$  connecting crossings of opposite signs.

We choose the complex orientations on the pseudoline components of  $\mathbb{R}A_{\text{sing}}$  as in Figure 2(a) and we perturb  $A_{\text{sing}}$  to  $A$  as in Figure 11 which can be done in two steps similarly to Figure 2. Namely, at the first step we perturb only one of the three curves so that each triple point splits into three double points and so that the appeared small triangles are placed on the two sides of any of the three curves in an alternating way. We may do it so that all the three curves are complex analytic near the double points. At the second step we perturb the double points according to the orientations (as in Figure 3). This can be done by a complex analytic perturbation near the double points which is smoothly extended to the remaining part of the curve respecting the symplecticity.

The perturbation can be chosen so that the ovals of  $C'_d$  and  $C''_d$  appear on either side of the corresponding oval of  $C_d$ , and the side can be chosen arbitrarily and independently for each oval of  $C_d$ . So, we assume that the ovals of  $C_d$  contributing to  $\Lambda_-^p + \Lambda_+^n$  (let us call them *good* ovals) are accompanied by two ovals of  $C'_d \cup C''_d$  from the both sides, but for odd positive ovals (*bad* ovals), the both close ovals of  $C'_d \cup C''_d$  appear from the interior side (see Figure 11).

Then we have  $\Lambda_-^n(A) = \Lambda_-^n(C_d) = 2p^2 - p - 1$  and  $\Lambda_+^p = 0$  as required. The number of ovals of  $C$  is  $d^2$  plus the tripled number of ovals of  $C_d$ , that is

$$l = 3(d-1)(d-2)/2 + d^2 = 40p^2 - 38p + 10$$

as required. Then  $(l - k^2 + 2k)/2 = 2p^2 - p + 1$  for  $k = (m-1)/2 = 6p-2$ , thus (2) fails for  $A$ . Hence the complex scheme of  $A$  is algebraically unrealizable (we point out that both sides of (2) are of order  $p^2$ , but their difference is just 1).  $\square$

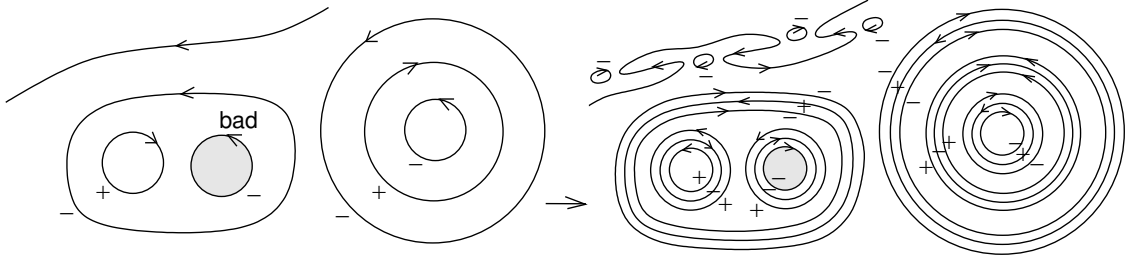


FIGURE 11. The final step of the construction

**Remark 5.3.** If we apply the same construction to the curves of degree  $4p + 1$  that appear as intermediate curves in the proof of Lemma 5.1, then we obtain the equality sign in the left inequality in (2).

## 6. SWAPPING OF PARALLEL OVALS

Let  $(X, \omega)$  be a symplectic 4-manifold and  $c : X \rightarrow X$  be a smooth involution such that  $c^*(\omega) = -\omega$ . We say that a smooth symplectic surface  $A$  in  $X$  is *real* if it is anti-invariant under  $c$ , i.e.,  $c(A) = A$  (as sets) and  $c_*([A]) = -[A]$  in  $H_2(X)$ . We denote the fixed point sets of  $X$  and  $A$  by  $\mathbb{R}X$  and  $\mathbb{R}A$  respectively. Then  $\mathbb{R}X$  and  $\mathbb{R}A$  are smooth submanifolds of  $X$  of respective dimensions 2 and 1. The condition  $c^*(\omega) = -\omega$  implies that  $\mathbb{R}X$  is Lagrangian. We say that  $A$  is *separating* if  $A \setminus \mathbb{R}A$  is not connected. In this case we define the complex orientations on  $\mathbb{R}A$  in the same way as for real algebraic curves.

