CRITERION OF HURWITZ EQUIVALENCE FOR QUASIPOSITIVE FACTORIZATIONS OF 3-BRAIDS

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The problem of Hurwitz equivalence of n-tuples of braids appears in the study of braid monodromy of algebraic curves in $\mathbb{C}^2$. It was considered by many authors, see [1, 3, 4] and references in [3]. We give an answer for the case mentioned in the title.

Let $\mathbf{B}_3 = \langle \mathcal{A} \mid \sigma_2\sigma_1 = \sigma_1\sigma_0 = \sigma_0\sigma_2 \rangle$, $\mathcal{A} = \{\sigma_0, \sigma_1, \sigma_2\}$ be the Birman–Ko–Lee presentation (see [2]) of the group of braids with three strings. A *quasipositive factorization* of a braid $X \in \mathbf{B}_3$ is a collection $(X_1, \dots, X_k) \in \mathbf{B}_3^k$ such that $X = X_1X_2\dots X_k$ and for each i, the braid X_i is conjugate to σ_1. Note that σ_0, σ_1, and σ_2 are conjugate to each other. We denote the set of quasipositive factorizations of X by $\mathcal{Q}(X)$. A braid X is called *quasipositive* if $\mathcal{Q}(X) \neq \varnothing$. The braid group $\mathbf{B}_k$ acts on $\mathcal{Q}(X)$ by $\Sigma_i : (X_1, \dots, X_k) \mapsto (Y_1, \dots, Y_k)$ where $(Y_i, Y_{i+1}) = (X_iX_{i+1}X_i^{-1}, X_i)$ and $Y_j = X_j$ for $j \notin \{i, i+1\}$. This action is called the *Hurwitz action.* Elements belonging to the same orbit are called *Hurwitz-equivalent.*

It is proven in [4] that each orbit of the Hurwitz action contains an element of a certain explicitly specified finite set. The purpose of the present paper is to give an easy criterion to decide if two given elements of this finite set belong to the same orbit. To give precise statements, we need to introduce some notation which slightly differs from that in [4]. Let us extend the alphabet $\mathcal{A}$ up to $\hat{\mathcal{A}} = \mathcal{A} \cup \{\hat\sigma_0, \hat\sigma_1, \hat\sigma_2\}$. Let $\mathcal{A}^*$ and $\hat{\mathcal{A}}^*$ be the free monoids generated by $\mathcal{A}$ and $\hat{\mathcal{A}}$ respectively. If $U, V \in \hat{\mathcal{A}}^*$, then $U \equiv V$ stands for equality in $\hat{\mathcal{A}}^*$ (i. e. letterwise coincidence of words) and $U = V$ (when $U, V \in \mathcal{A}^*$) stands for equality of the corresponding elements of $\mathbf{B}_3$. For $U \in \hat{\mathcal{A}}^*$, we denote the word obtained from U by erasing of all letters $\hat\sigma_i$ (resp. by replacing each $\hat\sigma_i$ with σ_i or by replacing each σ_i with $\hat\sigma_i$) by $\bar U$ (resp. U' or $\hat U$). For example, if $U \equiv \sigma_0\hat\sigma_1\sigma_2\hat\sigma_1$, then $U' \equiv \sigma_0\sigma_2$, $\bar U \equiv \sigma_0\sigma_1\sigma_2\sigma_1$, and $\hat U \equiv \hat\sigma_0\hat\sigma_1\hat\sigma_2\hat\sigma_1$.

We set $\delta = \sigma_2\sigma_1 = \sigma_1\sigma_0 = \sigma_0\sigma_2$. It is easy to check that any $X \in \mathbf{B}_3$ can be written as

$$X = U\delta^{-p}, \qquad U \in \mathcal{A}^*, \quad p \in \mathbb{Z}. \tag{1}$$

If, moreover, U does not contain any subword which is equal (in $\mathbf{B}_3$) to δ, then the presentation of X in the form (1) is unique and it is called the *right Garside normal form.*

Let

$$W = W_1\hat x_1 W_2 \hat x_2 \dots W_k \hat x_k W_{k+1} \in \hat{\mathcal{A}}^*, \quad W_i \in \mathcal{A}^*, \quad x_i \in \mathcal{A}. \tag{2}$$

If $W' = \delta^p$, then $[W]$ stands for the quasipositive factorization of $\bar W \delta^{-p}$ of the form $(X_1, \dots, X_k)$ where $X_i = A_i x_i A_i^{-1}$, $A_i = W_1 \dots W_i$.

