MARKOV THEOREM FOR TRANSVERSAL LINKS

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Abstract. It is shown that two braids represent transversally isotopic links if and only if one can pass from one braid to another by conjugations in braid groups, *positive* Markov moves, and their inverses.

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By a well-known theorem of Alexander [1], any oriented link in $\mathbb{R}^3$ is isotopic to the closure of a braid. The question when two braids represent isotopic links is answered by Markov's theorem [11] (see [3], [2], or [13] for proofs): It is so if and only if one can pass from one braid to another by conjugations in braid groups B_n, the transformations $M_n^\pm : B_n \to B_{n+1}$, $M_n^+ : b \mapsto b \cdot \sigma_n$, $M_n^- : b \mapsto b \cdot \sigma_n^{-1}$ called *positive/negative Markov moves* or *stabilizations*, and their inverses (*destabilizations*).

In the seminal paper [2] Bennequin established, among other very important results, the analogue of Alexander's theorem for transversal links (i.e., links transverse to the standard contact structure; see below). Namely, any transversal link is transversally isotopic to the closure of a braid. The purpose of this paper is to prove the corresponding analogue of Markov's theorem.

Theorem. *Two braids represent transversally isotopic links if and only if one can pass from one braid to another by conjugations in braid groups, positive Markov moves, and their inverses.*

When this paper had been already finished, we learned from Victor Ginzburg that he had announced this result around 1992. However, his proof has never been published. Another proof of the theorem based on completely different ideas was independently obtained by Nancy Wrinkle in her PhD thesis [14].

Let us recall the standard definitions (see e.g. [2]). Consider the 1-form $\alpha = dz + x\,dy - y\,dx$ in $\mathbb{R}^3$ with coordinates x, y, z. It defines the standard contact structure in $\mathbb{R}^3$. In the cylindric coordinates r, θ, z with $x = r\cos\theta$, $y = r\sin\theta$ one has $\alpha = dz + r^2 d\theta$.

A link L in $\mathbb{R}^3$ is *transversal* if the restriction $\alpha|_L$ nowhere vanishes. In this case $\alpha|_L$ defines a canonical orientation on L.

A *geometric braid* in $\mathbb{R}^3$ is an oriented link L such that the restriction $d\theta|_L$ is positive. In particular, L is disjoint from the z-axis Oz. The *degree* of L, also called the number of strings of L, is the degree of the projection $(r, \theta, z) \mapsto \theta$ restricted to L. There is a canonical one-to-one correspondence between isotopy classes of geometric braids of degree n and conjugacy classes in the braid group B_n.

Any conjugacy class in B_n defines a transversal isotopy class of transversal links. Indeed, any braid $b \in B_n$ can be realized as a geometric braid sufficiently C^1-close to the standard circle $r = 1$, $z = 0$, which is clearly transversal.

The rest of the paper is devoted to the proof of Theorem. Essentially, our proof is a parametric version of Bennequin's proof of his result cited above.

Let L_0 and L_1 be two transversal geometric braids and $\{L_t\}_{t\in[0,1]}$ a transversal isotopy between L_0 and L_1. Denote the interval $[0,1]$ by I, the number of components of L_0 by m, and the disjoint union of m abstract circles by S. Abusing notation, we shall denote by s a positively oriented (local) coordinate on S, as also a current point of S. The isotopy $\{L_t\}_{t\in I}$ can be parameterized by a smooth map $\mathcal{L} : S \times I \to \mathbb{R}^3$ such that for every $t \in I$ the map $\mathcal{L}_t : s \mapsto \mathcal{L}(s,t)$ is a parameterization of L_t.

Definition 1. Let $\{L_t\}_{t\in I}$ be a transversal isotopy parameterized by a map $\mathcal{L} : S \times I \to \mathbb{R}^3$. It is called *monotone near the axis* if there exists a finite number of parameters $0 < t_1 < \cdots < t_k < 1$ such that the following holds:

(1) For any t_i there exists a unique $s_i \in S$ such that $\mathcal{L}(s_i, t_i)$ lies on the z-axis Oz, and $\mathcal{L}^{-1}(Oz) = \{(s_1,t_1),\dots,(s_k,t_k)\}$.
(2) Up to a rotation of $\mathbb{R}^3$ around Oz, the mapping $\mathcal{L}$ is given in a neighborhood of every (s_i,t_i) by $x = \tau - 3s^2$, $y = s\tau - s^3$, $z = z_i + s$, where s is a positively oriented coordinate on S centered at s_i and τ is a coordinate on I centered at t_i and oriented either positively or negatively.

