TOPOLOGY OF MAXIMALLY WRITHED REAL ALGEBRAIC KNOTS

G. B. Mikhalkin and S. Yu. Orevkov

Abstract. Oleg Viro introduced an invariant of rigid isotopy for real algebraic knots in $\mathbb{RP}^3$ which can be viewed as a first order Vassiliev invariant. In this paper we look at real algebraic knots of degree d with the maximal possible value of this invariant. We show that for a given d all such knots are topologically isotopic and explicitly identify their knot type.

Introduction

A *real algebraic curve* in $\mathbb{P}^3$ is a (complex) one-dimensional subvariety L in $\mathbb{P}^3 = \mathbb{CP}^3$ invariant under the involution of complex conjugation $\mathrm{conj} : \mathbb{P}^3 \to \mathbb{P}^3$, $(x_0 : x_1 : x_2 : x_3) \mapsto (\bar{x}_0 : \bar{x}_1 : \bar{x}_2 : \bar{x}_3)$. The conj-invariance is equivalent to the fact that L can be defined by a system of homogeneous polynomial equations with real coefficients. The degree of L is defined as its homological degree, i. e. the number d such that $[L] = d[\mathbb{P}^1] \in H_2(\mathbb{P}^3) \cong \mathbb{Z}$. A curve of degree d intersects a generic complex plane in d points.

We denote the set of real points of L by $\mathbb{R}L$. We say that a real curve L is *smooth* if it is a smooth complex submanifold of $\mathbb{P}^3$. In this case, $\mathbb{R}L$ is a smooth real submanifold of $\mathbb{RP}^3$ and if it is non-empty, we call it a *real algebraic link* or, more specifically, a *real algebraic knot* in the case when $\mathbb{R}L$ is connected.

Two real algebraic links are called *rigidly isotopic* if they belong to the same connected component of the space of smooth real curves of the same degree. A rigid isotopy classification of real algebraic rational curves in $\mathbb{P}^3$ is obtained in [1] up to degree 5 and in [2] up to degree 6. Also we gave in [2] a rigid isotopy classification for genus one knots and links up to degree 6 (here we speak of the genus of the complex curve L rather than the minimal genus of a Seifert surface of $\mathbb{R}L$).

In all the above-mentioned cases, a rigid isotopy class is completely determined by the usual (topological) isotopy class, the complex orientation (for genus one links), and the invariant of rigid isotopy w introduced by Viro [4] (called in [4] *encomplexed writhe*). This invariant is defined as the sum of signs of crossings of a generic projection but the crossings with non-real branches are also counted with appropriate signs; see details in [4] (the definition of w is also reproduced in [2]).

Let $T(p,q) = \{(z,w) \mid z^p = w^q\} \cap \mathbb{S}^3$, $p \geq q \geq 0$, be the (p,q)-torus link in the 3-sphere $\mathbb{S}^3 \subset \mathbb{C}^2$. If $p \equiv q \mod 2$, we define the *projective torus link* $T_{proj}(p,q) = T(p,q)/(-1) \subset \mathbb{S}^3/(-1) = \mathbb{RP}^3$.

Research is supported in part by the SNSF-grants 159240, 159581 and the NCCR SwissMAP project (G.M.) and by RSF grant, project 14-21-00053 dated 11.08.14. (S.O.)

Let $N_d = (d-1)(d-2)/2$. By the genus formula, this is the maximal possible value of w for irreducible curves of degree d which can be attained on rational curves only. So, if a real algebraic curve K in $\mathbb{P}^3$ is smooth, irreducible, and $|w(K)| = N_d$ where $d = \deg K$ (and hence the genus of K is zero), then we call it *maximally writhed* or *MW-curve.* The main result is the following.

Theorem 1. *Let K be an MW-curve of degree $d \geq 3$, and $w(K) = N_d$. Then $\mathbb{R}K$ is isotopic to $T_{proj}(d, d-2)$.*

Corollary 1. *A plane projection of an MW-curve from any generic real point has N_d or $N_d - 1$ real double points with real local branches.*

Proof. Follows from Murasugi's result [3; Proposition 7.5] which states that any projection of a torus link $T(p, q)$, $1 \leq q \leq p$, has at least $p(q-1)$ crossings. □

In Proposition 2 (see the end of the paper) we give a precision and a self-contained (i.e., not using [3]) proof to Corollary 1.

