MARKOV THEOREM FOR TRANSVERSAL LINKS

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Abstract. It is shown that two braids represent transversally isotopic links if and only if one can pass from one braid to another by conjugations in braid groups, *positive* Markov moves, and their inverses.

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By a well-known theorem of Alexander [1], any oriented link in $\mathbb{R}^3$ is isotopic to the closure of a braid. The question when two braids represent isotopic links is answered by Markov's theorem [6] (see [3], [2], or [7] for proofs): It is so if and only if one can pass from one braid to another by conjugations in braid groups B_n, the transformations $M_n^\pm : B_n \to B_{n+1}$, $M_n^+ : b \mapsto b\cdot\sigma_n$, $M_n^- : b \mapsto b\cdot\sigma_n^{-1}$ called *positive/negative Markov moves* or *stabilizations*, and their inverses (*destabilizations*).

In the seminal paper [2] Bennequin established, among other very important results, the analogue of the Alexander theorem for transversal links (i.e., links transverse to the standard contact structure; see below). Namely, any transversal link is transversally isotopic to the closure of a braid. The purpose of this paper is to prove the corresponding analogue of Markov's theorem.

Theorem. *Two braids represent transversally isotopic links if and only if one can pass from one braid to another by conjugations in braid groups, positive Markov moves, and their inverses.*

When this paper had been already finished, we learned from Victor Ginzburg that he had announced this result around 1992. However, his proof has never been published. Another proof of the theorem based on completely different ideas was independently obtained by Nancy Wrinkle in her PhD thesis [8].

Let us recall the standard definitions (see e.g. [2]). Consider the 1-form $\alpha = dz + x\,dy - y\,dx$ in $\mathbb{R}^3$ with coordinates x, y, z. It defines the standard contact structure in $\mathbb{R}^3$. In the cylindric coordinates r, θ, z with $x = r\cos\theta$, $y = r\sin\theta$ one has $\alpha = dz + r^2 d\theta$.

A link L in $\mathbb{R}^3$ is *transversal* if the restriction $\alpha|_L$ nowhere vanishes. In this case $\alpha|_L$ defines a canonical orientation on L.

A *geometric braid* in $\mathbb{R}^3$ is an oriented link L such that the restriction $d\theta|_L$ is positive. In particular, L is disjoint from the z-axis Oz. The *degree* of L, also called the number of strings of L, is the degree of the projection $(r,\theta,z)\mapsto\theta$ restricted to L. There is a canonical one-to-one correspondence between isotopy classes of geometric braids of degree n and conjugacy classes in the braid group B_n.

Any conjugacy class in B_n defines a transversal isotopy class of transversal links. Indeed, any braid $b \in B_n$ can be realized as a geometric braid sufficiently C^1-close to the standard circle $r = 1$, $z = 0$, which is clearly transversal.

The rest of the paper is devoted to the proof of Theorem. Essentially, our proof is a parametric version of Bennequin's proof of his result cited above.

Let L_0 and L_1 be two transversal geometric braids and $\{L_t\}_{t\in[0,1]}$ a transversal isotopy between L_0 and L_1. Denote the interval $[0,1]$ by I, the number of components of L_0 by m, and the disjoint union of m abstract circles by S. Abusing notation, we shall denote by s a positively oriented (local) coordinate on S, as also a current point of S. The isotopy $\{L_t\}_{t\in I}$ can be parameterized by a smooth map $\mathcal{L} : S \times I \to \mathbb{R}^3$ such that for every $t \in I$ the map $\mathcal{L}_t : s \mapsto \mathcal{L}(s,t)$ is a parameterization of L_t.

Definition 1. Let $\{L_t\}_{t\in I}$ be a transversal isotopy parameterized by a map $\mathcal{L} : S \times I \to \mathbb{R}^3$. It is called *monotone near the axis* if there exists a finite number of parameters $0 < t_1 < \cdots < t_k < 1$ such that the following holds:

(1) For any t_i there exists a unique $s_i \in S$ such that $\mathcal{L}(s_i,t_i)$ lies on the z-axis Oz, and $\mathcal{L}^{-1}(Oz) = \{(s_1,t_1),\dots,(s_k,t_k)\}$.
(2) Up to a rotation of $\mathbb{R}^3$ around Oz, the mapping $\mathcal{L}$ is given in a neighborhood of every (s_i,t_i) by $x = \tau - 3s^2$, $y = s\tau - s^3$, $z = z_i + s$, where s is a positively oriented coordinate on S centered at s_i and τ is a coordinate on I centered at t_i and oriented either positively or negatively.

