Projective M-cubics and M-quartics in general position with a maximally intersecting pair of ovals

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An isotopy classification of curves mentioned in the title is obtained. A new aproach to construction of curves is suggested. The theory of complex orientations and the link theory (Murasugi-Tristram inequality) is used to prove the prohibitions.

Introduction

Increasing of the number of known examples of plane algebraic curves with a known topology usually induces a general progress in topics related to the first part of the 16-th Hilbert's problem. In particular, obtained in [1], [2], the classification of plane real curves of degree 6 decomposing into a product of two transversal factors has found various applications, main of them being listed in [3] (now this list can be updated by construction of curves of degree 6 on a cubic surface in a recent paper [4]). The nature of the mentioned applications allows to hope that an analogous classification of decomposing curves of degree 7 will be useful as well. However, one has "$7 \gg 6$" in this problem: even not speaking that much more topological types is to be studied, the technique of [1], [2] certainly is not enough. The latter is clear after a rather detailed "preleminary" study of decomposing curves made by the second author; part of the results is published in [3], [5]. Let us remark that even more powerful methods were not sufficient for the classification of arrangements of an M-curve of degree 6 and a line which meets it transversally [6], [7]. Recently, some progress in this problem was acheived by the first author involving methods of the link theory (see [8], [9]; in particular, a classification of flexible affine M-curves of degree 6 was obtained). In the present paper, we obtain an isotopic classification of curves mentioned in the title, using the methods of link theory. The methods used here are applyable also to other classes of curves — for instance, to conics and quintics in general position — and one may hope that they will allow to complete the classification of decomposing curves of degree 7 (under natural assumptions of maximality and transversality).

1. Statement of the problem and the result

From now on, C_n denotes the set of points of an M-curve of degree n in the real projective plane $\mathbb{R}P^2$. Recall that C_3 (an M-cubic) in $\mathbb{R}P^2$ consists of two disjoint topological circles one of which divides $\mathbb{R}P^2$ and the other does not. The former is called the oval and we denote it by O_3. The latter is called the odd branch and we

Supported by grant RFBI 98-01-00794

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denote it by J_3. An M-quartic C_4 in $\mathbb{R}P^2$ consists of four ovals, each being outside another. We assume that one of these ovals — denote it by O_4 — meets O_3 at 12 distinct real points. The other three ovals of the quartic and the branch J_3 will be called *free* (of intersection points). The main result of this paper is a topological classification of triples

$$(\mathbb{R}P^2, C_3 \bigsqcup C_4, C_3), \tag{1.1}$$

where $\bigsqcup$ denotes the union under the above assumptions (transversality of the intersections, maximality of the number of common real points, freedom of the three ovals of the quartic and of J_3) can be formulated as follows

Theorem 1. *Any triple* $(\mathbb{R}P^2, C_3 \bigsqcup C_4, C_3)$ *is homeomorphic to one of the* 31 *topological models presented in Fig. 1 where standard circles are models of* O_3*, odd branches* J_3 *are not depicted, and the components of the complement of* $O_3 \cup O_4$ *with the symbols* α *and* β *contains respectively* α *and* $\beta = 3 - \alpha$ *free ovals of the quartic for the values of* α *listed under the pictures.*

As usually in such problems, the general scheme of the proof is naturally divided into three steps: (i) enumerating of admissible topological models; (ii) constructions; (iii) prohibitions. Here "admissible models" mean arrangements of sets A and B consisting of disjoint circles in $\mathbb{R}P^2$ such that A, B, and $A \cup B$ satisfy the same combinatorial conditions as C_3, C_4, and $C_3 \cup C_4$ in (1.1) including the restrictions on the mutual arrangement of ovals provided by Bezout theorem for the intersections with lines.

The step (i) is described in details in [5]. Namely, an enumerating algorithm for admissible models for $O_3 \bigsqcup O_4$ is presented there. It gives a list of 20 arrangements shown in Fig. 1; Then, using Brusotti theorem (see, e.g. [10]) on independent smoothing of non-degenerated double points, one can prove that each of the arrangements admits a unique (up to symmetry) region α inside the oval O_3 (see Fig. 1) where α free ovals of C_4 may occure, $\alpha \in \{0, 1, 2, 3\}$. Thus, *the total number of admissible models for triples (1.1) is* $20 \times 4 = 80$.[1]

The sections 2 and 3 are devoted to the steps (ii) and (iii) respectively.

2. Constructions

Let us describe the encoding of admissible models used below. A word (a sequence of symbols) $w = {<}s_1 s_2 ... s_n{>}$, $s_k \in \{+, -\}$, will codify the mutual arrangement of the ovals O_3 and O_4 composed of the blocks $B(s_1), \dots, B(s_n)$ according to Fig. 2.1; the blocks $B(+)$ and $B(-)$ are shown in Fig. 2.2. We shall use a convention that the symbols "<" and ">" are also included into the word w; the corresponding blocks $B(<)$ and $B(>)$ being the most left and the most right blocks in Fig. 2.1.

Remark. It is clear that 32 different words w are possible. Identifying the words which coincide after reversing the order of reading — obviously, such words codify isotopic arrangements — we obtain 20 words written under the corresponding models in Fig. 1.

To encode the arrangements $C_3 \bigsqcup C_4$, we shall equip the word w by the parameter α in the braces after w.

[1] It is written erroneously "16" instead of "20" and "64" instead of "80" in Sect. VIIIa) [3].