Conjecture 1. In Theorem 1, K is rigidly isotopic to $T_{proj}(d, d-2)$.

Conjecture 2. If an algebraic knot $\mathbb{R}K$ of degree d in $\mathbb{RP}^3$ is isotopic to $T_{proj}(d, d-2)$, then $w(K) = N_d$.

In a forthcoming paper we are going to give a proof of Conjecture 2 as well as a generalization of Theorem 1 for links of arbitrary genus.

The following differential geometric property of maximally writhed algebraic knots was communicated to us by Oleg Viro.

Proposition 1. *Let K be as in Theorem 1. Then the torsion of $\mathbb{R}K$ is everywhere positive.*

1. *MW*-curves have everywhere positive torsion (proof of Proposition 1)

Recall that the sign of the (differential geometric) torsion of a curve $t \mapsto r(t) \in \mathbb{R}^3$, $t \in \mathbb{R}$, coincides with the sign of $\det(r', r'', r''')$ and it does not depend on the parametrization if $r' \neq 0$. The sign of the torsion of a curve in $\mathbb{RP}^3$ does not depend on a choice of positively oriented affine chart.

Lemma 1. *Let K be a real algebraic knot in $\mathbb{P}^3$ of genus 0 which is not contained in any plane. If the torsion of $\mathbb{R}K$ vanishes at a point p, then there exists an arbitrarily small deformation of K (in the class of real algebraic knots) which has points with negative torsion.*

Proof. We can always choose affine coordinates (x_1, x_2, x_3) centered at p such that a parametrization $t \mapsto (x_1(t), x_2(t), x_3(t))$ of K at p satisfies the condition $\operatorname{ord}_t x_1 < \operatorname{ord}_t x_2 < \operatorname{ord}_t x_3$. If $\operatorname{ord}_t x_k > k$ for $k = 1, 2$, or 3, then x_k can be perturbed so that $\operatorname{ord}_t x_k = k$ and the k-th derivative $x_k^{(k)}(0)$ has any sign we want. Indeed, let $(y_0 : y_1 : y_2 : y_3)$ be homogeneous coordinates such that $x_i = y_i/y_0$, $i = 1, 2, 3$. Then the parametrization can be chosen so that $x_i(t) = y_i(t)/y_0(t)$, $i = 1, 2, 3$, where $y_0(t), \dots, y_3(t)$ are real polynomials of degree $d = \deg K$, and $y_0(0) > 0$. Then the desired perturbation of $x_k(t)$ is just $(c_k t^k + y_k(t))/y_0(t)$ where $0 < |c_k| \ll 1$ and c_k has the prescribed sign. □

Proof of Proposition 1. By Lemma 1, it is enough to show that $\mathbb{R}K$ does not have points with negative torsion. Suppose it does. Then, in an appropriate affine chart, $\mathbb{R}K$ admits a parametrization of the form $t \mapsto (t, t^2 + O(t^3), -t^3 + O(t^4))$. This means that in a sufficiently small neighbourhood of the origin, the curve is approximated by a negatively twisted rational cubic curve. Hence there is a projection with a negative crossing (see [4; Section 1.4]). Since $w(K)$ is the sum of the signs of all real crossings and the number of them is at most N_d, a single negative crossing makes impossible to attain the equality $w(K) = N_d$. □

2. Uniqueness of MW-curves up to isotopy (proof of Theorem 1)

Let K be as in Theorem 1. So, K is a smooth rational curve in $\mathbb{P}^3$ of degree $d \geq 3$ and $w(K) = N_d$.

Given a point $p \in \mathbb{P}^3$, let $\pi_p : \mathbb{P}^3 \setminus \{p\} \to \mathbb{P}^2$ be the projection from p and let $\hat{\pi}_p : K \to \mathbb{P}^2$ be the restriction of π_p to K. If $p \in K$, then we extend $\hat{\pi}_p$ to p by continuity, thus $\pi_p^{-1}(\hat{\pi}_p(p)) = T_p$ where T_p is the tangent line to K at p.

