

ON THE HURWITZ ACTION ON QUASIPOSITIVE FACTORIZATIONS OF 3-BRAIDS

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Let \mathbf{B}_3 be the group of braids with three strings: $\mathbf{B}_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle$. A *quasipositive factorization* of $X \in \mathbf{B}_3$ is a collection $(X_1, \dots, X_k) \in \mathbf{B}_3^k$ such that $X_1 \dots X_k = X$, and each of the X_i 's is conjugate in \mathbf{B}_3 to the standard generator σ_1 . Note that σ_1 and σ_2 are conjugate to each other in \mathbf{B}_3 . A braid X is called *quasipositive* if it admits at least one quasipositive factorization.

Let G be a group. We define the mappings $\Sigma_i : G^k \rightarrow G^k$, $1 \leq i < k$, by setting $\Sigma_i(X_1, \dots, X_k) = (Y_1, \dots, Y_k)$ where $(Y_i, Y_{i+1}) = (X_i X_{i+1} X_i^{-1}, X_i)$ and $Y_j = X_j$ for $j \notin \{i, i+1\}$. These mappings are invertible because $(X_i, X_{i+1}) = (Y_{i+1}, Y_{i+1}^{-1} Y_i Y_{i+1})$. If (X_1, \dots, X_k) is a quasipositive factorization of a braid X , then it is easy to see that $\Sigma_i(X_1, \dots, X_k)$ is also a quasipositive factorization of the same braid. The correspondence $\sigma_i \mapsto \Sigma_i$ is an action of the braid group \mathbf{B}_k on the set G^k . This action is called the *Hurwitz action*. Factorizations belonging to the same orbit of the Hurwitz action are called *Hurwitz equivalent*.

If U, V are words over the alphabet $\mathcal{A} = \{\sigma_1, \sigma_2\}$ (such words, as well as the braids represented by them, are called *positive*), then $U \equiv V$ stands for letterwise coincidence and $U = V$ stands for equality in \mathbf{B}_3 . We set $\Delta = \sigma_1 \sigma_2 \sigma_1$.

If $W \equiv a_1 \dots a_n$ is a positive word and $I = \{i_1, \dots, i_k\}$, $1 \leq i_1 < \dots < i_k \leq n$, then we set $W \setminus I \equiv W_1 \dots W_{k-1}$ and $W_I = (A_1 x_1 A_1^{-1}, \dots, A_k x_k A_k^{-1})$ where $x_j = a_{i_j}$, W_1, \dots, W_{k+1} are pieces into which W is split by x_1, \dots, x_k (i. e., $W_j \equiv a_l \dots a_m$ where $l = i_{j-1} + 1$, $m = i_j - 1$, $i_0 = 0$, $i_{k+1} = n + 1$) and $A_j = W_1 \dots W_j$. It is easy to see that W_I is a quasipositive factorization of $W \Delta^{-p}$ if and only if $W \setminus I = \Delta^p$. If $i - 1 \notin I$, $i \in I$, and $a_{i-1} = a_i$, then $W_I = W_{\{i-1\} \cup (I \setminus \{i\})}$. When I does not contain such i , we say that I is *W-minimal*.

Theorem 1. *Let W be a positive word and $p \geq 0$. Suppose that the braid $X = W \Delta^{-p}$ is quasipositive. Then every orbit of the Hurwitz action on quasipositive factorizations of X contains an element of the form W_I with W -minimal I .*

Corollary 1. (see [3]). *A braid $X \in \mathbf{B}_3$ is quasipositive if and only if any positive word W such that $X = W \Delta^{-p}$ admits a removal of some letters so that the remaining word is equal to Δ^p in \mathbf{B}_3 . \square*

Since any 3-braid can be presented as $W \Delta^{-p}$, Theorem 1 provides an algorithm for finding representatives of all orbits of the Hurwitz action. The algorithm can be optimized in the 'branch and bound' spirit by analogy with [4; §6].

Corollary 2. *The number of orbits of the Hurwitz action for any 3-braid is finite.¹ \square*

¹(Added in 2019) There is a mistake here in the published version of this article.