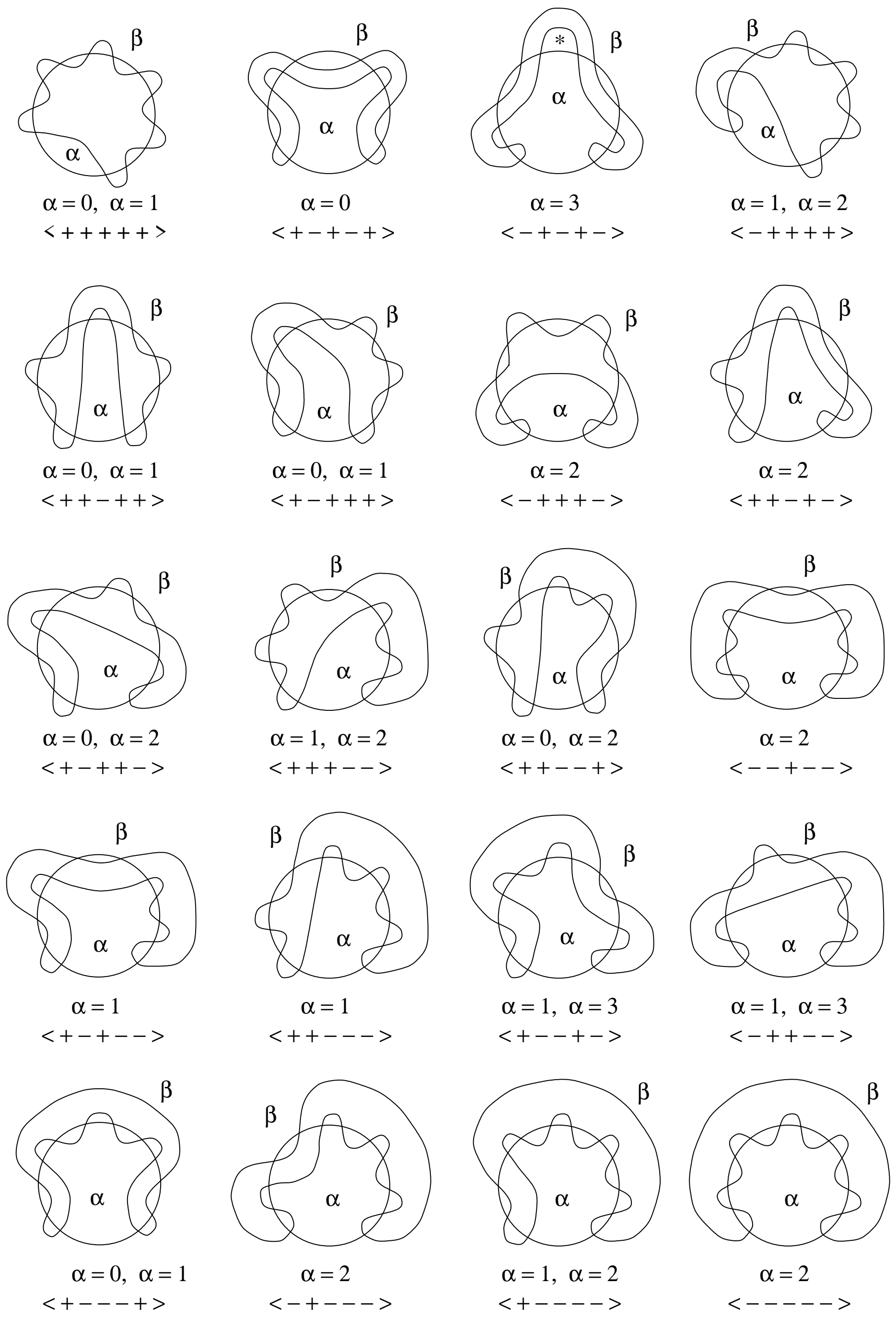

FIG. 1

2.1. Elementary constructions using the method of a small parame-

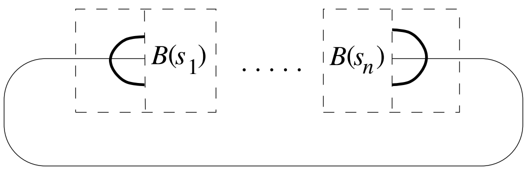

FIG. 2.1

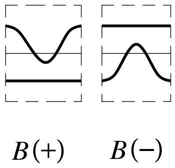

FIG. 2.2

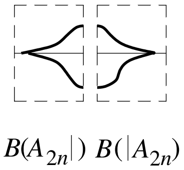

FIG. 2.3

ter. The model <+++++>$\{0\}$ is realized by the curve $C_3 \cup C_4$ where C_4 is an M-quartic which has 6 flex points on the oval O_4 (such a quartic was constructed in [11]) and C_3 is an M-cubic obtained by a small perturbation of the union of three sections of this oval, sufficiently close to the three its bitangents, such that each section meets the oval at four points.

To realize the model <++-++>$\{0\}$, let us consider an M-quartic C_4 obtained by a perturbation of the union of two ellipses. Let $L = 0$ be the bitangent to O_4, and $E = 0$ an ellips touching O_4 at three points as it is depicted in Fig. 3.1. Suppose that $L < 0$ on O_4 and $E < 0$ inside the ellips. Then $C_4 \cup \{EL = \varepsilon\}$ for $0 < \varepsilon \ll 1$ yields the required arrangement (see Fig. 3.2).

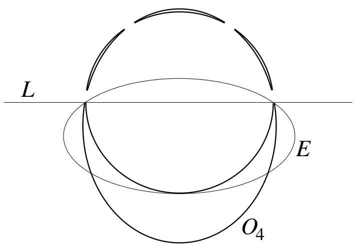

FIG. 3.1

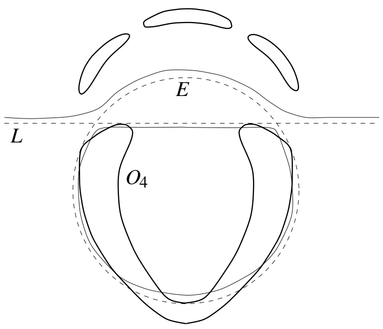

FIG. 3.2

Remarks. 1. Starting with the union of a conic and a quartic appearing in the method of Harnack (see, e.g., [10]), analogous elementary constructions allow to realize also the models <+-+-+>$\{0\}$ and <-+-+->$\{3\}$. We omit the details because these two models will be realized below by the same method as the others. 2. The realization of the model <++-++>$\{0\}$ and all the constructions of the next subsection were obtained by the first author.

2.2. Constructions using smoothing of cusps.

Constructions of singular curves. Let one curve be non-singular at a point p and another one have a singularity A_{2n} at this point. Let us say that these curves have the *maximal tangency* at the point p if they can be defined respectively by the equations $y = 0$ and $y^2 = x^{2n+1}$ at some local analytic coordinates. Similarly to the encoding of the mutual arrangements of ovals, we shall denote by a word $w = A_{2n}|s_1 \dots s_k|A_{2m}$ the mutual arrangement of an oval and an even singular branch which has the singularities A_{2n}, A_{2m} obtained from Fig. 2a by replacing the blocks $B(<)$ and $B(>)$ with the block $B(A_{2n}|)$ and $B(|A_{2m})$ depicted in Fig. 2.3. This notation assumes the maximal tangency at the singular points (this follows from the picture). Put also $B(A_0|) = B(<)$, $B(|A_0) = B(>)$. As it was in the non-singular case, $w\{\alpha\}$ will denote the mutual arrngement of an M-cubic and a quartic where α free ovals of the quartic are inside O_3 and $\beta = 3 - \alpha - n - m$ are outside. Since only $\alpha = \beta = 0$ is possible when $m + n = 3$, we shall omit "$\{0\}$" in this case.