To each $X \in \mathbf{B}_3$ we associate a graph $\mathcal{G}_0(X)$ in the following way. Let (1) be the right Garside normal form of X. We define the set of vertices of $\mathcal{G}_0(X)$ as

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$\mathcal{V}_0(X) = \{W \in \hat{\mathcal{A}}^* \mid W' = \delta^p, \bar{W} = U\}$. Two vertices are connected by an edge if they are of the form $A\hat{x}ByC$ and $AxB\hat{y}C$, $x, y \in \mathcal{A}$ where $xB' = B'y$ and, for some $q \ge 0$, either $B' = \delta^q$ (an edge of type (h1)), or $xB' = \delta^q$ (an edge of type (h2)).

Theorem 1. *Suppose that the right Garside normal form of a braid $X \in \mathbf{B}_3$ is* (1) *with $p \ge 0$. Then each orbit of the Hurwitz action on $\mathcal{Q}(X)$ contains an element of the form $[W]$, $W \in \mathcal{V}_0(X)$. Two such elements belong to the same orbit if and only if the corresponding vertices belong to the same connected component of $\mathcal{G}_0(X)$.*

Remark 1. The condition $p \ge 0$ in Theorem 1 is not very restrictive. Indeed, if $p < 0$, then the Hurwitz action has a single orbit (see [4; Corollary 4]).

The rest of the paper is devoted to the proof of Theorem 1. Let $\tau : \hat{\mathcal{A}}^* \to \hat{\mathcal{A}}^*$ be the monoid homomorphism defined on the generators by $\sigma_0 \mapsto \sigma_1 \mapsto \sigma_2 \mapsto \sigma_0$, $\hat{\sigma}_0 \mapsto \hat{\sigma}_1 \mapsto \hat{\sigma}_2 \mapsto \hat{\sigma}_0$. It induces the inner automorphism $\tau : X \mapsto \delta^{-1}X\delta$ of $\mathbf{B}_3$.

Lemma 1. *If $A, B \in \mathcal{A}^*$, $A = B$, and $A \not\equiv B$, then $A = C\delta = \delta\tau(C)$, $C \in \mathcal{A}^*$.*

Proof. It is known (see [2; Theorem 2.7]) that if $A = B$, then B is obtained from A by applying the relations without inserting the inverses of the generators. If $A \not\equiv B$, then a relation can be applied to A, hence $A = D\delta E$, $D, E \in \mathcal{A}^*$. Then $A = C\delta = \delta\tau(C)$ for $C = \tau(D)E$. $\square$

Definition. Let $W \equiv a_1 \dots a_n$, $a_i \in \hat{\mathcal{A}}$, $W' = \delta^q$, $q > 0$. We say that letters a_i and a_j $(i < j)$ match each other in W if $a_i, a_j \in \mathcal{A}$, $(a_i \dots a_j)' \equiv a_iBa_j = \delta^{r+1}$ and $B = \delta^r$, $r \ge 0$.

Formally speaking, the indices i and j rather than the letters a_i and a_j match each other in the above definition. However, abusing the language, we shall speak about matching letters to avoid cumbersome notation.

Lemma 2. (follows from [4; Lemma 5]) *Let $W \in \mathcal{A}^*$, $W = \delta^q$, $q > 0$. Then one can associate parentheses to all letters of W so that the parentheses are balanced and matching pairs of parentheses correspond to matching pairs of letters.*

Lemma 3. *Let $W \equiv AuBCvD \in \mathcal{A}^*$, $W = \delta^q$, and let v match u in W. Let $W_1 \equiv CvD\tau^q(AuB)$. Then $W_1 = \delta^q$ and $\tau^q(u)$ matches v in W_1.*

Proof. Let $E = AuB$, $F = CvD$. Then $\delta^q = EF$, hence $W_1 = F\delta^{-q}E\delta^q = F(F^{-1}E^{-1})E(EF) = EF = \delta^q$. The same computation with $E = Au$, $F = BCvD$ yields $BCvD\tau^q(Au) = \delta^q$. Since $BC = \delta^r$, we obtain $vD\tau^q(Au) = \delta^{q-r}$. Similarly, by setting $E = A$ and $F = uBCvD$, we obtain $D\tau^q(A) = \delta^{q-r-1}$. $\square$