The isotopy $\{L_t\}_{t\in I}$ is *monotone everywhere* if additionally

(3) L_t is a transversal geometric braid for every $t \notin \{t_1,\dots,t_k\}$.

Note, that if we fix $t \neq 0$ and substitute $x = \tau - 3s^2$, $y = s\tau - s^3$ into the 1-form $r^2 d\theta = x\,dy - y\,dx$, we get $r^2 d\theta = (\tau^2 + 3s^4)ds > 0$. Thus, conditions (2) and (3) of Definition 1 are consistent.

We shall always assume that isotopies we consider are sufficiently generic outside a small neighborhood of the axis Oz.

Lemma 1. *Let b_0 and b_1 be two braids, L_0 and L_1 the transversal geometric braids defined by them. Assume that there exists an everywhere monotone isotopy between L_0 and L_1. Then one can pass from b_0 to b_1 by conjugations in braid groups, positive Markov moves, and their inverses.*

Proof. When passing through a critical value $t = t_i$, the projection of L_t onto the horizontal plane Oxy transforms near the origin as in Figure 1. This is a positive Markov move. $\square$

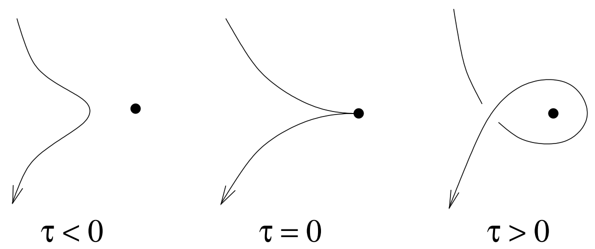

FIGURE 1. THE CURVE $s \mapsto (\tau - 3s^2, s\tau - s^3)$

Lemma 2. *Let $\{L_t\}_{t\in I}$ be a transversal isotopy between transversal geometric braids L_0 and L_1. Then it can be perturbed into an isotopy $\{L'_t\}_{t\in I}$ which is monotone near the axis.*

Proof. Let $\mathcal{L} : S\times I \to \mathbb{R}^3$ be a smooth mapping which parameterizes $\{L_t\}$. Perturbing it if necessary, we can suppose that it is transverse to the axis Oz. Let us consider a point $p = (s_0, t_0) \in S\times I$ such that $\mathcal{L}(p)$ lies on Oz. Let s and t be coordinates on S and I near s_0 and t_0 respectively (with $ds > 0$). Set $\mathcal{L}(s,t) = \big(x(s,t), y(s,t), z(s,t)\big)$. Since all L_t's are transversal braids, we have $\partial z/\partial s > 0$ at p. Hence, there exists a neighborhood U of p such that $\partial z/\partial s > \varepsilon > 0$ in U. Let us modify $\big(x(s,t), y(s,t)\big)$ in U replacing it by the homotopy in Figure 2 (the shaded zone corresponds to the homotopy described in Part (2) of Definition 1 and shown in Figure 1; we assume here that before the modification the homotopy looked as a parallel motion of a vertical line). If U is sufficiently small, then we can achieve that $|r^2\theta'_s| < \varepsilon$ in U, which provides that $\mathcal{L}_t^*\alpha > 0$. $\square$

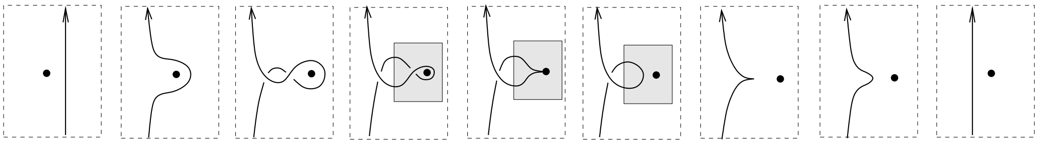

FIGURE 2. MAKING THE ISOTOPY MONOTONE NEAR Oz

Definition 2. Let $\{L_t\}_{t\in I}$ be a transversal isotopy parameterized by a map $\mathcal{L} : S \times I \to \mathbb{R}^3$. A *bad zone* of $\mathcal{L}$ is a connected component of the set of those points of $S \times I$ in which $\partial\theta/\partial s \leqslant 0$, where $\theta(s,t)$ is the θ-component of $\mathcal{L}(s,t)$.

