Projective conics and M-quintics in general position with a maximally intersecting pair of ovals

S.Yu. Orevkov

An isotopic classification of curves mentioned in the title is obtained. The link theory (Murasugi-Tristram inequality) is used for prohibitions.

This paper continues the study of plane real reducible curves of degree 7 started in [1–3]. We give here an isotopic classification of curves mentioned in the title. We use the same methods that where used in [4, 3]. It happens that the case of a conic and a quintic is more "regular" than the case of a cubic and a quartic: all the prohibitions can be proved in one way. The same is for the constructions. We shall use often the same notation as in [3], and we shall omit often the details which can be found in [3].

The author thanks G.M. Polotovskii for useful discussions. The author thanks Max-Planck-Institut für Mathematik for the hospitality.

1. Statement of the result. Below, C_n denotes the set of real points of an M-curve of degree m on $\mathbb{R}P^2$. Recall that an M-quintic C_5 consists of 7 smooth circles. Six of them are *ovals* (isotopic to a conic), they are one outside another, and one is the *odd branch* J_3 (isotopic to a line). In this paper we always suppose that one of the ovals of the M-quintic C_5 (denote it by O_5) has 10 distinct intrsection points with the conic C_2. Then, by Bezout theorem, all the intersections must be transversal and the other 5 ovals may not intersect C_2 — we shall call these ovals *free.*

Choosing in different ways two points in different components of $\mathbb{R}P^2 \setminus (C_2 \cup C_5)$ and noting that the line through them can not meet C_n at more than n points, it is not difficult to check (see [1, 2]) that up to an isotopy, C_5 must be arranged with respect to C_2 in one of 10 ways depicted in Fig. 1 where the odd branch is not shown and α, β a priori are arbitrary integers such that $\alpha + \beta = 5$ (the numbers of free ovals in the corresponding regions).

The purpose of this note is to show that those and only those arrangements are realizable which correspont to the values α listed in Fig. 1.

2. Constructions. Analogousely to [3], arrangements of C_5 with respect to C_2 (satisfying the conditions listed in §1) will be encoded as follows. A word $w = \langle s_1 \dots s_n \rangle$, $s_k \in \{+,-\}$ will codify the arrangement of C_5 with respect to C_2 composed of the blocks $B(s_1), \dots, B(s_n)$ according to Fig. 2.1; see the blocks $B(+)$

Partially supported by the grant RFFI 98-01-00794

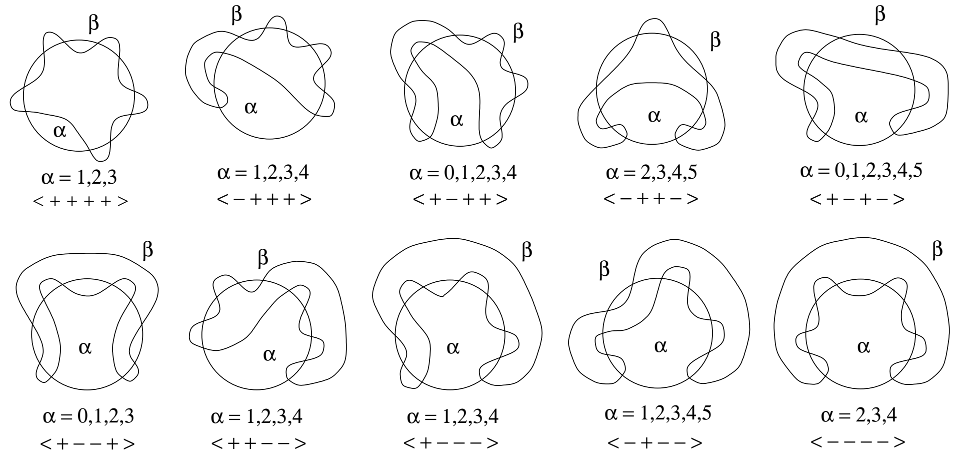

FIG. 1

and $B(-)$ in Fig. 2.2. The parameter α will be indicated in the braces after the word w.

All the realizable arrangements are constructed as smoothings of suitable singular curves. Denote by $2n\{\alpha\}$, $n = 0, 1, 2$, $\alpha = 0, 1$ the arrangement of a smooth three-component quintic with respect to the conic C_2 depicted in Fig. 2.3: the quintic has two singular points of the types A_{2n}, A_{8-2n} (by convention, A_0 is a smooth point) lying on the same even branch. The free oval of the quintic is inside C_2 for $\alpha = 1$ and outside for $\alpha = 0$.