Lemma 4. *Let $W \equiv ABC \in \mathcal{A}^*$, $W = \delta^q$, $B = \delta^r$. Then, for any letter u in A there is a letter in A or in C which matches u in W.*

Proof. Follows from Lemma 3 combined with Lemma 2 applied to $\tau^q(C)A$. $\square$

Let a braid X satisfies the hypothesis of Theorem 1. We extend the graph $\mathcal{G}_0(X)$ up to $\mathcal{G}(X)$ as follows. We define the set of vertices as $\mathcal{V}(X) = \bigcup_{j\ge 0}\mathcal{V}_j$ where $\mathcal{V}_j = \{W \in \hat{\mathcal{A}}^* \mid W' = \delta^{p+j}, \bar{W}\delta^{-p-j} = X\}$. The weight of a vertex $W \in \mathcal{V}_j$ is defined as $\mathrm{wt}(W) = j$. We define edges of types (h1) and (h2) in the same way as in $\mathcal{G}_0(X)$ and we add more edges (W, V) in the following cases:

(h3) $W \equiv A\hat{B}_1 C$, $V \equiv A\hat{B}_2 C$, $B_1, B_2 \in \mathcal{A}^*$, $B_1 = B_2$;
(v1) $W \equiv APB$, $V \equiv A\tau^{-1}(B)$, $P \in \mathcal{A}^*$, $P = \delta$;
(v2) either $W \equiv APByC$ and $V \equiv A\tau^{-1}(B\hat{y}C)$, or $W \equiv AyBPC$ and $V \equiv A\hat{y}B\tau^{-1}(C)$ where $P \in \{u\hat{x}, \hat{x}u\}$, $\bar{P} = \delta$, and y matches u in W;
(v3) $\operatorname{wt}(V) = \operatorname{wt}(W) - 1$, W is connected by an edge of type (h3) to a vertex which is connected to V by an edge of type (v2).

We call edges of types (h1–h3) *horizontal* and edges of types (v1–v3) *vertical*. The notation (W, V) for vertical edges assumes that $\operatorname{wt}(W) > \operatorname{wt}(V)$. If two vertices W and V belong to the same connected component of $\mathcal{G}(W)$, then we write $W \sim V$. If either $W \equiv V$ or there exists an edge (W, V) of type, say, (h1), then we write $W \sim_{\mathrm{h1}} V$. If $\operatorname{wt} W = \operatorname{wt} V$ and W is connected to V by a path in $\mathcal{G}(X)$, which passes through vertices of weight $\le \operatorname{wt}(W)$ only, then we write $W \sim_{\mathrm{h}} V$.

We define $f : (\hat{\mathcal{A}} \cup \mathcal{A}^{-1})^* \to \hat{\mathcal{A}}^*$ by $f(1) = 1$, $f(xA) \equiv xf(A)$ for $x \in \hat{\mathcal{A}}$ and $f(x^{-1}A) \equiv \tau(\tau(x)f(A))$ for $x \in \mathcal{A}$. Note that the braid $A^{-1}f(A)$ is a power of δ.

Lemma 5. *(a). Let $W \in \mathcal{V}(X)$. Then $W \sim f(Y_1 \dots Y_k)$, $Y_i \equiv A_i\hat{x}_iA_i^{-1}$ for some $A_i \in \mathcal{A}^*$, $x_i \in \mathcal{A}$, $i = 1, \dots, k$. (b). If X is conjugate to σ_1, then $\mathcal{V}_0(X)$ contains a single element and it has the form $f(AxA^{-1})$, $A \in \mathcal{A}^*$, $x \in \mathcal{A}$.*

Proof. (a). Let W be as in (2) and set $A_i \equiv W_1 \dots W_i$. Then $f(Y_1 \dots Y_k)$ is connected to W by a chain of edges of type (v1). (b). Follows from Lemma 2. □

Lemma 6. [4; Lemmas 6 and 7]. *If $W \in \mathcal{V}(X)$ then $W \sim V$ for some $V \in \mathcal{V}_0(X)$.* □

Lemma 7. *Let $W, V \in \mathcal{V}(X)$. Then $[W] \sim [V]$ is equivalent to $W \sim V$.*