Corollary 3. *Any two quasipositive factorizations of a positive 3-braid are Hurwitz equivalent.*

*Proof.*² By Theorem 1, it is enough to check that the factorizations $(\sigma_1, \sigma_2, \sigma_1)$ and $(\sigma_2, \sigma_1, \sigma_2)$ are Hurwitz equivalent. Indeed, $(\sigma_1, \sigma_2, \sigma_1) \xrightarrow{\Sigma_2} (\sigma_1, \sigma_2\sigma_1\sigma_2^{-1}, \sigma_2) \xrightarrow{\Sigma_1} (\sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}, \sigma_1, \sigma_2) = (\sigma_2, \sigma_1, \sigma_2)$. \square

The Birman–Ko–Lee (BKL) presentation for \mathbf{B}_3 is

$$\mathbf{B}_3 = \langle \sigma_0, \sigma_1, \sigma_2 \mid \sigma_2\sigma_1 = \sigma_1\sigma_0 = \sigma_0\sigma_2 \rangle \quad (2)$$

where σ_1 and σ_2 are the same as above, and hence $\sigma_0 = \sigma_1^{-1}\sigma_2\sigma_1$. Words over the alphabet $\{\sigma_0, \sigma_1, \sigma_2\}$ and the braids represented by them will be called *BKL-positive*. Any 3-braid can be written in the form $W\delta^{-p}$ with a BKL-positive W and $\delta = \sigma_2\sigma_1$.

Theorem 2. *Let W be a BKL-positive word and $p \geq 0$. Suppose that the braid $X = W\delta^{-p}$ is quasipositive. Then every orbit of the Hurwitz action on quasipositive factorizations of X contains an element of the form W_I with W -minimal I .*

Corollary 4. *Any two quasipositive factorizations of a BKL-positive 3-braid are Hurwitz equivalent.*

Of course, a BKL analog of Corollary 1 holds as well. In spite of the similarity between Theorems 1 and 2, their proofs are very different. Our proof of Theorem 1 is more geometric. It is inspired by the proof of the main result in [2]. The proof of Theorem 2 is purely combinatorial, it is in the spirit of [3].

Let $e : \mathbf{B}_3 \rightarrow \mathbb{Z}$ be the group homomorphism such that $e(\sigma_1) = e(\sigma_2) = 1$.

Theorem 3. *If $X \in \mathbf{B}_3$ and $e(X) = 2$, then X has at most two orbits of the Hurwitz action.*

Example. Let $W \equiv \sigma_1^2\sigma_2^2\sigma_1^2\sigma_2^2$. Then $W_{\{1,5\}} \not\sim W_{\{3,7\}}$. Hence, by Theorems 1 and 3, the braid $W\Delta^{-2}$ has exactly two orbits of the Hurwitz action.

Remark. Theorem 2 and its proof extend without changes to the case of Artin–Tits groups of type $I_2(p)$ if one defines BKL-positive words as positive words in the generators of the presentation $\langle a_1, \dots, a_p \mid a_p a_{p-1} = a_{p-1} a_{p-2} = \dots = a_2 a_1 = a_1 a_p \rangle$.

§1. Admissible graphs and quasipositive factorizations. We fix a disk D and a point q on its boundary ∂D . Let Γ be an oriented graph embedded in $D \setminus \{q\}$ and let the numbers 1 and 2 be assigned to every edge of Γ . Let $V(\Gamma)$ be the set of all vertices of Γ and let $V_n(\Gamma)$ be the set of the vertices adjacent to n edges. We set $\partial\Gamma = \Gamma \cap \partial D$, $R(\Gamma) = V_6(\Gamma)$, and $B(\Gamma) = V_1(\Gamma) \setminus \partial\Gamma$. Elements of $B(\Gamma)$ will be called *branch points*.

We say that Γ is an *admissible graph* if $\partial\Gamma \subset V_1(\Gamma)$, $V(\Gamma) = V_1(\Gamma) \cup R(\Gamma)$, and the edges incident to any vertex $v \in R(\Gamma)$ are oriented and labeled as in Fig. 1. An admissible graph Γ will be called *quasipositive* if the edge incident to any branch point b is oriented towards b (see Fig. 1).