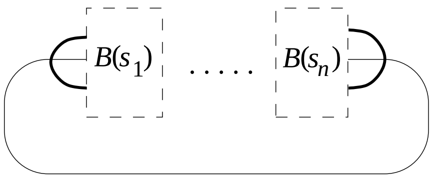

FIG. 2.1

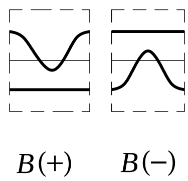

FIG. 2.2

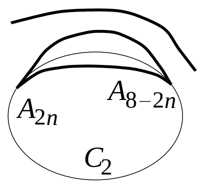

FIG. 2.3

Lemma. *All the six arrangements* $2n\{\alpha\}$, $n = 0, 1, 2$, $\alpha = 0, 1$ *are realizable by algebraic curves.*

Proof. If a curve C at a point p is non-singular and is not a flex point then we denote by $f_{C,p}$ the quadratic birational transformation $(u, v) \mapsto (u, v - u^2)$ where (u, v) is an affine coordinate system such that the infinite line is tangent to C at p and the power series expansion of C near p has form $v = u^2 + O(u)$, $u \to \infty$.

$0\{\alpha\}$ $2\{\alpha\}$. Let $Q := L^2L_0^2 + \varepsilon l_1 l_2 l_3 l_4$ where $|\varepsilon| \ll 1$ is a cuspidal three-component quartic and p, q are its tangency points with two of the lines l_i (see Fig. 3.1, 3.2). Then $f_{Q,p}(Q) \cup f_{Q,p}(L)$ realizes the arrangement $0\{1\}$ in the case 3.1 and $0\{0\}$ in the case 3.2 and $f_{Q,q}(Q) \cup f_{Q,q}(L)$ realizes the arrangement $2\{0\}$ in the case 3.1 and $2\{1\}$ in the case 3.2.

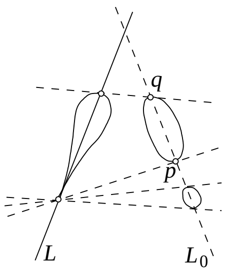

FIG. 3.1

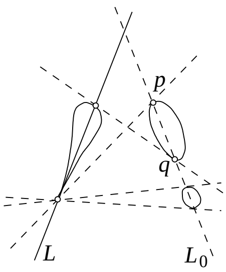

FIG. 3.2

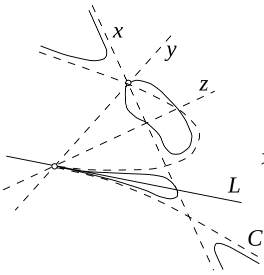

FIG. 3.3

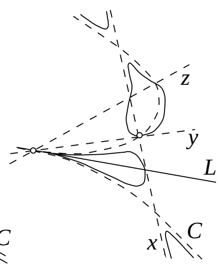

FIG. 3.4

$4\{\alpha\}$. Let C be a cuspidal cubic arranged with respect to the coordinate axes as it is shown in Fig. 3.3, 3.4 and $Q := xC + \varepsilon yzL^2$ be a cuspidal quartic. Let hy be the hyperbolism, i.e. the birational transformation $\mathrm{hy}(x:y:z) = (x^2 : xy : yz)$. Then $\mathrm{hy}(Q) \cup \mathrm{hy}(L)$ realizes the arrangement $4\{1\}$ in the case 3.3 and $4\{0\}$ in the case 3.4.

It is shown in [3] that for any choice of signs $s_1, \dots, s_n, s'_1, \dots, s'_{4-n}$ there exists a perturbation of the curve $2n\{\alpha\}$ of the form $\langle s_1 \dots s_n s'_{4-n} \dots s'_1 \rangle \{\alpha + \alpha_0\}$ where α_0 is the number of pluses in the sequence $-s_1, s_2, \dots, (-1)^n s_n, (-1)^n s'_{4-n}, \dots, s'_2, -s'_1$. It is not difficult to check that this procedure yields all the arrangements in Fig. 1.

Remark. The four arrangements <++++>$\{\alpha\}$ <---->$\{\alpha+1\}$, $\alpha = 2, 3$ can be constructed elementary: choose C_5 as a suitable perturbation of two conics (one of them will be C_2) and a line.

3. Prohibitions. The values of α in the case <++++> which were not realized above, are prohibited using the method of complex orientations ([1, VII]; see also [3, 3.1]). In all the other cases there exists a point p lying in the region α in the convex hull of O_5, and hence, outside free ovals of the quintic (cp. the projection III in [3, 3.2.4]). Let us choose affine coordinates (x, y) such that p is the infinite point of the lines $x = \text{cosnt}$ and the infilite line meets O_5 at 5 points. Then C_2, O_5 and J_5 are arranged with respect to the coordinates as it is shown in Fig. 4 where the blocks $B(s_i)$ are the same as in Fig. 2.2.