Lemma 2.2.1. *There exist the following arrangements of a non-singular cubic and singular quartic:*

$$A_6|{+}{-}{>};\ A_6|{-}{+}{>};\ A_4|{+}{-}|A_2;\ A_4|{-}{+}|A_2;\ A_2|{-}{-}{-}|A_2\{0\};\ A_4|{+}{+}{+}{>}\{0\}. \tag{2.1}$$

Proof. Let $(x : y : z)$ be homogeneous coordinated in $\mathbb{R}P^2$ and C_3 be an M-cubic whose oval O_3 is tangent to the axis $y = 0$ at the point $(0 : 0 : 1)$ having the curvature k at this point (i.e. O_3 is defined near this point by an equation of the form $y = kx^2 + \dots$) and whose odd branch J_3 meets this axis at $(0 : -1 : 1)$. Let a conic C_2 be tangent to the axis $y = 0$ at this point and has five more distinct common points with J_3. There are two possibilities for such an arrangement of C_3 and C_2 shown in Fig. 4.1, 4.2. Easy to check that the image $f_k(C_3)$ of C_3 under the quadratic transformation $f_k(x : y : z) = (xy : y^2 : yz - kx^2)$ is a cubic with the oval $f_k(J_3)$ and the odd branch $f_k(O_3)$ and $f_k(C_3 \cup C_2)$, and $f_k(C_3 \cup C_2)$ realizes the arrangement $A_6|{-}{+}{>}$ in the case of Fig. 4.1 and $A_6|{+}{-}{>}$ in the case of Fig. 4.2.

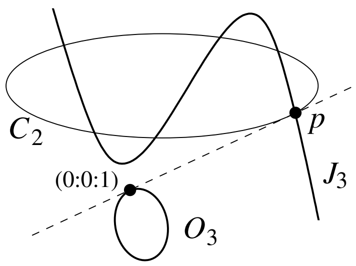

FIG. 4.1

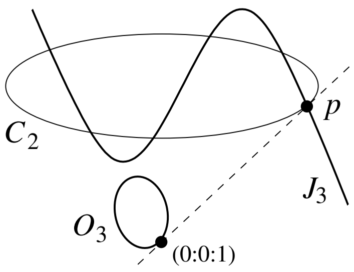

FIG. 4.2

Now let C_3 be an M-cubic arranged with respect to the coordinate axes as it is shown in Fig. 5.1 (the axis $y = 0$ is a flex tangent to the odd branch at the point $(0 : 0 : 1)$). Let us define a new cubic C_3' by the equation $(yz - kx^2)(x - \varepsilon z) + \delta y^3 = 0$. For $0 < \delta \ll \varepsilon \ll k \ll 1$, we obtain the arrangement of C_3 and C_3' depicted in Fig. 5.2. The arrangement $A_4|{+}{+}{+}{>}\{0\}$ is obtained as $f_k(C_3) \cup f_k(C_3')$.

Two more arrangements of (2.1) can be constructed applying the hyperbolism — the quadratic transformation $\mathrm{hy}(x : y : z) = (x^2 : xy : yz)$. Let the oval of a cubic C_3 be tangent to the axis $y = 0$ at $(0 : 0 : 1)$ and pass through $(0 : 1 : 0)$. Then $A_4|{+}{-}|A_2$ and $A_4|{-}{+}|A_2$ are realized as $\mathrm{hy}(C_3) \cup \mathrm{hy}(C_2)$ where C_2 is a conic which meets the odd branch of C_6 at six distinct real points, C_2 being tangent to

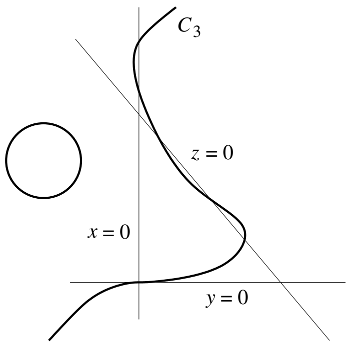

FIG. 5.1

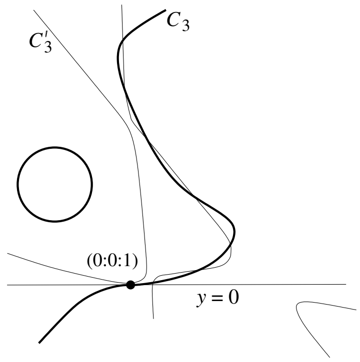

FIG. 5.2

the axes $x = 0$ and $y = 0$ at two of them. As above, C_2 can be chosen in two different ways.

Finally, let us construct the curve $A_2| - - - |A_2$. Let us consider ellipses E, C' meeting each other at points p, p' and touching each other at a point q. Let us trace the line L, tangent to C' at p. Let r be the other intersection point of L and E and let L' be the line tangent to E passing through r (see Fig. 6.1). Put $C = C' + \varepsilon L^2$, $H = {L'}^2 + \delta L L''$; here we use the same notation for a curve and a polynomial which defines it, we assume also that LL' is positive and C' is negative at the point marked by the asterisk in Fig. 6.2. Cutting $\mathbb{R}P^2$ along L, we obtain the disk depicted in Fig. 6.3.

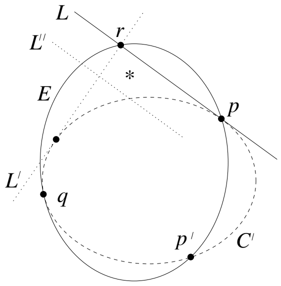

FIG. 6.1

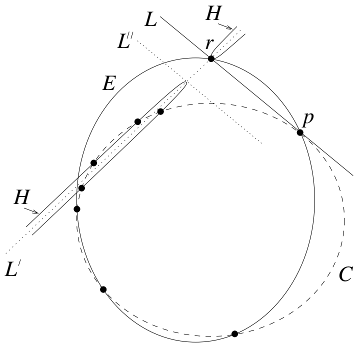

FIG. 6.2

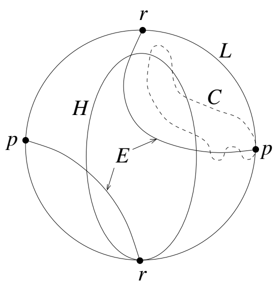

FIG. 6.3

Let us blow up $\mathbb{R}P^2$ at the point r (denote the pasted line by R) and then we blow up the common point of R and L (denote the second pasted line by F). The obtained surface is glued from the octagon in Fig. 6.4 according to the arrows on the boundary (F_1 and F_2 are the pieces into which F is divided by R and L). The self-intersection number of L in the complexification of this surface is -1. Blowing down L, we obtain a (minimal rational) ruled surface Σ_2 (see Fig. 6.5) where F is a fiber and R, H are sections of the fibration $\Sigma_2 \to \mathbb{P}^2$; the self-intersection numbers are $R^2 = -2$, $H^2 = 2$. Consider the double covering $\xi : \Sigma_1 \to \Sigma_2$ branched along