The isotopy $\{L_t\}_{t\in I}$ is *monotone everywhere* if additionally

(3) L_t is a transversal geometric braid for every $t \notin \{t_1,\dots,t_k\}$.

Note, that if we fix $t \neq 0$ and substitute $x = \tau - 3s^2$, $y = s\tau - s^3$ into the 1-form $r^2 d\theta = x\,dy - y\,dx$, we get $r^2 d\theta = (\tau^2 + 3s^4)ds > 0$. Thus, conditions (2) and (3) of Definition 1 are consistent.

We shall always assume that isotopies we consider are sufficiently generic outside a small neighborhood of the axis Oz.

Lemma 1. *Let b_0 and b_1 be two braids, L_0 and L_1 the transversal geometric braids defined by them. Assume that there exists an everywhere monotone isotopy between L_0 and L_1. Then one can pass from b_0 to b_1 by conjugations in braid groups, positive Markov moves, and their inverses.*

Proof. When passing through a critical value $t = t_i$, the projection of L_t onto the horizontal plane Oxy transforms near the origin as in Figure 1. This is a positive Markov move. □

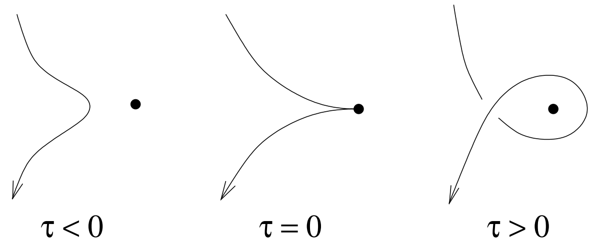

FIGURE 1. THE CURVE $s \mapsto (\tau - 3s^2, s\tau - s^3)$

Lemma 2. *Let $\{L_t\}_{t\in I}$ be a transversal isotopy between transversal geometric braids L_0 and L_1. Then it can be perturbed into an isotopy $\{L'_t\}_{t\in I}$ which is monotone near the axis.*

Proof. Let $\mathcal{L} : S\times I \to \mathbb{R}^3$ be a smooth mapping which parameterizes $\{L_t\}$. Perturbing it if necessary, we can suppose that it is transverse to the axis Oz. Let us consider a point $p = (s_0, t_0) \in S\times I$ such that $\mathcal{L}(p)$ lies on Oz. Let s and t be coordinates on S and I near s_0 and t_0 respectively (with $ds > 0$). Set $\mathcal{L}(s,t) = \big(x(s,t), y(s,t), z(s,t)\big)$. Since all L_t's are transversal braids, we have $\partial z/\partial s > 0$ at p. Hence, there exists a neighborhood U of p such that $\partial z/\partial s > \varepsilon > 0$ in U. Let us modify $\big(x(s,t), y(s,t)\big)$ in U replacing it by the homotopy in Figure 2 (the shaded zone corresponds to the homotopy described in Part (2) of Definition 1 and shown in Figure 1; we assume here that before the modification the homotopy looked as a parallel motion of a vertical line). If U is sufficiently small, then we can achieve that $|r^2\theta'_s| < \varepsilon$ in U, which provides that $\mathcal{L}_t^*\alpha > 0$. $\square$

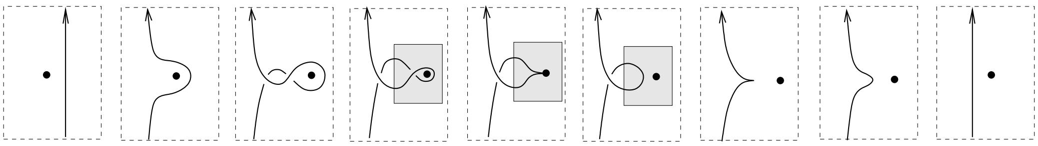

FIGURE 2. MAKING THE ISOTOPY MONOTONE NEAR Oz

Definition 2. Let $\{L_t\}_{t\in I}$ be a transversal isotopy parameterized by a map $\mathcal{L}$: $S \times I \to \mathbb{R}^3$. A *bad zone* of $\mathcal{L}$ is a connected component of the set of those points of $S \times I$ in which $\partial\theta/\partial s \leqslant 0$, where $\theta(s,t)$ is the θ-component of $\mathcal{L}(s,t)$.