If $p \in \mathbb{RP}^3$, then we set $C_p = \hat{\pi}_p(K)$. Note that

$$\deg C_p = \begin{cases} d, & p \notin K \\ d-1, & p \in K. \end{cases}$$

Recall that an algebraic curve C (maybe, singular) of degree m in $\mathbb{RP}^2$ is called *hyperbolic* with respect to a point $q \in \mathbb{RP}^2$ (which may or may not belong to C), if any real line through q intersects C at m real points counting the multiplicities. We denote:

$$\operatorname{hyp}(C) = \{q \mid C \text{ is hyperbolic with respect to } q\}. \tag{1}$$

It is easy to check that $\operatorname{hyp}(C)$ is either empty or a convex closed set. It is possible that $\operatorname{hyp}(C)$ contains only one point. In this case, the point should be singular. For example, if C is a cuspidal cubic, then $\operatorname{hyp}(C)$ consists of the cusp only.

Similarly, we say that K is hyperbolic with respect to a real line L if, for any real plane P passing through L, each intersection point of K and $P \setminus L$ is real.

The following two properties of hyperbolic curves are immediate from the definition:

Lemma 2. *Let C be a real plane curve, $q \in \operatorname{hyp}(C)$, and $q_1 \in \mathbb{R}C \setminus \{q\}$. Then each local branch of C at q_1 is smooth, real, and transverse to the line (qq_1). The projection from q defines a covering $\mathbb{R}\tilde{C} \to \mathbb{RP}^1$ where $\tilde{C}$ is the normalization of C.* □

Lemma 3. *Let L be a real line in $\mathbb{P}^3$ and $p \in \mathbb{R}L$. Then K is hyperbolic with respect to L if and only if $\pi_p(L) \in \operatorname{hyp}(C_p)$.* □

Lemma 4. *If $p \in \mathbb{R}K$, then $\hat{\pi}_p(p)$ is a smooth point of $\mathbb{R}C_p$ and $T_p \cap K = \{p\}$.*

Proof. By Proposition 1 the torsion of $\mathbb{R}K$ does not vanish at p, hence the image of the germ (K, p) under the projection $\hat{\pi}_p$ is a smooth local branch of C_p at $q = \hat{\pi}_p(p)$. Suppose that C_p has another local branch at q which is the projection of a germ (K, p_1). Let p_0 be a real point on the line (pp_1), $p_0 \notin \{p, p_1\}$. Then C_{p_0} has (at least) two branches at $\pi_{p_0}(p) = \pi_{p_0}(p_1)$ one of whom is cuspidal. In this case by perturbing K we may obtain either a non-real crossings or a pair of real crossings of opposite signs which contradicts the maximality of $w(K)$ (cp. the end of the proof of Proposition 1). □

Lemma 5. *If $p \in \mathbb{R}K$, then* $\operatorname{hyp}(C_p)$ *is the closure of a component of* $\mathbb{RP}^2 \setminus \mathbb{R}C_p$ *and* $\hat\pi_p(p)$ *is a smooth point of its boundary.*

Proof. Let $q = \hat\pi_p(p)$. It is a smooth point of C_p by Lemma 4. For each ordinary node u of C_p let $\sigma(u)$ be its contribution to $w(K)$, i.e. the sum of the signs of crossings of a nodal perturbation of u. (see Remark 1 below). Then, similarly to [2; Proposition 21], we have

$$w(K) = i(q') + i(q'') + \sum_{u \in \operatorname{Sing}(\mathbb{R}C_p)} \sigma(u), \tag{2}$$

where $q', q'' \notin C_p$ are points close to q on different sides of $\mathbb{R}C_p$, and $i(x)$ for $x \notin C_p$ is one half of the image of $[\mathbb{R}C_p]$ under the isomorphism $H_1(\mathbb{RP}^2 \setminus \{x\}) \cong \mathbb{Z}$ (see [2; §6] for the choice of the orientations). It is clear that $i(q') + i(q'') \le \deg C_p - 1 = d - 2$ and $\sum_u \sigma(u) \le \operatorname{Card}\operatorname{Sing}(\mathbb{R}C_p) \le (d-2)(d-3)/2$. The sum of these two upper bounds is N_d, thus $w(K) = N_d$ implies the equality sign in the both estimates. It remains to note that $i(q') + i(q'') = d - 2$ implies $q \in \operatorname{hyp}(C_p)$.