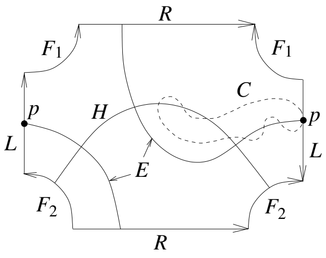

FIG. 6.4

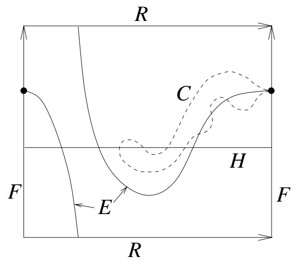

FIG. 6.5

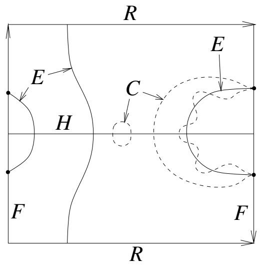

FIG. 6.6

$H \cup R$. Then Σ_1 is the ruled surface with the self-intersections of the sections $R^2 = -1$, $H^2 = 1$. The complex conjugation can be lifted from Σ_2 to Σ_1 in two different ways. We chose the lifting whose fixed points set is $\mathbb{R}\Sigma_1 := \xi^{-1}(A)$ where A is the upper (according to Fig. 6.5) component of $\mathbb{R}\Sigma_2 \setminus (R \cup H)$. The surface $\mathbb{R}\Sigma_1$ is depicted in Fig. 6.6 (we use the same notation for curves on $\mathbb{R}\Sigma_2$ and their transforms on $\mathbb{R}\Sigma_1$). Blowing down R, we obtain $\mathbb{R}P^2$ with the required arrangement of the quartic C and the cubic E.

The last two operations (the double covering and the blow down) in terms of equations, mean the following. One can choose affine coordinates (x, y) on $U := \mathbb{R}\Sigma_2 \setminus (R \cup F) \cong \mathbb{R}^2$ such that H has form $y = 0$ and the fibers of the fibration $\Sigma_2 \to \mathbb{R}P^2$ have form $x = \text{const}$. Let us choose the sign of the coordinate y so that $y > 0$ above H in Fig. 6.5. Then the equations of the required curves are obtained from the equations of the curves on U (Fig. 6.5) by the substitution $y = z^2$ followed by the projectivization. For example, if $f(x, y) = 0$ is the equation of C on U then the Newton polygon of f is the triangle with vertices (0,0), (4,0), (2,0) and $f(x, z^2)$ is the equation of the quartic involved in the arrangement $A_2| - - - |A_2$. $\square$

Construction of perturbations. Now let us construct perturbations of singularities A_{2n} taking into account there position with respect to a maximally tangent smooth branch. Let (x, y) be local coordinates such that this branch has form $y = 0$. According to our encoding system, let us denote by $A_{2n}|$ (resp. by $A_{2n}|$) the singularity $y^2 = x^{n+1}$ (resp. $y^2 = -x^{n+1}$), and let $<s_1 \dots s_n\{\alpha\}$ denote the sequence of blocks $B(<)$, $B(s_1), \dots, B(s_n)$ attached consequently one to the other along the axis $y = 0$, outside of them being α ovals in the lower half-plane and $\beta = n - \alpha$ ovals in the upper half-plane. We are going to describe only pertubations of $A_{2n}|$ because those of $|A_{2n}$ are obtained from them by the reflection in the vertical axis.

Proposition. *For any sequence of signs $s_1, \dots, s_n$, there exists a perturbation $A_{2n}| \to <s_1 \dots s_n\{\alpha\}$ where α is the number of pluses in the sequence $-s_1, s_2, -s_3, \dots, (-1)^n s_n$.*

Proof. The required perturbations can be constructen by the Viro T-curves method (see [12]). Let Δ be the triangle whose vertices are $(0,0)$, $(0,2)$, $(2n+1, 0)$. Divide it into triangles by the segments $[(0,2), (2k+1, 0)]$, $k = 1, \dots, n$ and continue this subdivision up to a primitive triangulation arbitrarily. Define the distribution of signs $s : \Delta \cap \mathbf{Z}^2 \to \{\pm 1\}$ by the formulas $s(k, 0) = (-1)^k$, $s(k, 1) = s_k$, $s(0, 2) = 1$. $\square$

Remark. It is proven in the paper of the first author [9] that one can obtain in this way all the M-arrangements of the singularity A_{2n} with respect to the horizontal axis up to "semi-rigid" isotopies, i.e. isotopies such that the curve has at most 2 intersections with any vertical line at any moment (less formally speaking, the isotopy classification taking into account the order of interchanging of the ovals). Similar results are obtained for M-curves on ruled surfaces which have intersection three with the fibers distinguishing their different arrangements with respect to the exceptional section.

Corollary 2.2.2. *The singularities A_2, A_4, A_6 admit the following perturbations:*

$A_2\vert \to$	<+{0}	$A_4\vert \to$	<++{1}	$A_6\vert \to$	<+++{1}	<-++{2}
	<-{1}		<+-{0}		<++-{2}	<-+-{3}
			<-+{2}		<+-+{0}	<--+{1}
			<--{1}		<+--{1}	<---{2}

Construction of arrangements $C_3 \bigsqcup C_4$. All the realizable models from Theorem 1 besides <+++++>{0} and <++-++>{0} (which were constructed in 2.1) can be obtained by perturbations of the singular quartics in the arrangements (2.1) using all the possibilities listed in Corollary 2.2.2. Concerning the proof of the fact that any perturbation of the list in Corollary 2.2.2 can be applied to any singularity in (2.1), see the remark in the end of the paper [13].

Using our encoding system, these constructions can be described just as formal replacing of expressions $A_n|$ and $|A_n$ in the codes from (2.1) according to the table in Corollary 2.2.2 followed by the addition of numbers appearing in the braces. For example, the singular arrangement A_2|---|A_2{0} yields <+---+>{0}, <+---->{1}, <----+>{1}, <----->{2} where the two middle arrangements are equivalent because they are symmetric to each other. All the results of the constructions (up to such equivalencies) are collected in Table 7 — see a reformulation of Theorem 1 in the end of the paper.