If α is a path in $D \setminus V(\Gamma)$ transversal to the edges of Γ , then we define the word $\Gamma(\alpha)$ as $\sigma_{i_1}^{\pm 1} \sigma_{i_2}^{\pm 1} \dots$ where i_1, i_2, \dots are the numbers assigned to the edges

²(Added in 2019) In fact it is enough to observe that I is unique.

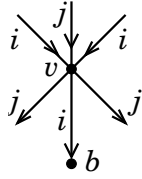


Fig. 1. $i \neq j$

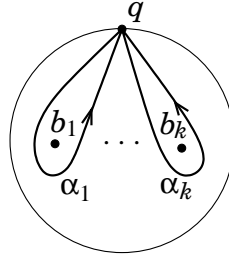


Fig. 2.

successively crossed by α and the signs are chosen according to the orientation of these edges, so that a positively oriented loop around a branch point labeled by i corresponds to σ_i . The word corresponding to the positive circuit along ∂D starting from q will be called the *boundary word* of Γ . We denote it by $\Gamma(\partial D)$. If Γ is an admissible graph, then it is easy to see that paths which are homotopic in $D \setminus B(\Gamma)$ define the same braid.

Every quasipositive graph Γ uniquely determines a Hurwitz equivalence class of quasipositive factorizations of the boundary braid as follows. Let $B(\Gamma) = \{b_1, \dots, b_k\}$. We choose pairwise distinct paths $\alpha_1, \dots, \alpha_k$ as in Fig. 2. Then $(\Gamma(\alpha_1), \dots, \Gamma(\alpha_k))$ is a quasipositive factorization of $\Gamma(\partial D)$. The collection of paths (α_i) is defined up to a diffeomorphism of the disk identical on the boundary, i. e., up to the action of the braid group \mathbf{B}_k . One can check that this is the Hurwitz action.

Lemma 1. *Any quasipositive factorization of a given word can be represented by a quasipositive graph.*

Proof. If words W_1 and W_2 are equal in \mathbf{B}_3 and a disk D_2 is inside D_1 , then there exists a branch point free admissible graph Γ in the annulus $D_1 \setminus D_2$ such that $\Gamma(\partial D_i) = W_i$, $i = 1, 2$. It is enough to check this fact when W_2 is obtained from W_1 either by a braid group relation or by inserting or removing $\sigma_i^{\pm 1} \sigma_i^{\mp 1}$. Let us construct a graph which defines a given quasipositive factorization (X_1, \dots, X_k) of a given word W , $X_i = a_i \sigma_1 a_i^{-1}$. We consider nested disks $D_3 \subset D_2 \subset D_1 \subset D$. In $D \setminus D_1$, we construct a graph realizing the equality $W = X_1 \dots X_k$. In $D_1 \setminus D_2$, we complete the edges corresponding to the central σ_1 's by adding branch points. Finally, in $D_2 \setminus D_3$, we realize the equality $(a_1 a_1^{-1}) \dots (a_k a_k^{-1}) = 1$. \square

Lemma 2. *If two quasipositive graphs coincide outside a disk $U \subset D$ and if each of them has at most one branch point in U , then the graphs define the same quasipositive factorization of the boundary braid.* \square



Fig. 3. $i \neq j$

In fact, we need Lemma 2 only in two cases: when one graph is obtained from the other by the modifications shown in Fig. 3.

§2. Proof of Theorem 1. Let Δ_{-p} be the word $\sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1} \dots$ ($3p$ alternating letters). It represents the braid Δ^{-p} . By Lemmas 1 and 2, it suffices to prove

that the modifications in Figure 3 allow us to transform any quasipositive graph with the boundary word $W\Delta_{-p}$ so that every branch point of the new graph is connected by an edge to a point on ∂D . We say that such branch points are *good* and the others are *bad*.