The fact that $\operatorname{hyp}(C_p)$ is the closure of a component of the complement of C_p follows from the discussion after (1) because q is smooth on C_p. □

Recall that the tangent line to K at $p \in K$ is denoted by T_p. Let us set

$$T = \bigcup_{p \in \mathbb{R}K} \mathbb{R}T_p$$

Lemma 6. *Suppose that K is hyperbolic with respect to a real line L and let $p \in (\mathbb{R}K) \setminus L$. Then $L \cap T_p = \varnothing$.*

Proof. Combine Lemma 2 and Lemma 3. □

Lemma 7. *Let p_1 and p_2 be two distinct points on $\mathbb{R}K$. Then $T_{p_1} \cap T_{p_2} = \varnothing$.*

Proof. Let $L = T_{p_1}$. Then K is hyperbolic with respect to L by Lemma 5 combined with Lemma 3, and we have $p_2 \notin L$ by Lemma 4. Hence the result follows from Lemma 6. □

Thus T is a disjoint union of a continuous family of real projective lines (topologically, circles) parametrized by $\mathbb{R}K$. We are going to show that the pair $(T, \mathbb{R}K)$ is isotopic in $\mathbb{RP}^3$ to a hyperboloid with a projective torus link $T_{proj}(d, d-2)$ sitting in it. Note that T is not smooth. It has a cuspidal edge along $\mathbb{R}K$.

Lemma 8. *There exist two real lines L_1 and L_2 such that K is hyperbolic with respect to each of them, $L_1 \cap K = \varnothing$, and L_2 crosses K without tangency at a pair of complex conjugated points.*

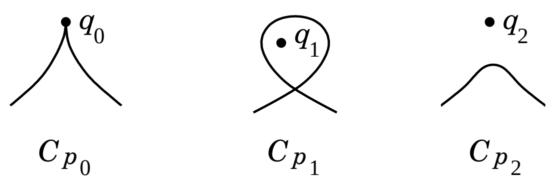

FIGURE 1. Two perturbations of a cusp in the proof of Lemma 8

Proof. Let us choose a point $p \in \mathbb{R}K$ and let $p_0 \in \mathbb{R}T_p \setminus \{p\}$. Then $\pi_p(T_p) = \hat{\pi}_p(p) \in \operatorname{hyp}(C_p)$ by Lemma 5 whence K is hyperbolic with respect to T_p by Lemma 3. Let $q_0 = \pi_{p_0}(p)$. Then, again by Lemma 3, we have $q_0 \in \operatorname{hyp}(C_{p_0})$. The curve C_{p_0} has a cusp at q_0 because the torsion at p is nonzero. Let p_1 and p_2 be points close to p_0 and chosen on different sides of the osculating plane of $\mathbb{R}K$ at p. Then C_{p_1} and C_{p_2} are obtained from C_{p_0} by a perturbation of the cusp as shown in Figure 1 where q_2 is a solitary node of C_{p_2} (a point where two complex conjugated branches cross). Then we set $L_j = \pi_{p_j}^{-1}(q_j)$, $j = 1, 2$, where the points q_1 and q_2 are chosen as in Figure 1. The fact that $q_0 \in \operatorname{hyp}(C_{p_0})$ implies $q_j \in \operatorname{hyp}(C_{p_j})$, $j = 1, 2$, whence the hyperbolicity of K with respect to L_j by Lemma 3. □

Proof of Theorem 1. Let L_1 and L_2 be as in Lemma 8.