3. Prohibitions

3.1. Application of the theory of complex orientations.

Let us recall necessary notions and facts (see details in [14]). If the set A of real points of a non-singular real curve of degree n separates the set of complex points of the curve then A separates it into two symmetric halfs. These halfs induce two opposite orientations on A as on the common boundary. They are called the *complex orientations.* A pair of ovals one of which surrounds the other is called an *injective pair.* An injective pair of oriented ovals is called *positive* if their orientations are induced by some orientation of the annulus in $\mathbb{R}P^2$ bounded by them, and *negative* otherwise. If n is odd then an oval O is called *positive* if $[O] = -2[J] \in H_1(\mathbb{R}P^2 \setminus \operatorname{Int} O)$ where J is the odd component of the curve, and *negative* otherwise. For a curve of degree $n = 2k$ there is the following Rokhlin's formula

$$2(\Pi^+ - \Pi^-) = l - k^2, \tag{3.1}$$

If $n = 2k+1$ then the following Rokhlin-Mishachev formula takes place:

$$\Lambda^+ - \Lambda^- + 2(\Pi^+ - \Pi^-) = l - k(k+1). \tag{3.2}$$

Here $\Pi^+(\Pi^-)$ and $\Lambda^+(\Lambda^-)$ denote the number of positive (negative) injective pairs and ovals of the curve, and l is the total number of ovals of the curve.

The theory of complex orientations allows one to prohibit certain values of (α, β) for each of the admissible models in Fig. 1. Let, for instance, the model <+++++>$\{\alpha\}$ be realized by a curve $C_3 \cup C_4$. The cubic C_3 and the quartic C_4 have complex orientations because they are M-curves, the oval O_3 being negative by (3.2) for $k = l = 1$. Let us fix a complex orientation of O_4 — for instance, as it is shown in Fig. 7. Suppose that among the α and β free ovals of the quartic, α^+ and β^+ are positive with respect to the odd branch J_3, and the other α^- and β^- are negative. Let us construct a non-singular curve of degree 7 as the result of a small perturbation of $C_3 \cup C_4$ removing all the 12 double points according to the chosen orientations (see the dashed lines in Fig. 7). This can be done in virtue of Brusotti theorem. By Fiedler's theorem (see [15] or [14]), the induced orientation of the obtained curve is its complex orientation, hence, it must satisfy (3.2) for $k = 3$. We have $l = 15$ in our example (12 ovals appeared after removing the double points, and three coming from the free ovals of the quartic), $\Lambda^+ = \alpha^+ + \beta^+ + 6$, $\Lambda^- = \alpha^- + \beta^- + 6$, $\Pi^+ = \alpha^+$, $\Pi^- = \alpha^-$, and the formula (3.2) together with the condition that the total number of free ovals is equal to three, yields us the following simultaneous equations

$$\begin{cases} 3(\alpha^+ - \alpha^-) + (\beta^+ - \beta^-) = 3, \\ \alpha^+ + \alpha^- + \beta^+ + \beta^- = 3. \end{cases} \tag{3.3}$$

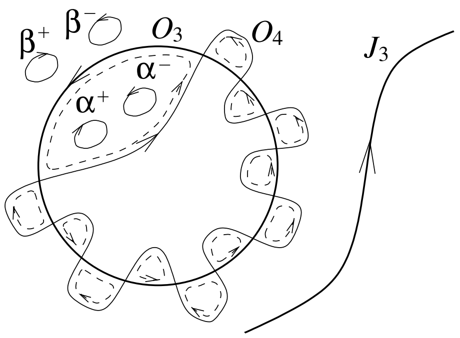

FIG. 7

It remains to find integral non-negative solutions of (3.3).

Now let us note that the form of the left hand side of the first equation in (3.3) is the same for all the models from the list of Fig. 1. The right hand side of this equation depends in general on the mutual arrangement of the ovals O_3 and O_4 and on the choice of the orientation of O_4, however, it is not difficult to check that if one reverse the orientation then only the sign of the right hand side is changed but

Table 1.

no.	α	α^+	α^-	β^+	β^-	l.h.s.	no.	α	α^+	α^-	β^+	β^-	l.h.s.
1	3	3	0	0	0	9	11	1	1	0	2	0	5
2	3	2	1	0	0	3	12	1	1	0	1	1	3
3	3	1	2	0	0	-3	13	1	1	0	0	2	1
4	3	0	3	0	0	-9	14	1	0	1	2	0	-1
5	2	2	0	1	0	7	15	1	0	1	1	1	-3
6	2	2	0	0	1	5	16	1	0	1	0	2	-5
7	2	1	1	1	0	1	17	0	0	0	3	0	3
8	2	1	1	0	1	-1	18	0	0	0	2	1	1
9	2	0	2	1	0	-5	19	0	0	0	1	2	-1
10	2	0	2	0	1	-7	20	0	0	0	0	3	-3

the absolute value is not.[2] Table 1 contains all the logically possible combinations of no-negative integers α^+, α^-, β^+, β^- satisfying the second equation of (3.3), and the corresponding values of the left hand side ("l.h.s.") of the first equation. Comparing them with possible values of the right hand side ("r.h.s.") of the first equation for each model in Fig. 1 (see Table 2), we obtain the prohibitions listed in the last column of this table. In particular, the model <+++++>$\{2\}$ can not be realized by a curve of degree 7.

Table 2.

Codes of arrangements $O_3 \bigsqcup O_4$	r.h.s.	Prohibition
<+++++> <-+-+-> <--+-+> <++---> <+--+-> <-++--> <+---+>	± 3	$\alpha \neq 2$
<--+--> <-+---> <+---->	± 5	$\alpha \neq 0;3$
<----->	± 7	$\alpha \neq 0;1;3$
The rest of the list in Fig. 1	± 1	$\alpha \neq 3$

3.2. Application of the link theory.

3.2.1. Arrangement of a curve with respect to a pencil of lines. Let $C \subset \mathbb{R}P^2$ be a model of a real curve of degree m all whose singulariies are simple double points. Let us fix a point $p \in \mathbb{R}P^2 \setminus C$ and denote by $\mathcal{L}_p$ the pencil of lines through this point. Let us choose the affine coordinates (x, y) so that (i) $\mathcal{L}_p = \{l_t\}$ where l_t is the line defined by $x = t$; (ii) C has m distinct common points with the infinite line; (iii) C is in *general position* with respect to $\mathcal{L}_p$, i.e., $\mathcal{L}_p$ contains a finite number of *critical lines* $l_{x_1}, \dots, l_{x_q}$, $x_1 < \dots < x_q$, and each of them has one double intersection point with C (i.e., each of these lines either is tangent to C or cuts C at its double point without tangency).

Like in [9], we shall encode the scheme of the arrangement of a curve C with respect to the pencil $\mathcal{L}_p$ by a word $u_1 \dots u_q$ where each symbol u_i characterizes the arrangement of C near the line l_{x_i}, and takes the value $\supset_k$, $\subset_k$, or $\times_k$ ($k \in \{1, \dots, m-1\}$) according to Fig. 8; a pair of successive characters $\subset_k \supset_k$ will be replaced with a single character $\circ_k$ ("free oval in the k-th strip").

[2] Therefore, the reversing the original orientation does not provide new prohibitions.