The *braiding construction* in [1] can be performed obeying the invariance under  $c$  and it provides the following result. Since the proof in [1] is too sketchy, we give here a more detailed self-contained proof.

**Proposition 6.1.** (Essentially, [1, §3]). *Let  $A$  be a smooth real symplectic surface in  $X$ . Let  $V_{-1}$  and  $V_1$  be two components of  $\mathbb{R}A$  bounding an annulus  $B$  on  $\mathbb{R}X$ . Then there exists a smooth real symplectic surface  $A'$  in  $X$  and a neighbourhood  $U$  of  $B$  in  $X$  such that  $\mathbb{R}A' = \mathbb{R}A$ ,  $A' \setminus U = A \setminus U$ , each of  $A \cap U$  and  $A' \cap U$  is a union of two annuli  $A_{-1} \cup A_1$  and  $A'_{-1} \cup A'_1$  respectively, and*

$$A_j \cap \mathbb{R}X = A'_{-j} \cap \mathbb{R}X = V_j \quad \text{and} \quad A_j \cap \partial U = A'_j \cap \partial U \quad \text{for } j = \pm 1.$$

*In particular, if  $A$  is separating and  $[V_{-1}] = -[V_1]$  in  $H_1(B)$  for some complex orientation of  $\mathbb{R}A$ , then  $A'$  is also separating and complex orientations on  $\mathbb{R}A$  and  $\mathbb{R}A'$  can be chosen so that they are opposite on  $V_{-1} \cup V_1$  and coincide on  $\mathbb{R}A \setminus (V_{-1} \cup V_1)$ .*

*Proof.* By Weinstein Neighbourhood Theorem, a symplectic structure near a Lagrangian submanifold is unique up to symplectomorphism. Analysing the proof of this result in [25], one can see that the same is true for  $c$ -anti-invariant symplectic structures near the fixed point set of a smooth involution  $c$ . Hence we may identify a neighbourhood of  $B$  with an open set in  $(\mathbb{C}/\mathbb{Z}) \times \mathbb{C}$  with coordinates  $z_k = x_k + iy_k$ ,  $k = 1, 2$ , where  $x_1$  is defined mod 1, so that

$$B = \{y_1 = y_2 = 0, |x_2| \leq 1\}, \quad V_j = \{y_1 = y_2 = 0, x_2 = j\} \quad (j = \pm 1),$$

$c$  is the usual complex conjugation  $(z_1, z_2) \mapsto (\bar{z}_1, \bar{z}_2)$ , and  $\omega$  is the standard affine symplectic form  $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ .

Let  $a, r > 0$  be such that the chosen coordinates are defined in the domain  $U_r = \{|y_1| < r, |x_2| < 1 + a, |y_2| < a\}$ . Since  $A$  is  $c$ -invariant, for any point  $p \in V_j$ , the tangent plane  $T_p A$  is generated by the  $c$ -invariant vector  $v_1 = \partial/\partial x_1$  and a  $c$ -anti-invariant vector  $v_2$  (a linear combination of  $\partial/\partial y_1$  and  $\partial/\partial y_2$ ). By combining this fact with  $\omega(v_1, v_2) \neq 0$  we conclude that, choosing a smaller  $r$  if necessary, we may assume that the projection  $(z_1, z_2) \mapsto z_1$  restricted to  $A_j \cap U_r$  is non-degenerate, that is  $A_j \cap U_r$  admits a parametrization

$$(u, v) \mapsto \varphi_j(u, v) = (u + iv, j + f(u, v) + ig(u, v))$$

(here  $f$  and  $g$  depend on  $j$ ). For any  $t < r/2$  we choose a smooth function  $h = h_t : [0, r] \rightarrow \mathbb{R}$  such that

$$h|_{[0,t]} = 0, \quad h|_{[2t,r]} = 1, \quad \text{and } 0 \leq h' < 3/t \text{ on } [t, 2t],$$

and we modify  $A$  in  $U_r$  by replacing each  $A_j \cap U_r$  with the annulus  $A_{j,t}$  parametrized by

$$(u, v) \mapsto \varphi_{j,t}(u, v) = (u + iv, j + h(|v|)(f(u, v) + ig(u, v))).$$