A bad zone V is *simple* if

(1) $V_t := (S \times t) \cap V$ is connected for all $t \in I$;
(2) the total increment of θ along V_t is less than 2π.

The *shadow of $\mathcal{L}$ on a bad zone* V is the set of those points $(s_0, t_0) \in V$ for which the shortest segment connecting $p_0 := \mathcal{L}(s_0, t_0)$ with the axis Oz meets L_{t_0} at some point $\mathcal{L}(s_1, t_0)$. The set of all such "shading" points (s_1, t_0) will be called the *inverse shadow of* V.

A bad zone V is called *non-shadowed* if the shadow of $\mathcal{L}$ on V is empty.

Lemma 3. *Let $\{L_t\}_{t\in I}$ be a transversal isotopy between transversal geometric braids L_0 and L_1 parameterized by $\mathcal{L} : S \times I \to \mathbb{R}^3$ which is monotone near the axis. Let V be a simple and non-shadowed bad zone and U an arbitrary open subset of $S \times I$ containing V.*

Then $\mathcal{L}$ can be deformed into a transversal isotopy $\tilde{\mathcal{L}} : S \times I \to \mathbb{R}^3$ which is monotone near the axis, coincides with $\mathcal{L}$ outside U, and such that no bad zone of $\tilde{\mathcal{L}}$ meets V.

Proof. Let us write in the cylindric coordinates $\mathcal{L}(s,t) = \big(r(s,t), \theta(s,t), z(s,t)\big)$. Then we have $z'_s + r^2\,\theta'_s > 0$. This implies that $z'_s > 0$ on V. Choose a neighborhood V^+ of V contained in U such that $z'_s \geqslant \varepsilon > 0$ in V^+.

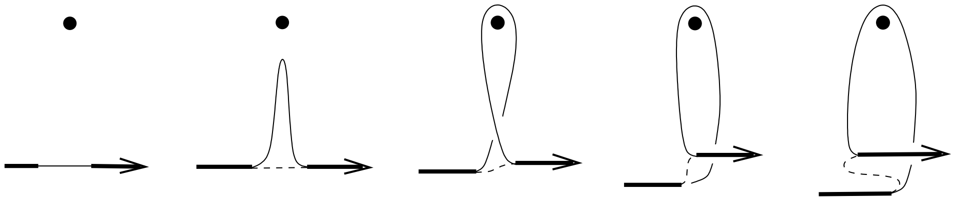

FIGURE 3. ELIMINATION OF A BAD ZONE (PROJECTION ONTO Oxy)

Let $[a, b]$ be the projection of V onto I. We replace the components $x(s,t)$ and $y(s,t)$ of $\mathcal{L}$ in V^+ by the homotopy shown in Figure 3, preserving the component $z(s,t)$.

In Figure 3, the bold lines represent the part of the homotopy which is not changed; the dashed and resp. thin solid lines depict the isotopy before and after the modification; the "$\bullet$" represents the origin of the plane Oxy. The first three steps in Figure 3 is a deformation of the homotopy described in Definition 1(2), see Figure 1.

Figure 3 depicts the modified homotopy for $t < c$ for some $c \in [a, b]$. To construct the modified homotopy for $t > c$ we perform the same operations in the reverse order. $\square$

Lemma 4. *Let $\{L_t\}_{t\in I}$ be a transversal isotopy between transversal geometric braids L_0 and L_1 parameterized by $\mathcal{L} : S \times I \to \mathbb{R}^3$, which is monotone near the axis. Let $\big(r(s,t), \theta(s,t), z(s,t)\big)$ be a representation of $\mathcal{L}$ in cylindric coordinate. Let V be a bad zone, l a generic smooth embedded curve in V which is the graph of a function $t = \varphi(s)$, and U a neighborhood of l in $S \times I$. Let $\varepsilon > 0$.*