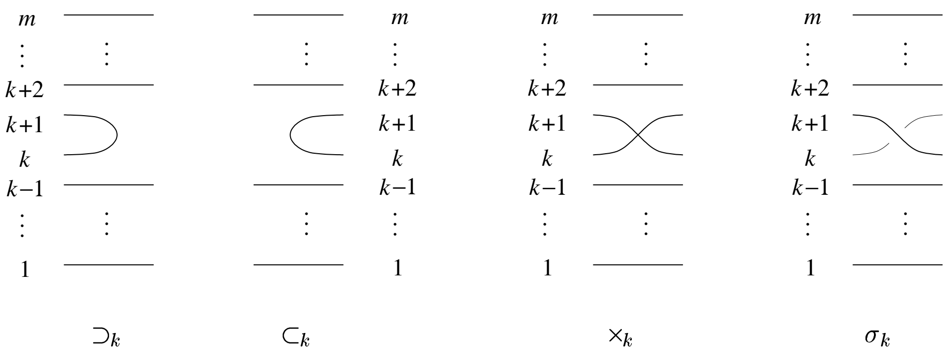

FIG. 8

Example. Suppose that the model `<+-+-+>`$\{2\}$ is realizable by a curve $C = C_3 \cup C_4$. Let O^*_α be one of the two free ovals of this curve lying in the region α, and p a point inside O^*_α. Then one can choose affine coordinates (x, y) satisfying the above conditions and such that C is arranged as in Fig. 9. By Bezout theorem the other two free ovals, O_α in the region α and O_β in the region β, should be contained in the vertical strip $D = \{x_7 < x < x_{12}\}$, where l_{x_7} and $l_{x_{12}}$ are critical lines tangent to the oval O_4. The set $J_3 \cup O_3 \cup O^*_\alpha$ divide D into six horizontal strips. We numerate them from 1 to 6 in the order of increasing the y-coordinate. By Bezout theorem, no line from $\mathcal{L}_p$ (vertical line in Fig. 9) can meet the both ovals O_α and O_β. Let the left of these two ovals lye in the i-th strip and the right of them in the j-th strip. Then the scheme of arrangement of the curve C with respect to $\mathcal{L}_p$ is encoded by the word

$$\times_3\times_2\times_2\times_3\times_3\times_2\supset_3\circ_i\circ_j\subset_4\times_5\times_4\times_4\times_5\times_5\times_4. \tag{3.4}$$

Since the strips numbered by 2 and 5 (resp. 3 and 4) belong to the same region α (resp. β), taking into account the evident symmetry, we see that it suffices to regard only the cases $(i, j) \in \{(2,3), (2,4), (5,3), (5,4)\}$.

Remark. To shorten the length of the codes in Table 3, we shall write the index k instead of $\times_k$. For instance, (3.4) will take the form $32^2 3^2 2 \supset_3 \circ_i \circ_j \subset_4 54^2 5^2 4$.

3.2.2. The braid of a real algebraic curve. Recall that *a braid on m strings* is the graph of an m-valued function $F : [0,1] \to \mathbb{C}$ taking pairwise distinct values at every point and considered up to a fiberwise isotopy being $F(0) = F(1) = \{y_1, \dots, y_m\}$, $\operatorname{Re} y_1 < \dots < \operatorname{Re} y_m$. The braid group B_m on m strings has the standard presentation

$$\langle \sigma_1, \dots, \sigma_{m-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1,\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ for } |i-j| = 1\rangle,$$

where the generator σ_k is shown in Fig. 8 and the orientation is chosen so that the two-valued function $F(t) = e^{\pm t\pi i}, t \in [0,1]$, represents the braid $\sigma_1 \in B_2$.

The closure of a braid $b \in B_m$ is the link in $\mathbb{R}^3$, obtained by connecting the ends of b with its beginnings using m arcs whose projections onto the plane are disjoint.

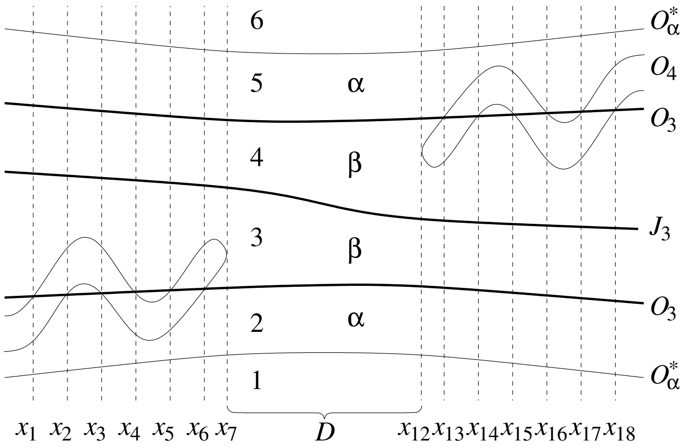

FIG. 9

Following [9], we associate a braid $b_{(C,p)} \in B_m$ to a curve C of degree m and a point $p \in \mathbb{R}P^2 \setminus C$ as follows. Let $y = F(x)$ be an m-valued function whose graph is the complexification of the curve C and let $x = \gamma(t)$, $t \in [0,1]$ be a simple closed curve in the upper half-plane of the complexification of the axis Ox surrounding all the values x with $\operatorname{Im} x > 0$ such that l_x is a critical line of the pencil $\mathcal{L}_p$. Then $b_{(C,p)}$ is the braid corresponding to the graph of $y = F(\gamma(t))$. This braid, in general, depends on the start point of the path γ but its conjugacy class and, hence, the closure do not. Put $\pi_{k,l} = \prod_{i=k}^{l} \sigma_i$ (the index i run from $i = k$ to $i = l$), $\tau_{k,l} = \pi^{-1}_{l,k-1}\pi_{k,l+1}$ and $\tau_{l,k} = \tau^{-1}_{k,l}$ for $k > l$, $\tau_{k,k} = 1$, $\Delta_m = \pi_{1,m-1}\pi_{1,m-2}\dots\pi_{1,1}$. Then $b_{(C,p)} = b_{\mathbb{R}}\Delta_m$ where $b_{\mathbb{R}} \in B_m$ is obtained from the encoding word $u_1 \dots u_q$ (see 3.2.1) by applying the following subword replacing rules:

(i) $\supset_k \subset_l \longrightarrow \sigma_k^{-1}\tau_{k,l}$;

(ii) $\supset_k \times_{i_1} \times_{i_2} \cdots \times_{i_p} \subset_l \longrightarrow \sigma_k^{-1}\delta_1 \dots \delta_p \tau_{k,l}$ where

$$\delta_j = \begin{cases} \sigma_{i_j}^{-1} & \text{for } i_j < k-1 \\ \sigma_{i_j+2}^{-1} & \text{for } i_j > k-1 \\ \tau_{k,k+1}\sigma_{k-1}^{-1}\tau_{k+1,k} & \text{for } i_j = k-1 \end{cases}$$

(iii) Each character $\times_i$ not replaced by the rule (ii) is replaced with σ_i^{-1}.