We prove this fact by induction on the weight of the graph which we define as $|R(\Gamma)|$ plus the number of bad branch points. If the weight is zero, then all branch points are good. Let us prove the required result for a graph Γ assuming that it is proven for all graphs of smaller weight. If there are no bad points, we are done. So, we assume that they exist. Suppose that no modification reduces the weight. Let b be a bad branch point. Without loss of generality, we may assume that the edge bv incident to b is labeled by 1. In a neighbourhood of this edge, Γ is oriented as in Fig. 1 with $i = 1$ and $j = 2$ because otherwise modification A in Fig. 3 would reduce $|R(\Gamma)|$. Let P be the closure of the component of $D \setminus \Gamma$ that contains b . Let e_1, \dots, e_n be the edges of Γ lying on ∂P , numbered in the order of a positive circuit along ∂P starting from v .

Let us prove by induction on i that if e_i has a positive (resp. negative) orientation with respect to ∂P , then it is labeled by 2 (resp. by 1). Since this fact contradicts the orientation of e_n in Fig. 1, this will complete the proof of Theorem 1. The assertion is true for $i = 1$. Suppose that it is true for some i . To deduce it for $i + 1$, it suffices to exclude all the cases shown in Fig. 4.

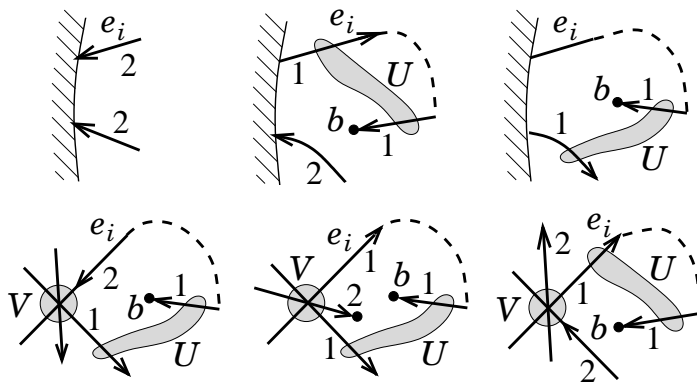


Fig. 4.

The left upper case is impossible due to our choice of Δ_{-p} . The two right upper cases are impossible because modification A applied to the disk U reduces the number of bad branch points. The others are impossible because modification A applied to U followed by modification B applied to V reduces $|R(\Gamma)|$. Theorem 1 is proven.

§3. Proof of Theorem 2. Let $\mathcal{A} = \{\sigma_0, \sigma_1, \sigma_2\}$ and let \mathcal{P} be the set of BKL-positive words. Recall that $U = V$ is the equality in \mathbf{B}_3 whereas $U \equiv V$ is the letterwise coincidence of words. We follow the convention that if the notation \equiv is used, then all factors on both sides of the equality belong to \mathcal{P} . We set $\tau : \mathbf{B}_3 \rightarrow \mathbf{B}_3$ and $\tau(X) = \delta^{-1}X\delta$. Since $\tau(\mathcal{A}) = \mathcal{A}$, we can define $\tau : \mathcal{P} \rightarrow \mathcal{P}$.

Lemma. (see [1; Theorem 2.7]). *If $U = V$ for $U, V \in \mathcal{P}$, then V is obtained from U by relations (2) (without inserting σ_i^{-1}).* \square

Lemma 4. *If $a_1 \dots a_n = b_1 \dots b_n$, $a_i, b_i \in \mathcal{A}$, then $(a_1, \dots, a_n) \sim (b_1, \dots, b_n)$.* \square

Lemma 5. *Let $\delta^p = W \equiv AuB$, $u \in \mathcal{A}$. Then there exist $v \in \mathcal{A}$ and $k \geq 0$ such that either $A \equiv A_1vA_2$, $A_2 = \delta^k$, $vA_2u = \delta^{k+1}$ or $B \equiv B_1vB_2$, $B_1 = \delta^k$, $uB_1v = \delta^{k+1}$.*