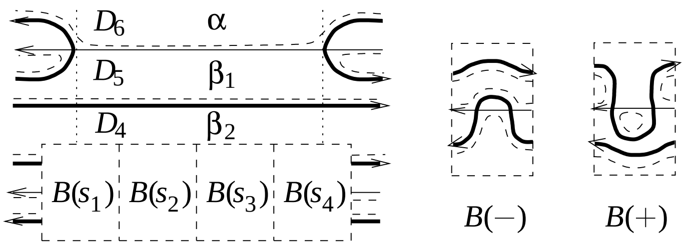

FIG. 4

Since a vertical line can not meet C_5 at more than 5 points, free ovals of the quintic must lye in the band between the vertical tangents to O_5 (denote it by D).

Let D_4, D_5, D_6 be the three upper components of $D \setminus (C_2 \cup O_5)$ (see Fig. 4). Then α ovals lye in D_6 and β ovals lye in $D_5 \cup D_4$. Let us numerate free ovals of the quintic from the left to the right and let D_{i_j} be the strip containing the j-th oval, $j = 1, \dots, 5$. Put $\delta\alpha = \sum_{i_j=6}(-1)^j$, $\delta\beta_k = \sum_{i_j=6-k}(-1)^j$, $k = 1, 2$.

The complex orientations formula for C_5 (see [5]) and the Fiedler theorem on the orientation interchangement [6] imply

$$\delta\alpha + \delta\beta_1 - \delta\beta_2 = 0. \tag{1}$$

Let us choose complex orientations on C_2 and C_5 so that C_2 and O_5 are oriented oppositely with respect to J_5. Let F be the curve of degree 7 obtained from $C_2 \cup C_5$ by smoothing the double points according to the chosen orientations (it is shown by a dashed line in Fig. 4). Then the number of ovals of the curve F is equal to $2P + 7$ where P is the number of pluses in the word $<s_1 s_2 s_3 s_4>$ encoding the given arrangement $C_2 \cup O_5$. Thus, the complex orientation formula for F yields

$$\varepsilon((1 + 2\varepsilon)\delta\alpha + \delta\beta_1 - \delta\beta_2) = 2P - 5 \tag{2}$$

where $\varepsilon = \pm 1$ is the sign of O_5 with respect to the odd branch J_5. Completing (1), (2) by the evident identity $\delta\alpha + \delta\beta_1 + \delta\beta_2 = -1$ and solving the obtained simultaneous equations, we find:

$$\delta\alpha = q - 2, \quad \delta\beta_1 = 1 - q, \quad \delta\beta_2 = 0, \qquad q = P + (\varepsilon - 1)/2. \tag{3}$$

For arrangements $<s_1 s_2 s_3 s_4>\{\alpha\}$ not realized in §2, all the values of $i_1 \dots i_5$ such that $\varepsilon = \pm 1$ satisfying (3) exists, are as follows:[1]

<-+++>{0}	<-++->{1}	<-++->{0}	<+--+>{4}	<++-->{5}	<++++>{0}
<--++>{0}				<+--+>{5}	<++++>{4}
				<+--->{5}	<++++>{5}
44445	44446				<+++->{5}
44544	44556				<++-+>{5}
44555	44644				<+--->{0}
45545	44655				<-+-->{0}
54444	45546	44445			<---->{0}
54455	45645	44544			<---->{1}
54554	54456	44555			<---->{5}
55445	55446	45545			
55544	55556	54455	56666		no
55555	55655	55555	66566	66666	solution

(The symmetricity of the words <-++->, <+--+> was used in the columns 2–4.) Checking the Murasugi-Tristram inequality in each of the 41 cases listed above, as it was done in [3], we see that all the cases are non-realizable.

[1]The last column of this table correspond to arrangements prohibited in [1, VII] by the complex orientations method. In particular, we see that Fig. 16 of [1] contained a misprint: $\alpha \neq 2$ for the 5-th model should be replaced by $\alpha \neq 5$ for the 2-nd model.

References

1. Polotovskii G.M., *On the classification of decomposing plane real algebraic curves*, Lect.Notes in Math. **1524** (1992), 52–74.
2. Polotovskii G.M., *On the classification of decomposable 7-th degree curves*, Contemp. Math. (1999) (to appear).
3. Orevkov S.Yu., Polotovskii G.M., *Projective M-cubics and M-quartics in general position with a maximally intersecting pair of ovals*, submitted to "St.Petersburg Math. J.".
4. Orevkov S.Yu., *Link theory and oval arrangements of real algebraic curves*, Topology (to appear).
5. Rokhlin V.A., *Complex topological characteristics of real algebraic curves*, Uspekhi Mat. Nauk **33:5** (1978), 77–89 (in Russian); English transl. Russ. Math. Surv. **33:5** (1978), 85–98.
6. Fiedler T., *Pencils of lines and the topology of real algebraic curves*, Izv. AN SSSR, ser. mat. **46** (1982), 853–863 (Russian); English transl. in Math. USSR-Izvestia **21** (1983), 161–170.

Steklov Math. Inst. Russ. Acad. Sci.
E-mail address: orevkov@mi.ras.ru