Example. For the word (3.4) with $(i,j) = (2,3)$ we obtain

$$b_{\mathbb{R}} = \underbrace{\sigma_3^{-1}\sigma_2^{-2}\sigma_3^{-2}\sigma_2^{-1}}_{\times_3\times_2\times_2\times_3\times_3\times_2} \cdot \underbrace{\sigma_3^{-1}(\sigma_2^{-1}\sigma_3)}_{\supset_3\subset_2} \cdot \underbrace{\sigma_2^{-1}(\sigma_3^{-1}\sigma_2)}_{\supset_2\subset_3} \cdot \underbrace{\sigma_3^{-1}(\sigma_4^{-1}\sigma_3)}_{\supset_3\subset_4} \cdot \underbrace{\sigma_5^{-1}\sigma_4^{-2}\sigma_5^{-2}\sigma_4^{-1}}_{\times_5\times_4\times_4\times_5\times_5\times_4}.$$

3.2.3. Murasugi-Tristram inequality. Recall some definitions. Let A be a real symmetric matrix and $B = QAQ^t$ its diagonalization. *The signature* $\sigma(A)$ and *nullity* $n(A)$ of A are defined as the sum of signs of the diagonal entries and the number of zeros on the diagonal of B.

A Seifert surface of an oriented link L in the 3-sphere S^3 is a conncted oriented surface $X \subset S^3$ whose boundary (taking into account the orientation) is L. Let $x_1, \ldots, x_n$ be a base in $H_1(X, \mathbb{Z})$. *A Seifert matrix* is the matrix of the linking numbers $\mathrm{lk}(x_i, x_j^+)$ where x_j^+ is the result of a small shift of the cycle x_j along the positive normal field to X. The *signature* $\sigma(L)$ and the *nullity* $n(L)$ of L are defined by $\sigma(L) = \sigma(V + V^t)$, $n(L) = 1 + n(V + V^t)$ where V is a Seifert matrix.

We define the signature $\sigma(b)$ and the nullity $n(b)$ of a braid b as the signature and the nullity of its closure and for $b = \prod \sigma_i^{k_i}$ we put $e(b) = \sum k_i$.

It is shown in [9] that Murasugi-Tristram inequality is a necessary condition for a scheme of the arrangement of a curve with respect to $\mathcal{L}_p$ to be realizable by a real algebraic curve of degree m. In this case Murasugi-Tristram inequality can be written in the form

$$|\sigma(b)| + m - e(b) - n(b) \leqslant 0, \qquad b = b_{(C,p)} \in B_m. \tag{3.5}$$

To compute the Seifert matrix of a given braid b, we used a simple computer program which were written by the first author for computations in [9]. This program is based on the standard algorithm of construction of a Seifert surface using so-called Seifert circles obtained by smoothing double points on the link diagram; see details in [9]. This algorithm produces a matrix $s \times s$ where $s = e(b) + m - 1$.

3.2.4. Results of computations.

Here we present the results of computations of the amounts involved into the inequality (3.5) and the prohibitions obtained in this way. All the models prohibited with use of the link theory are listed in the second and the third columns of the Table 3. The column "N" contains the number of the row; below, the reference to the k-th row of Table 3 has form "Nk". The column "p" indicates the choise of the point p (the center of the pencil of lines) as follows:

I — inside a free oval of C_4 in the region α;
II — inside the intersection of the intriorities of non-free ovals O_3 and O_4;
III — in the region α, outside the ovals of C_4, but in the convex hull of O_4;
IV — in the region pointed by the asterisk on the third model in Fig. 1 (this choise of p is used only for this model).

The next column of Table 3 contains the codes of mutual arrangements of the pencil $\mathcal{L}_p$ and the curve $C_3 \cup C_4$. Note that the choice of the place for p determines the code up to the following subword replacements:

$$\times_j \supset_{j\pm 1} \leftrightarrow \times_{j\pm 1} \supset_j; \quad \subset_{j\pm 1} \times_j \leftrightarrow \subset_j \times_{j\pm 1}; \quad \times_j u_k \leftrightarrow u_k \times_j; \tag{3.6}$$

$$\subset_j \supset_{j\pm 1} \;\leftrightarrow\; \varnothing \tag{3.7}$$

where $|k - j| > 1$ and u is any of $\times$, $\subset$, $\supset$. The replacements (3.6) do not affect the result of the computations. In the case (3.7) we shall always choose "$\varnothing$" (see Proposition 5.3.1[9]). Then $e(b) = 9 - \#(\mathrm{o}) - \#(\supset)$ where $\#(\cdot)$ is the number of entries of this character in the code (see the last column in Table 3).

Table 3.

N	Model	α	p	Code of arrangement	$e(b)$
1	<+++++>$\{\alpha\}$	3	II	$\supset_2 o_2 o_2 1^{12} \subset_5$	6
2	<+-+-+>$\{\alpha\}$	1,2	I	$32^2 3^2 2 \supset_3 o_i o_j \subset_4 54^2 5^2 4$	6
3	<-+-+->$\{\alpha\}$	0,1	IV	$2312^3 \supset_3 o_i o_j o_k \subset_3 2^3 132 \supset_3 \subset_4$	4
4	<-++++>$\{\alpha\}$	0	II	$2^2 \supset_3 o_i o_j o_k 1^8 4 \subset_5 6$	5
5	<++-++>$\{\alpha\}$	2	III	$5^2 \supset_6 o_i o_j o_k 2^3 1^2 2^3 \subset_6 5^2$	5
6	<+++-+>$\{\alpha\}$	2	III	$6 \supset_5 o_i o_j o_k 2^2 1^2 2^6 \subset_5 6$	5
7	<-+++->$\{\alpha\}$	0,1	III	$6^2 \supset_5 o_i o_j o_k 12^6 1 \subset_5 6^2$	5
8	<-+-++>$\{\alpha\}$	0,1	III	$6 \supset_5 o_i o_j o_k 2^4 1^2 2^2 1^2 \subset_5 6$	5
9	<-++-+>$\{\alpha\}$	1	III	$6 \supset_5 o_i o_j o_k 2^2 1^2 2^4 1^2 \subset_5 6$	5
10	<--+++>$\{\alpha\}$	0	III	$6 \supset_5 o_i o_j o_k 2^6 1^4 \subset_5 6$	5
11	<+--++>$\{\alpha\}$	1	III	$6 \supset_5 o_i o_j o_k 2^4 1^4 2^2 \subset_5 6$	5
12	<--+-->$\{\alpha\}$	1	III	$6^2 \supset_5 o_i o_j o_k 1^3 2^2 1^3 \subset_5 6^2$	5
13	<--+-+>$\{\alpha\}$	0,3	III	$6 \supset_5 o_i o_j o_k 2^2 1^2 2^2 1^4 \subset_5 6$	5
14	<++--->$\{\alpha\}$	0,3	III	$6 \supset_5 o_i o_j o_k 1^6 2^4 \subset_5 6$	5
15	<+--+->$\{\alpha\}$	0	III	$6 \supset_5 o_i o_j o_k 1^2 2^2 1^4 2^2 \subset_5 6$	5
16	<-++-->$\{\alpha\}$	0	III	$6 \supset_5 o_i o_j o_k 1^4 2^4 1^2 \subset_5 6$	5
17	<+---+>$\{\alpha\}$	3	III	$5^2 \supset_6 o_i o_j o_k 21^6 2 \subset_6 5^2$	5
18	<-+--->$\{\alpha\}$	1	III	$6 \supset_5 o_i o_j o_k 1^6 2^2 1^2 \subset_5 6$	5