Then there exist a sufficiently small open tubular neighborhood U^- of l in $S \times I$ and a perturbation $\tilde{\mathcal{L}}$ of $\mathcal{L}$ of the form $\tilde{\mathcal{L}} = \big(r(s,t), \tilde\theta(s,t), z(s,t)\big)$ (i.e., only the θ-component is changed), such that

1. *$\partial(V \backslash U^-)$ is smooth.*
2. *$\tilde{\mathcal{L}}$ is monotone near the axis and coincides with $\mathcal{L}$ outside U.*
3. *$\partial\tilde\theta/\partial s$ is positive in $U^- \cap V$ for $\tilde{\mathcal{L}}$.*
4. *the signs of $\partial\theta/\partial s$ and $\partial\tilde\theta/\partial s$ coincide outside $U^- \cap V$.*
5. $\max\limits_{U^-}\left(\frac{\partial\tilde\theta}{\partial s}\Big/\frac{\partial z}{\partial s}\right) < \varepsilon$.

Informally speaking, this means that a bad zone can be cut along any smooth curve. The operation described in the proof of Lemma 4 will be called *wrinkling along the curve l*. The left hand side of (5) will be called the *maximal slope of the wrinkling.* The assertion of the lemma in the manifestation of the Gromov's h-principle in this setting.

Proof. In a neighborhood of every point (s_0, t_0) of l we perturb $\theta(s,t)$ by making a small wrinkle on the graph of $\theta(s, t_0)$ at s_0 as it is shown in Figure 4, cf. [2], pp.143–144. $\square$

Let $\{L_t\}_{t\in I}$ be a transversal isotopy. Assume that $\{L_t\}$ is monotone near the Oz-axis and generic outside a small neighborhood of the axis Oz. Then for a generic value t_0 of the parameter t the projection of the link L_{t_0} on the cylinder $S^1 \times \mathbb{R}$ with the coordinates (θ, z) is an immersion and the only singularities of the image

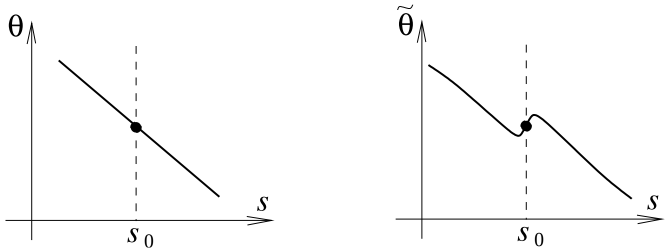

FIGURE 4. WRINKLING

are crossings, i.e., ordinary double points. Moreover, there exist only finitely many values $0 < t_1 < \cdots < t_k < 1$ for which the projection of L_{t_i} on θz-cylinder has a unique singularity of one of the following types:

(I) L_{t_i} meets the axis Oz at some point in the way described in Definition 1.
(II) The projection of L_{t_i} on θz-cylinder has a unique ordinary tangency point.
(III) The projection of L_{t_i} on θz-cylinder has a unique ordinary triple point.

The singularities of types (II) and (III), respectively, are the second and third Reidemeister moves in coordinates (θ, z). The first Reidemeister move in coordinates (θ, z) is impossible for *transversal* links since the derivatives $\frac{\partial z}{\partial s}$ and $\frac{\partial \theta}{\partial s}$ can not both vanish. Instead, a single Reidemeister move of the first kind occurs in every type (I) singularity of a transversal isotopy provided we consider the *projection on Oxy-plane*, see Figure 1.

When we depict a crossing of the θz-projection of a link L_t, we assume that we look from the axis Oz, i.e. the overpass (resp. underpass) corresponds to the arc with a smaller (resp. bigger) value of r. So, we say that an arc with a smaller value of r *passes over* or *shadows* an arc with a bigger value of r (compare with Definition 2).

A singularity of the type (II) or (III) is called *positive* if $\frac{\partial \theta}{\partial s} > 0$ at every point of L_{t_i} which projects on the singularity, and *non-positive* otherwise. A non-positive singularity of the type (II) is called *bad* if there is a negative arc (with $\frac{\partial \theta}{\partial s} > 0$) which is shadowed by another arc at the singularity.