Proof. If $p = 1$, the statement is obvious. Assume that it is true for $p - 1$. By Lemma , either we have $W \equiv (\sigma_2\sigma_1)^p$, or at least one relation can be applied to W . In both cases we have $W \equiv CxyD$ where $xy = \delta$. If $A \equiv C$ or $A \equiv Cx$, then we set $v = y$ or $v = x$ respectively and we are done. Otherwise we have either $C \equiv AuC_1$, or $D \equiv D_1uB$. We consider only the former case (the latter one is similar). Let $E = \tau^{-1}(D)$. Then $AuC_1E = CE = C\delta D\delta^{-1} = W\delta^{-1} = \delta^{p-1}$. Hence, by the induction hypothesis, we have either 1) $A \equiv A_1vA_2$, $A_2 = \delta^k$, $vA_2u = \delta^{k+1}$, or 2) $C_1 \equiv B_1vC_2$, $B_1 = \delta^k$, $uB_1v = \delta^{k+1}$, or 3) $E \equiv E_1wE_2$, $C_1E_1 = \delta^m$, $uC_1E_1w = \delta^{m+1}$. In Cases 1) and 2), the statement of the lemma is evident. In Case 3), we set $B_1 = C_1xy\tau(E_1)$, $v = \tau(w)$, $B_2 = \tau(E_2)$, $k = m + 1$. \square

Let $\mathcal{I}_k(W) = \{I \mid W \setminus I = \delta^p, p \geq 0, |I| = k\}$.

Lemma 6. *Let $W = a_1 \dots a_n \in \mathcal{P}$, $I \in \mathcal{I}_k(W)$. Suppose that $a_i a_{i+1} = \delta$ for some i and that one of the numbers $i, i+1$ belongs to I and the other one does not. Then there exists $J \in \mathcal{I}_k(W)$ such that $W_I \sim W_J$ and $\{i, i+1\} \cap J = \emptyset$.*

Proof. Let $i \notin I$ and $i+1 \in I$ (in the case $i \in I$ and $i+1 \notin I$ the proof is similar). Let $W \equiv W_1x_1 \dots W_kx_kW_{k+1}$ where x_1, \dots, x_k are the letters whose positions belong to I . Then, for some m , we have $x_m = a_{i+1}$ and $W_m \equiv Au$ where $u = a_i$. Let $A_j = W_1 \dots W_j$ and $X_j = A_jx_jA_j^{-1}$, $j = 1, \dots, k$. Let $v = a_l$ be the letter in $W \setminus I$ matching u whose existence is proven in Lemma 5, and let W_s be the subword that contains it. Let $J = \{l\} \cup (I \setminus \{i+1\})$. We are going to prove that $W_J \sim W_I$. Let $W_J = (Y_1, \dots, Y_k)$. It is clear that $Y_j = X_j$ when $j < \min(m, s)$.

Case 1. $s \leq m$. Then $W_s \dots W_m \equiv BvDu$, $D = \delta^q$, $vDu = \delta^{q+1}$, and if, moreover, $s < m$, then $W_s \equiv BvC$ and $D \equiv CW_{s+1} \dots W_{m-1}A$. Since $ux_m = \delta$, it follows that $vDu = Dux_m$, hence $X_m = A_{s-1}Bv(Dux_m)(vDu)^{-1}B^{-1}A_{s-1}^{-1} = A_{s-1}BvB^{-1}A_{s-1}^{-1} = Y_s$. If $j > m$, then we have $Y_j = B_jx_jB_j^{-1}$ where $B_j = A_{s-1}B(Dux_m)W_{m+1} \dots W_j = A_{s-1}B(vDu)W_{m+1} \dots W_j = A_j$, whence $Y_j = X_j$. If $s \leq j < m$, then $Y_sY_{j+1}Y_s^{-1} = B_jx_jB_j^{-1}$ where $B_j = Y_sA_{s-1}BCW_{s+1} \dots W_j = (A_{s-1}BvB^{-1}A_{s-1}^{-1})A_{s-1}BCW_{s+1} \dots W_j = A_{s-1}BvCW_{s+1} \dots W_j = A_j$, whence $Y_sY_{j+1}Y_s^{-1} = X_j$. Thus $W_J = (X_1, \dots, X_{s-1}, X_m, X_m^{-1}X_sX_m, \dots, X_m^{-1}X_{m-1}X_m, X_{m+1}, \dots, X_k) = \Sigma_s^{-1} \dots \Sigma_{m-1}^{-1}(W_I)$.