Proof. ($\Rightarrow$) Let $[W] = (X_1, \dots, X_i)$. Lemma 5(a) combined with Lemmas 6 and 5(b) applied to Y_i's imply $W \sim f(Y_1 \dots Y_k)$ where $\mathcal{V}_0(X_i) = \{f(Y_i)\}$. Thus $[W] = [V] \Rightarrow W \sim V$. It remains to note that if $W \equiv f(Y_1 \dots Y_k)$, $Y_i \equiv A_i\hat{x}_iA_i^{-1}$, and

$$V \equiv f(\dots Y_{i-1}\bar{Y}_iY_{i+1}\bar{Y}_i^{-1}Y_iY_{i+2}\dots), \qquad W_1 \equiv f(\dots Y_{i-1}Y_iY_{i+1}\bar{Y}_i^{-1}\bar{Y}_iY_{i+2}\dots),$$

then $\Sigma_i([W]) = [V]$, $W_1 \sim_{\mathrm{h2}} V$, and W_1 is connected to W by a chain of (v1)-edges.

($\Leftarrow$) (cp. [4; Lemma 6]). It suffices to consider the cases $W \sim_{\mathrm{h1}} V$ and $W \sim_{\mathrm{h2}} V$. Let W be as in (2). Then, for some m, s, $1 \le m < s \le k+1$, we have $W \equiv A\hat{x}_mByC$ and $V \equiv Ax_mB\hat{y}C$ where $A \equiv W_1\hat{x}_1W_2\hat{x}_2 \dots W_m$, $B \equiv W_{m+1}\hat{x}_{m+1} \dots W_{s-1}\hat{x}_{s-1}D$, $C \equiv E\hat{x}_sW_{s+1}\hat{x}_{s+1} \dots \hat{x}_kW_{k+1}$ (if $s = k+1$, then $C \equiv E$), $W_s \equiv DyE$. Let $[W] = (X_1, \dots, X_k)$ and $[V] = (Y_1, \dots, Y_k)$.

It is clear that $Y_i = X_i$ for $i < m$.

If $m \le i < s-1$, then $Y_i = B_ix_{i+1}B_i^{-1}$ where $B_i = A'x_mW_{m+1}W_{m+2} \dots W_{i+1}$ $= (A'x_m(A')^{-1})A'W_{m+1} \dots W_{i+1} = X_mW_1 \dots W_{i+1}$, whence $Y_i = X_mX_{i+1}X_m^{-1}$.

If $i = s-1$, then $Y_i = A'x_mB'y(A'x_mB')^{-1}$. Using $x_mB' = B'y$, we obtain $Y_i = A'x_m(x_mB')(A'x_mB')^{-1} = A'x_m(A')^{-1} = X_m$.

If $i \ge s$, then $Y_i = B_ix_iB_i^{-1}$ where $B_i = A'x_mB'EF$ and $F = W_{s+1} \dots W_i$. Using $x_mB' = B'y$, we obtain $B_i = A'B'yEF = W_1W_2 \dots W_i$, whence $Y_i = X_i$.

Thus, $[V] = (X_1, \dots, X_{m-1}, X_mX_{m+1}X_m^{-1}, \dots, X_mX_{s-1}X_m^{-1}, X_m, X_s, \dots, X_k)$ $= \Sigma_{s-2} \dots \Sigma_{m+1}\Sigma_m[W]$. □

Let $\mathbf{c}_s : \hat{\mathcal{A}}^* \to \hat{\mathcal{A}}^*$ be defined as $aW \mapsto W\tau^s(a)$, $a \in \hat{\mathcal{A}}$, $W \in \hat{\mathcal{A}}^*$.

Lemma 8. (follows from Lemma 3) *(a). Let $e = (W, V) \equiv (aA, bB)$, $a, b \in \hat{\mathcal{A}}$, be an edge of $\mathcal{G}(X)$ of type (h1) or (h2). Let $s = p + \mathrm{wt}(W)$. Then $e_1 = (\mathbf{c}_s(W), \mathbf{c}_s(V))$ is an edge of $\mathcal{G}(\mathbf{c}_s(\bar{W})\delta^{-s})$ of type (h1) or (h2). If $a = b$, then e and e_1 are of the same type. If $a \neq b$, then they are not.*

(b). Let $e = (W, V)$ be an edge of $\mathcal{G}(X)$ of type (v1) or (v2). Let $s = p + \mathrm{wt}(W)$. Let $e_1 = (\mathbf{c}_s^m(W), \mathbf{c}_{s-1}^m(V))$ where $m = 2$ if W starts with the word P used in the definition of e, and $m = 1$ otherwise. Then e_1 is an edge of $\mathcal{G}(\mathbf{c}_s^m(\bar{W})\delta^{-s})$ of the same type as e. $\square$

Lemma 9 (Diamond Lemma). *Let $e_1 = (W, V_1)$ and $e_2 = (W, V_2)$ be vertical edges in $\mathcal{G}(X)$. Then $V_1 \sim_{\mathrm{h}} V_2$.*

Proof. By Lemma 8, mutual arrangements of subwords of W used in the definition of the edges can be considered up to cyclic permutations.