A bad zone V is *simple* if

(1) $V_t := (S \times t) \cap V$ is connected for all $t \in I$;
(2) the total increment of θ along V_t is less than 2π.

The *shadow of $\mathcal{L}$ on a bad zone* V is the set of those points $(s_0, t_0) \in V$ for which the shortest segment connecting $p_0 := \mathcal{L}(s_0, t_0)$ with the axis Oz meets L_{t_0} at some point $\mathcal{L}(s_1, t_0)$. The set of all such "shading" points (s_1, t_0) will be called the *inverse shadow of* V.

A bad zone V is called *non-shadowed* if the shadow of $\mathcal{L}$ on V is empty.

Lemma 3. *Let $\{L_t\}_{t\in I}$ be a transversal isotopy between transversal geometric braids L_0 and L_1 parameterized by $\mathcal{L} : S \times I \to \mathbb{R}^3$ which is monotone near the axis. Let V be a simple and non-shadowed bad zone and U an arbitrary open subset of $S \times I$ containing V.*

Then $\mathcal{L}$ can be deformed into a transversal isotopy $\tilde{\mathcal{L}} : S \times I \to \mathbb{R}^3$ which is monotone near the axis, coincides with $\mathcal{L}$ outside U, and such that no bad zone of $\tilde{\mathcal{L}}$ meets V.

Proof. Let us write in the cylindric coordinates $\mathcal{L}(s,t) = \big(r(s,t), \theta(s,t), z(s,t)\big)$. Then we have $z'_s + r^2\,\theta'_s > 0$. This implies that $z'_s > 0$ on V. Choose a neighborhood V^+ of V contained in U such that $z'_s \geqslant \varepsilon > 0$ in V^+.

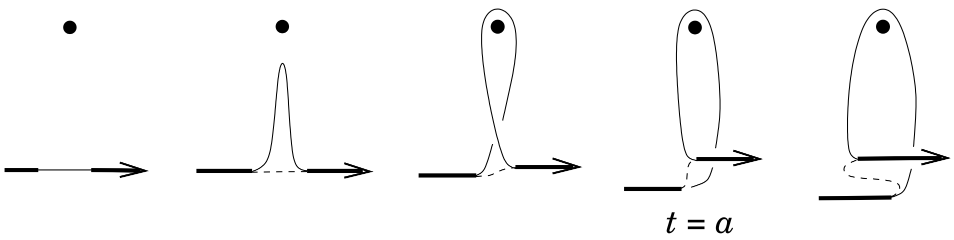

FIGURE 3. ELIMINATION OF A BAD ZONE (PROJECTION ONTO Oxy)

Let $[a, b]$ be the projection of V onto I. We replace the components $x(s, t)$ and $y(s, t)$ of $\mathcal{L}$ in V^+ by the homotopy shown in Figure 3, preserving the component $z(s, t)$.

In Figure 3, the bold lines represent the part of the homotopy which is not changed; the dashed and resp. thin solid lines depict the isotopy before and after the modification; the "•" represents the origin of the plane Oxy. The first three steps in Figure 3 is a deformation of the homotopy described in Definition 1(2), see Figure 1.

Figure 3 depicts the modified homotopy for $t < c$ for some $c \in [a, b]$. To construct the modified homotopy for $t > c$ we perform the same operations in the reverse order. □

Lemma 4. *Let $\{L_t\}_{t\in I}$ be a transversal isotopy between transversal geometric braids L_0 and L_1 parameterized by $\mathcal{L} : S \times I \to \mathbb{R}^3$, which is monotone near the axis. Let $\big(r(s,t), \theta(s,t), z(s,t)\big)$ be a representation of $\mathcal{L}$ in cylindric coordinate. Let V be a bad zone, l a generic smooth embedded curve in V which is the graph of a function $t = \varphi(s)$, and U a neighborhood of l in $S \times I$. Let $\varepsilon > 0$.*

Then there exist a sufficiently small open tubular neighborhood U^- of l in $S \times I$ and a perturbation $\tilde{\mathcal{L}}$ of $\mathcal{L}$ of the form $\tilde{\mathcal{L}} = \big(r(s,t), \tilde\theta(s,t), z(s,t)\big)$ (i.e., only the θ-component is changed), such that