Case 2. $s > m$. Then $W_s \equiv BvC$, $ux_m = \delta$, $D = \delta^q$, $uDv = \delta^{q+1}$ where $D \equiv W_{m+1} \dots W_{s-1}B$. Hence $uDv = ux_mD$ and, by canceling u , we obtain $Dv = x_mD$. Therefore, $Y_{s-1} = A_mx_m(Dv)(x_mD)^{-1}A_m^{-1} = A_mx_mA_m^{-1} = X_m$. If $j \geq s$, then $Y_j = B_jx_jB_j^{-1}$ where $B_j = A_m(x_mD)CW_{s+1} \dots W_j = A_m(Dv)CW_{s+1} \dots W_j = A_j$, whence $Y_j = X_j$. If $m < j < s$, then $Y_{j-1} = B_jx_jB_j^{-1}$ where $B_j = A_mx_mW_{m+1} \dots W_j = (A_mx_mA_m^{-1})A_j = X_mA_j$, whence $Y_{j-1} = X_mX_jX_m^{-1}$. Thus $W_J = (X_1, \dots, X_{m-1}, X_mX_{m+1}X_m^{-1}, \dots, X_mX_{s-1}X_m^{-1}, X_m, X_s, \dots, X_k) = \Sigma_{s-1} \dots \Sigma_m(W_I)$. \square

Lemma 7. *Let $xy = \delta$, $x, y \in \mathcal{A}$, and either (i) $W \equiv AxyB$ and $V \equiv AuvB$ where $uv = \delta$, or (ii) $W \equiv AB$ and $V \equiv Axy\tau(B)$, or (iii) $W \setminus I \neq 1$, $W \equiv AxyB$, and $V \equiv A\tau^{-1}(B)$. Suppose that $I \in \mathcal{I}_k(W)$. Then there exists $J \in \mathcal{I}_k(V)$ such that $V_J \sim W_I$.*

Proof. Let $i - 1$ be the letter length of A and let $m = |\{i, i + 1\} \cap I|$.

(i). If $m = 0$, then $V_I = W_I$. If $m = 2$, then the result follows from Lemma 4. The case when $m = 1$ reduces to the case when $m = 0$ by Lemma 6.

(ii). $V_J = W_I$ for $J = (I \cap [1, i - 1]) \cup (2 + (I \cap [i, k]))$.

(iii). The case when $m = 0$ follows from (ii). The case when $m = 1$ reduces to the case when $m = 0$ by Lemma 6. Let us consider the case when $m = 2$, i. e., $\{i, i + 1\} \subset I$. Let $W \equiv a_1 \dots a_n$. The condition $W \setminus I \neq 1$ means that either $\{1, \dots, i - 1\} \not\subset I$, or $\{i + 2, \dots, n\} \not\subset I$. We consider only the former case (the latter is similar). Let $l = \max(\{1, \dots, i - 1\} \setminus I)$. Without loss of generality we may assume that $a_l = \sigma_2$. Set $C \equiv a_1 \dots a_{l-1}$, $D \equiv a_{l+1} \dots, a_{i-1}$, $E \equiv \tau(D)$. Since $Dxy = D\delta = \delta E = \sigma_1 \sigma_0 E$, we have $W_I \sim U_I$ by Lemma 4 where $U \equiv C\sigma_2\sigma_1\sigma_0EB$. By Lemma 6, we have $U_I \sim U_K$ where $\{l, l + 1\} \cap K = \emptyset$. Since $C\sigma_2\tau^{-1}(EB) \equiv C\sigma_2D\tau^{-1}(B) \equiv V$, we have reduced the case when $m = 2$ to the case when $m = 1$ with $(A, xy, B; I)$ being replaced by $(C\sigma_2, \sigma_1\sigma_0, EB; K)$. \square