Case 1. Both e_1 and e_2 are of type (v1). The statement is evident.

Case 2. Both e_1 and e_2 are of type (v2). Let $P_i \in \{u_i\hat{x}_i, \hat{x}_i u_i\}$, and y_i, $i = 1, 2$, be the subwords of W used in the definition of e_i.

Case 2.1. P_1 coincides with P_2, i. e., $W \equiv Ay_1By_2CP$, $V_1 \equiv A\hat{y}_1By_2C$, $V_2 \equiv Ay_1B\hat{y}_2C$, $P \in \{\hat{x}u, u\hat{x}\}$, $\bar{P} = \delta$. By definition of edges of type (v2), we have $B'y_2C' = \delta^q$, $C' = \delta^r$, $y_1B'y_2C'u = \delta^{q+1}$, $y_2C'u = \delta^{r+1}$. The former two identities imply $B'y_2 = \delta^{q-r}$. The latter two identities imply $y_1B' = \delta^{q-r}$. Hence $V_1 \sim_{\mathrm{h}2} V_2$.

Case 2.2. P_1 and P_2 have a common letter $\hat{x}_1 = \hat{x}_2 = \hat{x}$. Since $u_1x = xu_2 = \delta$, it follows that u_1 cannot match u_2. Hence y_1 and y_2 cannot coincide with them.

Case 2.2.1. $y_1 = y_2 = y$. $W \equiv AyBu_1\hat{x}u_2$, $V_1 \equiv A\hat{y}B\tau^{-1}(u_2)$, $V_2 \equiv A\hat{y}Bu_1$. Then $B'u_1 = \delta^q$ and $B' = \delta^r$, whence $u_1 = \delta^{q-r}$. Contradiction.

Case 2.2.2. $W \equiv Ay_2By_1Cu_1\hat{x}u_2$, $V_1 \equiv Ay_2B\hat{y}_1C\tau^{-1}(u_2)$, $V_2 \equiv A\hat{y}_2By_1Cu_1$. Since $u_1\delta = u_1xu_2 = \delta u_2$, we have $u_2 = \tau(u_1)$, whence $V_1 \equiv Ay_2B\hat{y}_1Cu_1$. Since $B'y_1C'u_1 = \delta^q$ (e_2 is of type (v2)) and $y_1C'u_1 = \delta^{r+1}$ (e_1 is of type (v2)), it follows that $B' = \delta^{q-r-1}$. To conclude that $V_1 \sim_{\mathrm{h}1} V_2$, it remains to check that $y_1 = \tau^{q-r-1}(y_2)$. Indeed, $y_1\delta^ru_1 = \delta^{r+1}$ implies $\tau^r(y_1)u_1 = \delta$, and $y_2\delta^qu_2 = \delta^{q+1}$ implies $\tau^q(y_2)u_2 = \delta$. Recall that $u_2 = \tau(u_1)$. So, we obtain $\delta = \tau(\delta) = \tau(\tau^r(y_1)u_1) = \tau^{r+1}(y_1)u_2$, whence $\tau^{r+1}(y_1) = \delta u_2^{-1} = \tau^q(y_2)$.

Case 2.2.3. $W \equiv Ay_1By_2Cu_1\hat{x}u_2$, $V_1 \equiv A\hat{y}_1By_2C\tau^{-1}(u_2)$, $V_2 \equiv Ay_1B\hat{y}_2Cu_1$. Since e_2 is of type (v2), we have $C'u_1 = \delta^q$. Hence, by Lemma 2, we have $Cu_1 \equiv C_1z_1C_2u_1$ where z_1 matches u_1 in Cu_1. Let $V_3 \equiv Ay_1By_2C_1\hat{z}_1C_2u_1$. Then $e_3 = (W, V_3)$ is an edge of type (v2) and the pair (e_1, e_3) (resp. (e_3, e_2)) satisfies the conditions of Case 2.1 (resp. by Case 2.2.2).