Lemma 5. *Let L be a transversal link. Suppose that the projection onto the θz-cylinder has a bad non-positive singularity of the type* (II). *Then the both branches are negative at this point.*

Proof. Let the branches be parametrized by $(r_\nu(s), \theta_\nu(s), z_\nu(s))$, $\nu = 1, 2$, so that $r_1 > r_2$ at the crossing point. The tangency means that $z_2'/z_1' = \theta_2'/\theta_1' = \lambda$. Since $\alpha|_L$ is positive, we have $z_j' + r_j^2\theta_j' > 0$, $j = 1, 2$. Since the singularity is bad, we have $\theta_1' < 0$. Suppose that $\theta_2' > 0$. Then $\lambda < 0$ and we have

$$0 < z_2' + r_2^2\theta_2' < z_2' + r_1^2\theta_2' = (z_1' + r_1^2\theta_1')\lambda < 0. \qquad \square$$

Lemma 6. *Any transversal isotopy $\{L_t\}$ monotone near the Oz-axis and generic outside it can be perturbed into a transversal isotopy $\tilde{\mathcal{L}}$ without non-positive singularities of type* (III) *and without bad non-positive singularities of the type* (II). *Moreover, such a perturbation can be made C^0-small and located in arbitrarily small neighborhoods of the points (s_j, t_j) for which the thread $\mathcal{L}(s, t_j)$ passes though a singularity of the type* (II) *or* (III) *with non-positive derivative $\frac{\partial \theta}{\partial s}$ at $s = s_j$.*

Proof. As in Lemma 4, it is sufficient to perturb only the coordinate θ.

Step 1. Elimination of non-positive triple points. At each non-positive triple point, we perturb all negative branches as in Figure 5a. This can be done by replacing $\theta(s,t)$ with $\tilde\theta(s,t) = \theta(s,t) + f(z(s,t),s)$ where the function $f(z,s)$ is the same for all the negative branches. In the case when there are exactly two negative branches, we take care that for any t the crossing point of the perturbed branches rests on the same place as it was before the perturbation. After such modification the triple point becomes positive and no other triple points apear (a priori, new singularities of the type (II) may appear).

Step 2. Elimination of bad tangencies. Consider a bad non-positive singularity of the type (II). By Lemma 5, the both branches are negative at this point. We perturb them in the same way as in Step 1 (see Figure 5b). $\square$

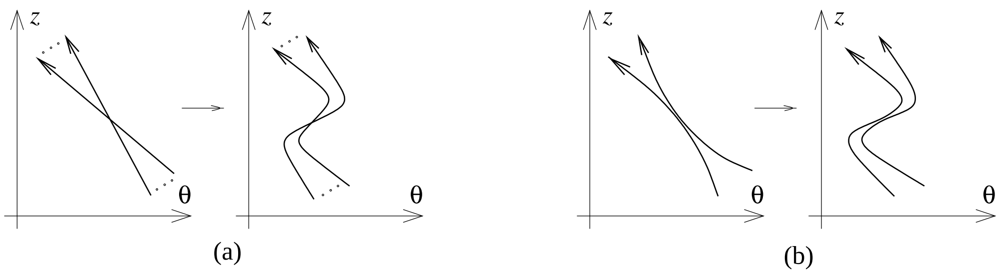

FIGURE 5. ELIMINATION OF BAD NON-POSITIVE SINGULARITIES

Proof of Theorem. By Lemma 1, it is sufficient to prove that any transversal isotopy $\mathcal{L}$ between transversal geometric braids L_0 and L_1 can be transformed into an everywhere monotone isotopy (see Definition 1). By Lemma 2, we may suppose that $\mathcal{L}$ is monotone near the axis Oz.

Wrinkling $\mathcal{L}$ along sufficiently many segments $s = const$ as in Lemma 4, we can assume that all the bad zones are simple. Let us denote them by $V_1, V_2, \dots, V_n$. Fix disjoint neighborhoods U_i's of V_i's. We are going to eliminate the bad zones one by one modifying $\mathcal{L}$ at the i-th step only in $U_i \cup \dots \cup U_n$. This insures that the procedure will terminate. The isotopy obtained after the i-th modification is denoted by $\mathcal{L}_i$ and $\mathcal{L}_0 = \mathcal{L}$ is the initial isotopy. Every $\mathcal{L}_i$ will be monotone near the axis Oz.