1. *$\partial(V \backslash U^-)$ is smooth.*
2. *$\tilde{\mathcal{L}}$ is monotone near the axis and coincides with $\mathcal{L}$ outside U.*
3. *$\partial\tilde\theta/\partial s$ is positive in $U^- \cap V$ for $\tilde{\mathcal{L}}$.*
4. *the signs of $\partial\theta/\partial s$ and $\partial\tilde\theta/\partial s$ coincide outside $U^- \cap V$.*
5. *$\max\limits_{U^-} \left(\frac{\partial\tilde\theta}{\partial s} / \frac{\partial z}{\partial s}\right) < \varepsilon$.*

Informally speaking, this means that a bad zone can be cut along any smooth curve. The operation described in the proof of Lemma 4 will be called *wrinkling along the curve l*. The left hand side of (5) will be called the *maximal slope of the wrinkling*. The assertion of the lemma in the manifestation of the Gromov's h-principle in this setting.

Proof. In a neighborhood of every point (s_0, t_0) of l we perturb $\theta(s, t)$ by making a small wrinkle on the graph of $\theta(s, t_0)$ at s_0 as it is shown in Figure 4, cf. [2], pp.143–144. □

Let $\{L_t\}_{t\in I}$ be a transversal isotopy. Assume that $\{L_t\}$ is monotone near the Oz-axis and generic outside a small neighborhood of the axis Oz. Then for a generic value t_0 of the parameter t the projection of the link L_{t_0} on the cylinder $S^1 \times \mathbb{R}$ with the coordinates (θ, z) is an immersion and the only singularities of the image

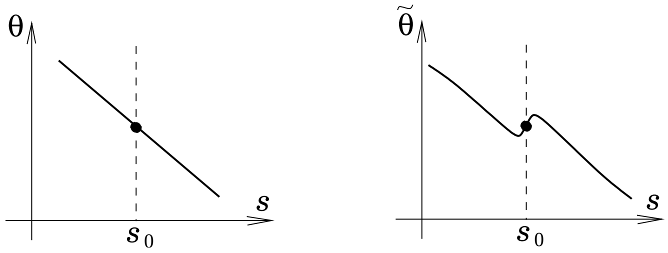

FIGURE 4. WRINKLING

are crossings, i.e., ordinary double points. Moreover, there exist only finitely many values $0 < t_1 < \cdots < t_k < 1$ for which the projection of L_{t_i} on θz-cylinder has a unique singularity of one of the following types:

(I) L_{t_i} meets the axis Oz at some point in the way described in Definition 1.
(II) The projection of L_{t_i} on θz-cylinder has a unique ordinary tangency point.
(III) The projection of L_{t_i} on θz-cylinder has a unique ordinary triple point.

The singularities of types (II) and (III), respectively, are the second and third Reidemeister moves in coordinates (θ, z). The first Reidemeister move in coordinates (θ, z) is impossible for *transversal* links since the derivatives $\frac{\partial z}{\partial s}$ and $\frac{\partial \theta}{\partial s}$ can not both vanish. Instead, a single Reidemeister move of the first kind occurs in every type (I) singularity of a transversal isotopy provided we consider the *projection on Oxy-plane*, see Figure 1.

When we depict a crossing of the θz-projection of a link L_t, we assume that we look from the axis Oz, i.e. the overpass (resp. underpass) corresponds to the arc with a smaller (resp. bigger) value of r. So, we say that an arc with a smaller value of r *passes over* or *shadows* an arc with a bigger value of r (compare with Definition 2).

A singularity of the type (II) or (III) is called *positive* if $\frac{\partial \theta}{\partial s} > 0$ at every point of L_{t_i} which projects on the singularity, and *non-positive* otherwise. A non-positive singularity of the type (II) is called *bad* if there is a negative arc (with $\frac{\partial \theta}{\partial s} > 0$) which is shadowed by another arc at the singularity.

Lemma 5. *Let L be a transversal link. Suppose that the projection onto the θz-cylinder has a bad non-positive singularity of the type* (II)*. Then the both branches are negative at this point.*

Proof. Let the branches be parametrized by $(r_\nu(s), \theta_\nu(s), z_\nu(s))$, $\nu = 1, 2$, so that $r_1 > r_2$ at the crossing point. The tangency means that $z_2'/z_1' = \theta_2'/\theta_1' = \lambda$. Since $\alpha|_L$ is positive, we have $z_j' + r_j^2\theta_j' > 0$, $j = 1, 2$. Since the singularity is bad, we have $\theta_1' < 0$. Suppose that $\theta_2' > 0$. Then $\lambda < 0$ and we have

$$0 < z_2' + r_2^2\theta_2' < z_2' + r_1^2\theta_2' = (z_1' + r_1^2\theta_1')\lambda < 0. \qquad \square$$