We denote the resulting surface by  $A_{(t)}$ . By construction we have  $c(A_{j,t}) = A_{j,t}$  for any  $t$ . Let us show that  $A_{j,t}$  is symplectic for  $t$  small enough. Indeed, for  $v > 0$  we have

$$\begin{aligned} \varphi_j^*(\omega) &= (1 + G)du \wedge dv, & G &= f'_u g'_v - g'_u f'_v, \\ \varphi_{j,t}^*(\omega) &= (1 + h(v)^2 G + h(v)h'(v)F)du \wedge dv, & F &= f'_u g - g'_u f. \end{aligned}$$

The smoothness of  $f$  and  $g$  combined with  $f(u, 0) = g(u, 0) = 0$  implies that

$$\max_{|v| < 2t} \max(|f|, |g|, |f'_u|, |g'_u|) = O(t).$$

We also have  $|h| < 1$  and  $|h'| < 3/t$  whence  $\max_{|v| < 2t} |h^2 G + h h' F| \rightarrow 0$  as  $t \rightarrow 0$ . Hence we may choose  $t$  small enough such that  $A_{j,t}$  is symplectic for  $j = \pm 1$  and hence so is  $A_{(t)}$ . Note that we have  $A_{j,t} \cap U_t = \{z_2 = j\}$  for  $j = \pm 1$ .

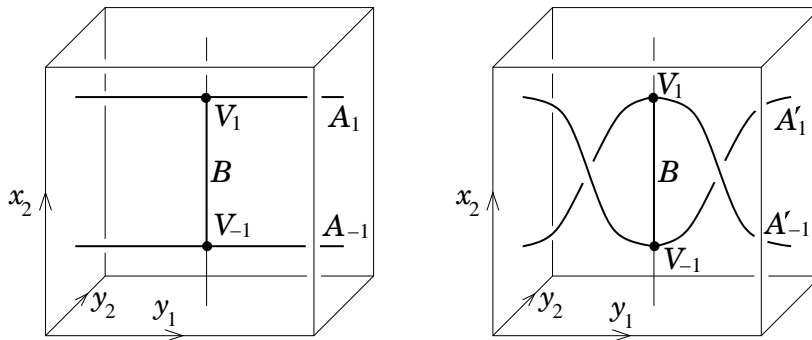


FIGURE 12. The sections  $x_1 = \text{const}$  of  $A \cup B$  and  $A' \cup B$  in  $U$

Finally, we define  $A'$  as an appropriate smoothing of the surface obtained from  $A_{(t)}$  by replacing  $A_{j,t} \cap U_t$  for  $j = \pm 1$  with the surface parametrized by

$$(u, v) \mapsto (u + iv, j \cos(\pi v/t) + jia \sin(\pi v/t))$$

(see Figure 12). It is straightforward to check that  $A'$  is symplectic,  $c$ -invariant, and that it satisfies all other requirements.