Case 2.3. P_1 and P_2 have a common letter $u_1 = u_2 = u$, i. e., $W \equiv A\hat{x}_1u\hat{x}_2$. Since $x_1\delta = x_1ux_2 = \delta x_2$, we have $x_2 = \tau^{-1}(x_1)$. Hence the edge of type (v2) defined by (P_1, y_2) coincides with e_2. The problem is reduced to Case 2.1.

Case 2.4. P_1 and P_2 are disjoint, $y_1 = u_2$, and $y_2 = u_1$. Then $W \equiv AP_1BP_2$, $V_1 \equiv A\tau^{-1}(B\hat{P}_2)$, $V_2 \equiv A\hat{P}_1B$, $B' = \delta^q$. If $q = 0$, then $V_1 \sim_{\mathrm{h}3} V_2$. Let $q > 0$. Let $B \equiv \hat{C}zD$, $z \in \mathcal{A}$ and let $xy = yz = \delta$. Since $P_1C = \tau^{-1}(C)xy$, we have $V_3 \equiv A\tau^{-1}(\hat{C})\hat{x}\hat{y}zD \sim_{\mathrm{h}3} V_2$. Then $V_1 \equiv E\tau^{-1}(zD\hat{P}_2)$ and $V_3 \equiv E\hat{x}\hat{y}zD$ for $E \equiv A\tau^{-1}(\hat{C})$. Since $zD' = (\hat{C}zD)' = B' = \delta^q$, it follows from Lemma 2 that zD includes a letter u which matches z. Let (V_3, U_3) be the edge of type (v2) defined by $(\hat{y}z, u)$. Then $D \equiv D_1uD_2$ and $U_3 \equiv E\hat{x}\tau^{-1}(D_1\hat{u}D_2)$. We have $D_1' = \delta^r$ and $zD_1'u = \delta^{r+1}$. Recall that $zD_1'uD_2' = \delta^q$ which implies $D_2' = \delta^{q-r-1}$. Let

$v = \tau^{q-r-1}(u)$ and $vw = \delta$. We may assume that $P_2 \equiv vw$ (otherwise we add an edge of type (h3)). Then $V_1 \sim_{\rm h1} V_4 \equiv E\tau^{-1}(zD_1\hat{u}D_2v\hat{w})$. Let us show that $\tau^{-1}(z)$ matches $\tau^{-1}(v)$ in V_4. Indeed, we have $D_1'D_2' = \delta^r\delta^{q-r-1} = \delta^{q-1}$ and $uD_2' = D_2'v$ (by the choice of v), whence $zD_1'D_2'v = zD_1'uD_2' = B' = \delta^q$. Let (V_4, U_4) be the edge of type (v2) defined by $(\tau^{-1}(v\hat{w}), \tau^{-1}(z))$. Then $U_4 \equiv E\tau^{-1}(\hat{z}D_1\hat{u}D_2)$. It remains to note that $x\delta = xyz = \delta z$, whence $x = \tau^{-1}(z)$ and therefore $U_3 \equiv U_4$.

Case 2.5. P_1, P_2, and y_2 are pairwise disjoint: $W \equiv D_1D_2$ where $D_1 \equiv AP_1B$, $D_2 \equiv y_2CP_2$, $C' = \delta^q$, and $D_2' = \delta^{q+1}$. Then $D_1' = W'\delta^{-q-1}$ is a power of δ. By Lemma 2, this implies that D_1 has a letter z_1 which matches u_1. Let (W, V_3) be the edge of type (v2) defined by (P_1, z_1). Then we have $V_1 \sim_{\rm h} V_3$ (see Case 2.1), $V_2 \equiv D_1E_2$, and $V_3 \equiv E_1\tau^{-1}(D_2)$ where $D_i \sim_{\rm v2} E_i$ in the corresponding graphs. Hence $V_2 \sim_{\rm v2} E_1\tau^{-1}(E_2) \sim_{\rm v2} V_3$.

Case 3. e_1 is of type (v3) and e_2 is of type (v2) or (v3). Let $W \sim_{\rm h3} W_i \sim_{\rm v2} V_i$, $i = 1, 2$. Let $P_i \in \{\hat{x}_iu_i, u_i\hat{x}_i\}$ and y_i be the subwords of W_i used in the definition of the edge (W_i, V_i). We may assume that $W_1 \equiv Eu_1\hat{x}_1\hat{F}_1u_2$ and $W_2 \equiv Eu_1\hat{F}_2\hat{x}_2u_2$ where $x_1F_1 = F_2x_2$ (otherwise the problem reduces to Case 2). By Lemma 1, we have $x_1F_1 = \delta\tau(D) = D\delta$, $D \in \mathcal{A}^*$. So, we may assume that $x_1F_1 \equiv x_1v_1\tau(D)$ and $F_2x_2 \equiv Dv_2x_2$ where $x_1v_1 = v_2x_2 = \delta$. Without loss of generality we may assume also that $u_1x_1v_1 \equiv \sigma_2\sigma_1\sigma_0$.