Lemma 6. *Any transversal isotopy $\{L_t\}$ monotone near the Oz-axis and generic outside it can be perturbed into a transversal isotopy $\tilde{\mathcal{L}}$ without non-positive singularities of type* (III) *and without bad non-positive singularities of the type* (II)*. Moreover, such a perturbation can be made C^0-small and located in arbitrarily small neighborhoods of the points (s_j, t_j) for which the thread $\mathcal{L}(s, t_j)$ passes though a singularity of the type* (II) *or* (III) *with non-positive derivative $\frac{\partial \theta}{\partial s}$ at $s = s_j$.*

Proof. As in Lemma 4, it is sufficient to perturb only the coordinate θ.

Step 1. Elimination of non-positive triple points. At each non-positive triple point, we perturb all negative branches as in Figure 5a. This can be done by replacing $\theta(s,t)$ with $\tilde{\theta}(s,t) = \theta(s,t) + f(z(s,t),s)$ where the function $f(z,s)$ is the same for all the negative branches. In the case when there are exactly two negative branches, we take care that for any t the crossing point of the perturbed branches rests on the same place as it was before the perturbation. After such modification the triple point becomes positive and no other triple points apear (a priori, new singularities of the type (II) may appear).

Step 2. Elimination of bad tangencies. Consider a bad non-positive singularity of the type (II). By Lemma 5, the both branches are negative at this point. We perturb them in the same way as in Step 1 (see Figure 5b). □

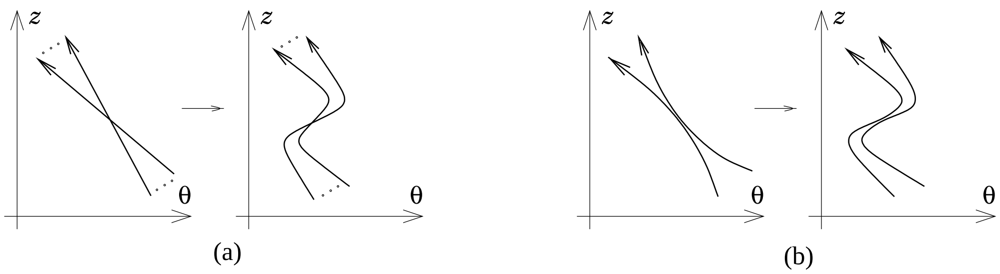

FIGURE 5. ELIMINATION OF BAD NON-POSITIVE SINGULARITIES

Proof of Theorem. By Lemma 1, it is sufficient to prove that any transversal isotopy $\mathcal{L}$ between transversal geometric braids L_0 and L_1 can be transformed into an everywhere monotone isotopy (see Definition 1). By Lemma 2, we may suppose that $\mathcal{L}$ is monotone near the axis Oz.

Wrinkling $\mathcal{L}$ along sufficiently many segments $s = const$ as in Lemma 4, we can assume that all the bad zones are simple. Let us denote them by $V_1, V_2, \dots, V_n$. Fix disjoint neighborhoods U_i's of V_i's. We are going to eliminate the bad zones one by one modifying $\mathcal{L}$ at the i-th step only in $U_i \cup \dots \cup U_n$. This insures that the procedure will terminate. The isotopy obtained after the i-th modification is denoted by $\mathcal{L}_i$ and $\mathcal{L}_0 = \mathcal{L}$ is the initial isotopy. Every $\mathcal{L}_i$ will be monotone near the axis Oz.

To pass from $\mathcal{L}_i$ to $\mathcal{L}_{i+1}$, we proceed as follows (compare with [2], Theorem 8, pp.142–144).

Step 1. Eliminate non-positive singularities of $\mathcal{L}_i$ of the type (III) and bad non-positive singularities of the type (II) applying Lemma 6.

Let us consider connected components $\ell_1, \ell_2, \dots$ of the inverse shadow of V_i on the other bad zones (a bad zone cannot shadow itself because $\partial z/\partial s > 0$ on it). Any point (s,t) of any ℓ_ν corresponds to a crossing of the projection of L_t onto the θz-cylinder. The crossing is either as in Figure 6a or as in Figure 6b.