Now Theorem 2 follows from Lemma 7. Indeed, any quasipositive factorization $(A_j x_j A_j^{-1})_{j=1}^k$ of a given braid can be represented in the form W_I where $W = W_1 x_1 \dots W_k x_k W_{k+1}$, $W_j \in \mathcal{P}$, $A_{j-1}^{-1} A_j = W_j \delta^{-3p_j}$ ($A_0 = A_{k+1} = 1$) and the replacements (i)–(iii) of Lemma 7 allow us to transform W to any given word.

§4. Proof of Theorem 3. Let $]1, \Delta[= \{\sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1\}$, $\tau(X) = \Delta^{-1}X\Delta$. For $u \in]1, \Delta[$, we denote the first and the last letter of u by $S(u)$ and $F(u)$ respectively. Any 3-braid can be written in the form $u_1 \dots u_n \Delta^{-p}$ with $u_i \in]1, \Delta[$ and $F(u_i) = S(u_{i+1})$ (the *right normal form*). Any conjugacy class except $\sigma_1 \Delta^{2m+1}$ and $\sigma_1 \sigma_2 \Delta^{2m}$ contains an element of this form such that, moreover, $\tau^p(F(u_n)) = S(u_1)$. Let X be such an element. We identify u_1, \dots, u_n (in this order) with the vertices of a regular polygon P . We define an *antisymmetry* of P as a reflection s such that its axis passes through the midpoints of two sides ab and cd , so that $e(u_i) = 2$ for $u_i \in \{a, b, c, d\}$, and $e(u_i) \neq e(s(u_i))$ otherwise. It is not difficult to derive the following fact from Theorem 1.

Lemma 8. *If X is as above and $e(X) = 2$, then the number of orbits of the Hurwitz action on quasipositive factorizations of X is equal to the number of antisymmetries of P .* \square

Thus, Theorem 3 follows from:

Lemma 9. *P has at most two antisymmetries.*

Proof. Suppose that P has three different antisymmetries s_1, s_2, s_3 . We number them so that the angle α between the axes of s_1 and s_2 is minimal, in particular, $\alpha \leq \pi/3$.

Case 1. $\alpha = 2\pi/n$. Let $u_{-1}u_0$ and u_0u_1 be the invariant sides for s_1 and s_2 . We assume here that the indices are defined mod n . Then the antisymmetry of s_1 and s_2 implies $e(u_0) = e(u_{\pm 1}) = 2$, $e(u_{\pm 2}) = 3 - e(u_{\mp 1}) = 1$, $e(u_{\pm 3}) = 3 - e(u_{\mp 2}) = 2$, etc. till the vertices antipodal to $u_{\pm 1}$. Since the values of e alternate, it follows that there is no room for the axis of s_3 .

Case 2. $\alpha > 2\pi/n$. Let $r = s_2 \circ s_1$ (a rotation by 2α) and let ab, cd be the sides invariant by s_3 . Let $a^\pm = r^{\pm 1}(a)$, $b^\pm = r^{\pm 1}(b)$. The condition $2\pi/n < \alpha \leq \pi/3$ implies that $\{a, b, c, d\} \cap \{a^\pm, b^\pm\} = \emptyset$. Since $s_3(a^+) = b^-$, it follows that $e(a^+) = 1$ or $e(b^-) = 1$. We assume that $e(b^-) = 1$ (the case when $e(a^+) = 1$ is similar).

Note that if $e(u_i) = 1$, then always $e(s_j(u_i)) = 2$. Hence $e(s_1(b^-)) = 2$. Since $s_2(b) = s_2(r(b^-)) = s_1(b^-)$ and $e(b) = 2$, it follows that $e(b) = e(s_2(b)) = 2$. Therefore, b and $s_2(b)$ are consecutive vertices and the axis of s_2 passes between them. Hence the angle between the axes s_2 and s_3 is equal to $2\pi/n$. This contradicts the minimality of α .

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