Case 3.1. $y_1 = u_2$ and $y_2 = u_1$. Then $u_1(\hat{x}_1\hat{F}_1)'u_2 = u_1u_2 = \delta$. Hence $W_1 \equiv E\sigma_2\hat{\sigma}_1\hat{\sigma}_0\tau(\hat{D})\sigma_1$ and $W_2 \equiv E\sigma_2\hat{D}\hat{\sigma}_0\hat{\sigma}_2\sigma_1$, whence $V_1 \equiv E\tau^{-1}(\hat{\sigma}_0\tau(\hat{D})\hat{\sigma}_1) \equiv E\hat{\sigma}_2\hat{D}\hat{\sigma}_0 \equiv V_2$.

Case 3.2. $E \equiv E_2y_1E_1$. Then $E_1' = \delta^q$ and $y_1E_1'u_1 = \delta^{q+1}$. Hence, by Lemma 4, there is a letter z_2 in E_2 which matches u_2. Let (W_2, V_3) be the edge of type (v2) defined by (P_2, z_2). Then $V_2 \sim_{\rm h} V_3$ (see Case 2.1). By passing to a cyclic permutation of W, we may assume that $W_1 \equiv Ay_1Bu_1\hat{x}_1\hat{v}_1\tau(\hat{D})u_2Cz_2$, $W_2 \equiv Ay_1Bu_1D\hat{v}_2\hat{x}_2u_2Cz_2$, $B' = \delta^q$, $y_1Bu_1 = \delta^{q+1}$, $C' = \delta^r$, $u_2C'z_2 = \delta^{r+1}$. For each possible value of u_2, we find V_4 such that $V_1 \sim_{\rm h} V_4 \sim_{\rm h} V_3$. (for $u_2 = \sigma_1, \sigma_2$ we just list the indices which should be inserted into the formulas for V_1, V_3, and V_4).

$u_2 = \sigma_0$:

$$V_1 \equiv A\tau^{-q}(\hat{\sigma}_0)B\tau^{-1}\big(\hat{\sigma}_0\tau(\hat{D})\sigma_0C\tau^r(\sigma_2)\big) \equiv A\tau^{-q}(\hat{\sigma}_0)B\hat{\sigma}_2\hat{D}\sigma_2\tau^{-1}(C)\tau^r(\sigma_1),$$

$$V_3 \equiv A\,\tau^{-q}(\sigma_0)\,B\,\sigma_2\,\hat{D}\,\hat{\sigma}_2\,\tau^{-1}\big(C\,\tau^r(\hat{\sigma}_2)\big) \equiv A\tau^{-q}(\sigma_0)B\sigma_2\hat{D}\hat{\sigma}_2\tau^{-1}(C)\tau^r(\hat{\sigma}_1),$$

$$V_4 \equiv A\tau^{-q}(\sigma_0)B\hat{\sigma}_2\hat{D}\sigma_2\tau^{-1}(C)\tau^r(\hat{\sigma}_1);$$

$u_2 = \sigma_1$: $V_1(\hat{0}\hat{2}02)$, $V_4(0\hat{2}\hat{0}2)$, $V_3(02\hat{0}\hat{2})$; $\quad u_2 = \sigma_2$: $V_1(\hat{0}\hat{2}10)$, $V_4(\hat{0}21\hat{0})$, $V_3(02\hat{1}\hat{0})$.

Case 4. e_1 is of type (v1) and e_2 is of type (v2). Let $(W, V_1) \equiv (AP_1, A)$, $P_1 = \delta$. Let P and y be the subwords of W used in the definition of e_2.

Case 4.1. P_1 has one common letter with P: $W \equiv AyB\hat{x}uv$, $xu = uv = \delta$, $B' = \delta^q$, $yB'u = \delta^{q+1}$, $V_1 \equiv AyB\hat{x}$, and $V_2 \equiv A\hat{y}B\tau^{-1}(v)$. Then $x\delta = xuv = \delta v$ implies $\tau^{-1}(v) = x$, and $y\delta^qu = \delta^{q+1}$ combined with $xu = \delta$ implies $x = \tau^q(y)$. Hence $V_1 \sim_{\rm v1} V_2$.