Step 2. For each component ℓ_ν corresponding to Figure 6b, we wrinkle the corresponding bad zone along it (see Figure 7).

Step 3. Wrinkle V_i along the shadow of $\mathcal{L}_i$ (see Figure 8).

Note that crossings as in Figure 6a are eliminated at Step 2 and the fact that crossings as in Figure 9 are impossible, is proved in [2, pp.142–144] (the proof is similar to that of Lemma 5). If the maximal slope of the wrinkling is small enough

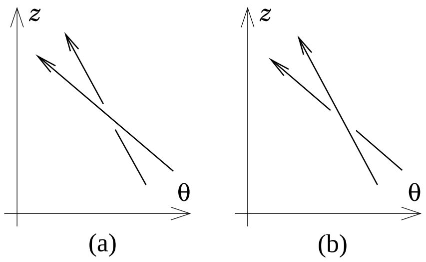

FIGURE 6.

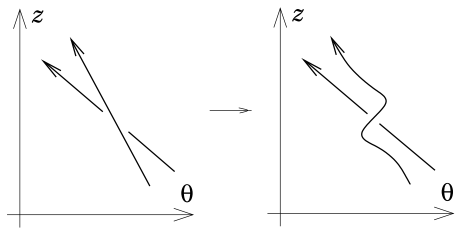

FIGURE 7. WRINKLING AT STEP 2

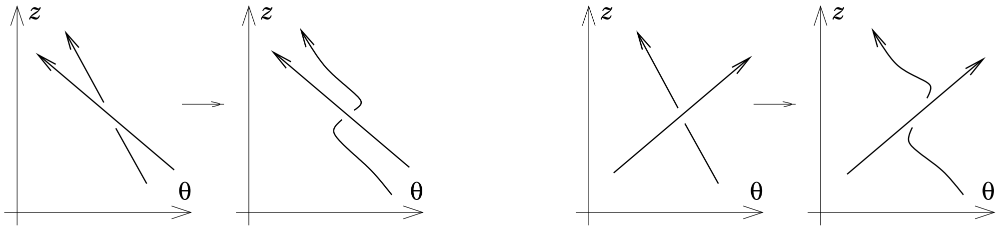

FIGURE 8. WRINKLING AT STEP 3

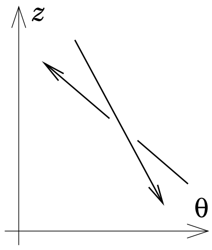

FIGURE 9. IMPOSSIBLE CROSSING

(see condition (5) of Lemma 4), then no new shadow appears because the wrinkling is performed away from tangencies and triple points.

Step 4. Wrinkle, if necessary, the obtained bad zones along segments $s = const$ to make all the bad zones simple.

Step 5. Apply Lemma 3 to all the newly obtained bad zones in U_i. □

Example. According to [5], two transversal unknots are transversally isotopic iff they have the same Bennequin index. The Bennequin index of a transversal geometric braid L corresponding to a braid $b \in B_n$ is equal to $(\sum_i k_i) - n$ for $b = \prod_i \sigma_{j_i}^{k_i}$ (see [2]). Therefore, by our theorem, any braid representing an unknot can be transformed by positive (de)stabilizations and conjugations into the braid $\sigma_1^{-1} \dots \sigma_{n-1}^{-1} \in B_n$ for some n. Here is the sequence of transformations for the braid $\sigma_1^{-1}\sigma_2\sigma_3^{-1}$ (k and $\bar{k}$ stand for σ_k and σ_k^{-1}; M_+^{-1} for a positive destabilization):

$$\bar{1}2\bar{3} = \bar{1}\bar{3}32\bar{3} = \bar{3}\bar{1}32\bar{3} = \bar{3}\bar{1}\bar{2}32 \overset{conj}{\longrightarrow} \bar{1}\bar{2}32\bar{3} = \bar{1}\bar{2}\bar{2}32 \overset{conj}{\longrightarrow} 2\bar{1}\bar{2}\bar{2}3 \overset{M_+^{-1}}{\longrightarrow} 2\bar{1}\bar{2}\bar{2} \overset{conj}{\longrightarrow} \bar{1}\bar{2}.$$

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Remark. It appears that our theorem combined with *Theorem 5.4* of the paper [4] by Fuchs and Tabachnikov implies the "usual" topological Markov theorem. To see this fact, one should only notice that the negative Markov move M^- is the "braid" realization of the destabilization operation ρ used in [4].

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