## REFERENCES

1. D. Auroux, S. K. Donaldson, and L. Katzarkov, *Luttinger surgery along Lagrangian tori and non-isotopy for singular symplectic plane curves*, Math. Ann. **326** (2003), 185–203.
2. E. Brugallé, *Symmetric plane curves of degree 7: pseudoholomorphic and algebraic classification*, J. Reine Angew. Math. **612** (2007), 129–171.
3. M. Coppens, *The separating gonality of a separating real curve*, Monatsh. Math. **170** (2013), no. 1, 1–10.
4. T. Fiedler, *Eine Beschränkung für die Lage von reellen ebenen algebraischen Kurven*, Beitr. Algebra Geom. **11** (1981), 7–19.
5. S. Fiedler-LeTouzé and S. Orevkov, *A flexible affine  $M$ -sextic which is algebraically unrealizable*, J. Alg. Geom. **11** (2002), 293–320.
6. S. Fiedler-LeTouzé, S. Orevkov, and E. Shustin, *Corrigendum to the paper “A flexible affine  $M$ -sextic which is algebraically unrealizable”*, J. of Alg. Geom. **29** (2020), 109–121.
7. A. Gabard, *Sur la représentation conforme des surfaces de Riemann à bord et une caractérisation des courbes séparantes*, Comment. Math. Helv. **81** (2006), no. 4, 945–964.
8. M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), 307–347.
9. D. Hilbert, *Über die reellen Züge der algebraischen Kurven*, Math. Ann. **38** (1891), 115–137.
10. S. Ivashkovich, V. Shevchishin, *Complex curves in almost complex manifolds and meromorphic hulls*, arxiv:math/9912046.
11. M. Kummer and K. Shaw, *The separating semigroup of a real curve*, Ann. de la fac. des sci. de Toulouse. Math. (6) **29** (2020), no. 1, 79–96.
12. A. Marin, *La transversalité topologique. Une extension d’un théorème de Rohlin et application au 16e problème de Hilbert*, Thèse d’Etat, Univ. Paris-Sud, 1979.
13. S. Yu. Orevkov, *Quasipositivity test via unitary representations of braid groups and its applications to real algebraic curves*, J. Knot Theory Ramif. **10** (2001), 1005–1023.
14. S. Yu. Orevkov, *Classification of flexible  $M$ -curves of degree 8 up to isotopy*, GAFA - Geom. and Funct. Anal. **12** (2002), 723–755.
15. S. Yu. Orevkov, *Construction of arrangements of an  $M$ -quartic and an  $M$ -cubic with a maximal intersection of an oval and the odd branch*, Bull. (vestnik) of Nizhni Novgorod State Univ., Ser. Math. Modeling and Optimal Control, issue 1(25) (2002), 12–48 (Russian); English transl. <https://www.math.univ-toulouse.fr/~orevkov/j3o4e.pdf>.
16. S. Yu. Orevkov, *Riemann existence theorem and construction of real algebraic curves*, Ann. Fac. Sci. de Toulouse. Math. (6), **12** (2003), 517–531.
17. S. Yu. Orevkov, *Arrangements of an  $M$ -quintic with respect to a conic that maximally intersects its odd branch*, Algebra i Analiz **19** (2007), no. 4, 174–242 (Russian); English transl. St. Petersburg Math. J. **19** (2008), 625–674.
18. S. Yu. Orevkov, *Some examples of real algebraic and real pseudoholomorphic curves*, In: Perspectives in Analysis, Geometry and Topology, Progr. in Math. 296, Birkhauser/Springer, N. Y., 2012, pp. 355–387.
19. S. Yu. Orevkov, *Separating semigroup of hyperelliptic curves and of genus 3 curves*, Algebra i analiz **31** (2019), no. 1, 108–113 (Russian); English transl., St. Petersburg Math. J. **31** (2020), 81–84.
20. S. Yu. Orevkov and E. I. Shustin, *Flexible algebraically unrealizable curves: rehabilitation of Hilbert-Rohn-Gudkov approach*, J. für die Reine und Angew. Math. **511** (2002), 145–172.
21. S. Yu. Orevkov and E. I. Shustin, *Pseudoholomorphic algebraically unrealizable curves*, Moscow Math. J. **3** (2003), 1053–1083.
22. S. Yu. Orevkov and E. I. Shustin, *Real algebraic and pseudoholomorphic curves on the quadratic cone and smoothings of singularity  $X_{21}$* , Algebra i Analiz **28** (2016), no. 2, 138–186 (Russian); English transl., St. Petersburg Math. J. **28** (2017), 225–257.
23. V. A. Rokhlin, *Complex topological characteristics of real algebraic curves*, Usp. mat. nauk **33** (1978), no. 5, 77–89 (Russian); English transl., Russian Math. Surveys **33** (1978), no. 5, 85–98.
24. O. Ya. Viro, *Real algebraic plane curves: constructions with controlled topology*, Algebra i analiz **1** (1989), no. 5, 1–73 (Russian); English transl., Leningrad J. Math. **1** (1990), 1059–1134.
25. A. Weinstein, *Symplectic manifolds and their Lagrangian submanifolds*, Adv. Math. **6** (1971), 329–346.

26. J.-Y. Welschinger, *Courbes algébriques réelles et courbes flexibles sur les surfaces réglées de base  $CP^1$* , Proc. London Math. Soc. (3) **85** (2002), 367–392.
27. J.-Y. Welschinger, *Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry*, Invent. Math. **162** (2005), 195–234.

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