The line L_1 is disjoint from T by Lemma 6. Let P be a real plane through L_1. Again by Lemma 6, P crosses each line T_p, $p \in \mathbb{R}K$, at a single point. Let us denote this point by $\xi_P(p)$. Then $\xi_P : \mathbb{R}K \to \mathbb{R}P$ is a continuous mapping. It is injective by Lemma 7 and its image (which is $T \cap \mathbb{R}P$) is disjoint from L_1. Hence $T \cap \mathbb{R}P$ is a Jordan curve in the affine real plane $\mathbb{R}P \setminus L_1$. Let D_P be the disk bounded by this Jordan curve and let $U_1 = \bigcup_P D_P$ where P runs through all the real planes through L_1. Then U_1 is fibered by disks over a circle which parametrizes the pencil of planes through L_1. Since $\mathbb{RP}^3$ is orientable, this fibration is trivial, thus U_1 is a solid torus and $\partial U_1 = T$. Each P transversally crosses K at d real points, thus $\mathbb{R}K$ sits in T and it realizes the homology class $d\alpha$ where α is a generator of $H_1(U_1)$.

The same arguments applied to the line L_2 show that T bounds a solid torus U_2 such that $\mathbb{R}K$ realizes the homology class $(d-2)\beta$ where β is a generator of $H_1(U_2)$. We conclude that the lift of K on $\mathbb{S}^3$ is $T(d, d-2)$ and the result follows. □

We see that T cuts $\mathbb{RP}^3$ into two solid tori U_1 and U_2 such that $\mathbb{R}L_1 \subset U_2$ and $\mathbb{R}L_2 \subset U_1$.

Proposition 2. (Compare with Corollary 1). *Let p be a generic point of $\mathbb{RP}^3$. Then C_p has only real double points. If $p \in U_1$, then all the double points have real local branches and the interior of* $\operatorname{hyp}(C_p)$ *is non-empty. If $p \in U_2$, then one double point q is solitary (i.e. has complex conjugated local branches), all the other double points have real local branches, and* $\operatorname{hyp}(C_p) = \{q\}$.

Proof. Let us consider a generic path $p(t)$ which relate the given point with a point on T. It defines a continuous deformation of the knot diagram which is a sequence of Reidemeister moves (R1) – (R3). However, (R2) is impossible because it involves a negative crossing and (R1) may occur only when $p(t)$ passes through T. Thus the number and the nature of double points does not change during the deformation. The projection from a point of T is cuspidal and it is hyperbolic with respect to the cusp, so the result follows from Lemma 2.

Non-emptiness of the interior of $\operatorname{hyp}(C_p)$ in case $p \in U_1$, follows from the fact that $\operatorname{hyp}(C_p)$ can disappear only by a move (R3). This is however impossible because all crossings are positive and the boundary orientation on $\partial(\operatorname{hyp}(C_p))$ agrees with an orientation of $\mathbb{R}C_p$ due to Lemma 2 (see Figure 2). □

Acknowledgement. We are grateful to Oleg Viro for very useful discussions.

References

1. J. Björklund, *Real algebraic knots of low degree*, J. Knot Theory Ramifications **20:9** (2011), 1285–1309.

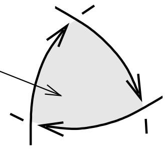

FIGURE 2. Impossibility of a move (R3) which eliminates hyp(C_p)

2. G. Mikhalkin and S. Orevkov, *Real algebraic knots and links of low degree*, J. Knot Theory Ramifications **26** (2016), 1642010, 34 pages.
3. K. Murasugi, *On the braid index of alternating links*, Trans. Amer. Math. Soc. **326:1** (1991), 237–260.
4. O. Viro, *Encomplexing the writhe*, in: Topology, Ergodic Theory, Real Algebraic Geometry, Amer. Math. Soc. Transl. Ser. 2, vol. 202, 2001, pp. 241–256; arxiv:math.0005162.

UNIVERSITÉ DE GENÉVE, SECTION DE MATHÉMATIQUES, BATTELLE VILLA, 1227 CAROUGE, SUISSE

E-mail address: grigory.mikhalkin@unige.ch

STEKLOV MATHEMATICAL INSTITUTE, GUBKINA 8, 119991, MOSCOW, RUSSIA;

IMT, UNIVERSITÉ PAUL SABATIER, 118 ROUTE DE NARBONNE, 31062, TOULOUSE, FRANCE;

NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS, RUSSIAN FEDERATION, USACHEVA 6, MOSCOW, 119048 RUSSIA

E-mail address: orevkov@math.ups-tlse.fr