We have $m = 7$ in this paper. Denote by h the left hand side of (3.5). Performing the computations for the row N1 we obtain $n = 1$, $\sigma = 2$, hence, $h = 2+7-6-1 = 2$. This contradicts inequality (3.5), i.e. proves the non-realizability of the model <+++++>$\{3\}$.

Table 4 contains all values of the parameters i and j which must be considered (i.e. all possible distributions of free ovals up to the symmetry — cp. the example in 3.2.1) for the row N2; Table 5 contains the values of (i, j, k) for the rows N4–N18; the possibilities for the row N3 are contained in Table 6.

Table 4.

Row N2				
α	i	j	σ	h
1	3	3	-2	2
1	3	4	-2	2
1	4	3	-4	4
2	2	3	-4	4
2	2	4	-2	2
2	5	3	-2	2
2	5	4	-2	2

Table 5.

Row	α	Considered values of (ijk)
N4	0	(333)(334)(343)(344)(433)(434)(443)(444)
N5	2	(665)(664)(656)(646)
N6	2	(665)(664)(656)(646)(566)(466)
N7	0	(555)(554)(545)(544)(444)
N8,N10,N13–N16	0	(555)(554)(545)(544)(455)(454)(445)(444)
N7,N12	1	(655)(654)(645)(644)(565)(564)(464)
N8,N9,N11,N18	1	(655)(654)(645)(644)(565)(564) (556)(546)(465)(464)(456)(446)
N13,N14,N17	3	(666)

The computations performed for the row N2 yeild $n = 1$ and the values of σ presented in Table 4. In all the cases we again obtain $h > 0$. For the rows N3–N18

Table 6.

Row	α	i	j	k	n	σ	h		Row	α	i	j	k	n	σ	h
N3	0	3	3	3	2	-1	2		N3	1	3	2	3	2	-3	4
N3	0	3	3	4	2	-3	4		N3	1	3	2	4	2	-3	4
N3	0	3	4	3	2	-3	4		N3	1	4	2	4	3	-2	2
N3	0	3	4	4	2	-3	4		N8	0	4	4	5	2	-2	2
N3	0	4	3	4	2	-3	4		N8	1	6	4	5	2	-2	2
N3	0	4	4	4	3	-2	2		N8	1	5	6	4	2	-2	2
N3	1	2	3	3	2	-1	2		N8	1	5	4	6	2	-2	2
N3	1	2	3	4	2	-3	4		N8	1	4	6	5	2	-2	2
N3	1	2	4	3	2	-3	4		N15	0	5	5	4	2	-2	2
N3	1	2	4	4	3	-2	2		N18	1	4	5	6	2	-2	2

we have $e(b) < 6$, hence, if $n = 1$ then (3.5) prohibites all these rows independent of the value of the signature σ. The results of computations for the cases where $n > 1$ are collected in Table 6.

Thus, in all the cases included into Table 3 we have $h > 0$. This completes the proof of Theorem 1.

Remarks. 1. A "lucky" choice of the region (component of $\mathbb{R}P^2 \setminus C$) for the point p was essential for our proofs: The computations show that not any choice leads to a contradiction with (3.5). On the other hand, the "lucky" choice is not unique: for example, the model <-+-+->{1} (see the row N3 in Table 3) can be prohibited using the projection of the type "I" as well.

2. The models non-realizable by Sect. 3.1 can be prohibited also by the methods of Sect. 3.2. We presented in Sect. 3.1 the proofs based on the complex orientations because they are considerably simpler and do not require any computer.

In conclusion, we give an equivalent formulation of Theorem 1 including references to the methods of proofs.

Table 7.

	{0}	{1}	{2}	{3}			{0}	{1}	{2}	{3}
<+++++>	$\exists$ e	$\exists$ 6	c.o.			<+--++>	$\exists$ 4		$\exists$ 2	c.o.
<+-+-+>	$\exists$ e,2,3			c.o.		<--+-->	c.o.		$\exists$ 3	c.o.
<-+-+->			c.o.	$\exists$ e,1,4		<--+-+>		$\exists$ 2,3	c.o.	
<++++->		$\exists$ 1	$\exists$ 6	c.o.		<++--->		$\exists$ 4	c.o.	
<++-++>	$\exists$ e	$\exists$ 4		c.o.		<+--+->		$\exists$ 1,4	c.o.	$\exists$ 2
<+++-+>	$\exists$ 6	$\exists$ 2,3		c.o.		<-++-->		$\exists$ 1	c.o.	$\exists$ 3
<-+++->			$\exists$ 1	c.o.		<+---+>	$\exists$ 5	$\exists$ 2	c.o.	
<-+-++>			$\exists$ 1,4	c.o.		<-+--->	c.o.		$\exists$ 1,4	c.o.
<+-++->	$\exists$ 1		$\exists$ 2,3	c.o.		<+---->	c.o.	$\exists$ 5	$\exists$ 2	c.o.
<--+++>		$\exists$ 6	$\exists$ 3	c.o.		<----->	c.o.	c.o.	$\exists$ 5	c.o.

Theorem 1′. *The topological classification of triples $(\mathbb{R}P^2, C_3 \bigsqcup C_4, C_3)$ can be described by Table 7 where the realizable models and only they are pointed by the* *symbol "$\exists$", the character "e" near which indicating an elementary construction (see Sect. 2.1) and a digit indicating the number of the curve being perturbed when the constructions of Sect. 2.2 are applied. The models prohibited by complex orientations (see Table 2) are pointed by "c.o.". The other models (see empty squares) are prohibited in Sect. 3.2.*

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