Case 4.2. P_1 is disjoint from P. Lemma 4 implies $A \equiv BzCPD$ where z matches u. Let (W, V_3) be the edge of type (v2) defined by (P, z). Then $V_2 \sim_{\rm h} V_3$ (see Case 2.1) and there exist vertical edges (V_1, V_4) and (V_3, V_4) where $V_4 \equiv B\hat{z}C\tau^{-1}(D)$.

Case 5. e_1 is of type (v1) and e_2 is of type (v3). Let $W \sim_{\rm h3} W_1 \sim_{\rm v2} V_2$. Then we have $W_1 \sim_{\rm v1} V_1$ which reduces the problem to Case 4. $\square$

Lemma 10. *Let* $e = (W_1, W_2)$ *be a horizontal edge in* $\mathcal{G}(X)$*. If* $\mathrm{wt}(W_1) > 0$*, then there exist vertical edges* (W_1, V_1) *and* (W_2, V_2) *such that* $V_1 \sim_{\mathrm{h}} V_2$*.*

Proof. The condition $\mathrm{wt}(W_1) > 0$ implies that $\bar{W}_1$ has a subword P which is equal to δ. By Lemma 8, it suffices to consider edges of type (h1) and (h3) only.

Case 1. e of type (h1). Let x and y be as in the definition of edges of type (h1). It is clear that $P \not\equiv xy$. If P, x, y are disjoint, then the statement is evident.

Case 1.1. $W_1 \equiv A\hat{y}Bux$, $W_2 \equiv AyBu\hat{x}$, $ux = \delta$. Let $V_1 = A\hat{y}B$. Then $W_1 \sim_{\mathrm{v1}} V_1$. Since e is an edge of type (h1), we have $B'u = \delta^q$, $x = \tau^q(y)$. Hence, by Lemma 2, there is a letter z in Bu which matches u, i. e., $B \equiv B_1 z B_2$, $B_2' = \delta^r$, $zB_2'u = \delta^{r+1}$. Hence $W_2 \sim_{\mathrm{v2}} V_2 \equiv AyB_1\hat{z}B_2$. Let us show that $V_1 \sim_{\mathrm{h1}} V_2$. Indeed, $B_1' z B_2' u = B'u = \delta^q$ combined with $zB_2'u = \delta^{r+1}$ imply $B_1' = \delta^{q-r-1}$, and $z\delta^r u = \delta^{r+1}$ combined with $ux = \delta$ imply $x = \tau^{r+1}(z)$. Since $x = \tau^q(y)$, we obtain $z = \tau^{q-r-1}(y)$.

Case 1.2. $W_1 \equiv A\hat{y}Bxu$, $W_2 \equiv AyB\hat{x}u$, $xu = \delta$, $B' = \delta^q$, and $yB' = B'x$. Then $yB'u = B'(xu) = \delta^{q+1}$, i. e., $(W_1, A\hat{y}B)$ and $(W_2, A\hat{y}B)$ are edges of type (v1) and (v2) respectively.

Case 2. e of type (h3). Without loss of generality we may assume that $W_1 \equiv Au\hat{B}_1$, $W_2 \equiv Au\hat{B}_2$, $u \in \mathcal{A}$, $B_1, B_2 \in \mathcal{A}^*$, and $B_1 = B_2$. Lemma 1 implies $B_1 = \delta B$. Let $W \equiv Au\hat{x}\hat{v}\hat{B}$ where $ux = xv = \delta$. Then $W_1 \sim_{\mathrm{h3}} W \sim_{\mathrm{h3}} W_2$. Let y matches u and let (W, V) be the edge of type (v2) defined by $(u\hat{x}, y)$. Then $W_1 \sim_{\mathrm{v3}} V \sim_{\mathrm{v3}} W_2$. $\square$

Proof of Theorem 1. By Lemma 7, it is enough to show that: if $W \sim V$ and $\mathrm{wt}(W) = \mathrm{wt}(V)$, then $W \sim_{\mathrm{h}} V$. Indeed, if a path from W to V in $\mathcal{G}(X)$ passes through vertices whose weight is greater than $\mathrm{wt}(W)$, then, by Lemmas 9 and 10, we can modify the path so that either the number of vertices of the maximal weight is reduced, or the number of horizontal edges incident to such vertices is reduced.

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