To pass from $\mathcal{L}_i$ to $\mathcal{L}_{i+1}$, we proceed as follows (compare with [2], Theorem 8, pp.142–144).

Step 1. Eliminate non-positive singularities of $\mathcal{L}_i$ of the type (III) and bad non-positive singularities of the type (II) applying Lemma 6.

Let us consider connected components $\ell_1, \ell_2, \dots$ of the inverse shadow of V_i on the other bad zones (a bad zone cannot shadow itself because $\partial z/\partial s > 0$ on it). Any point (s,t) of any ℓ_ν corresponds to a crossing of the projection of L_t onto the θz-cylinder. The crossing is either as in Figure 6a or as in Figure 6b.

Step 2. For each component ℓ_ν corresponding to Figure 6b, we wrinkle the corresponding bad zone along it (see Figure 7).

Step 3. Wrinkle V_i along the shadow of $\mathcal{L}_i$ (see Figure 8).

Note that crossings as in Figure 6a are eliminated at Step 2 and the fact that crossings as in Figure 9 are impossible, is proved in [2, pp.142–144] (the proof is

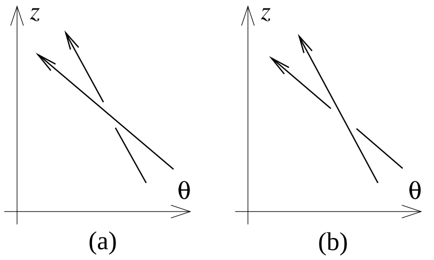

FIGURE 6.

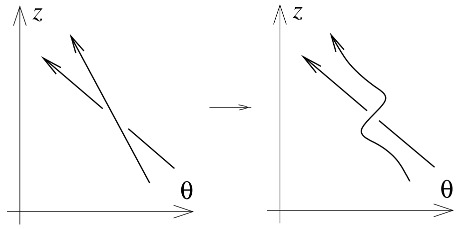

FIGURE 7. WRINKLING AT STEP 2

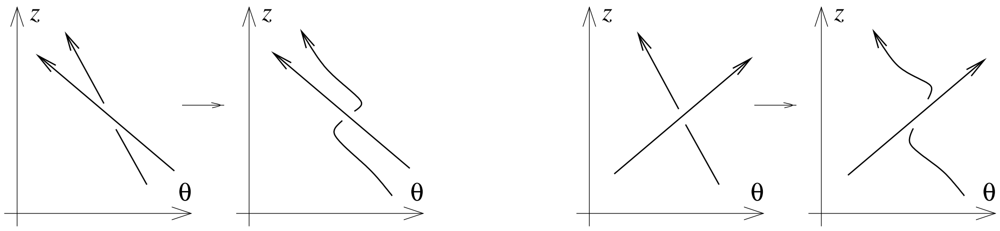

FIGURE 8. WRINKLING AT STEP 3

similar to that of Lemma 5). If the maximal slope of the wrinkling is small enough (see condition (5) of Lemma 4), then no new shadow appears because the wrinkling is performed away from tangencies and triple points.

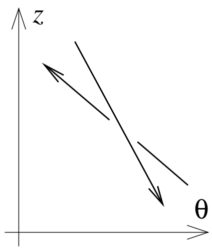

FIGURE 9. IMPOSSIBLE CROSSING

Step 4. Wrinkle, if necessary, the obtained bad zones along segments $s = const$ to make all the bad zones simple.

Step 5. Apply Lemma 3 to all the newly obtained bad zones in U_i. □

Example. According to [7], two transversal unknots are transversally isotopic iff they have the same Bennequin index. The Bennequin index of a transversal geometric braid L corresponding to a braid $b \in B_n$ is equal to $(\sum_i k_i) - n$ for $b = \prod_i \sigma_{j_i}^{k_i}$ (see [2]). Therefore, by our Theorem, any braid representing an unknot can be transformed by positive (de)stabilizations and conjugations into the braid $\sigma_1^{-1} \dots \sigma_{n-1}^{-1} \in B_n$ for some n. Here is the sequence of transformations for the braid $\sigma_1^{-1}\sigma_2\sigma_3^{-1}$ (k and $\bar{k}$ stand for σ_k and σ_k^{-1}; M_+^{-1} for a positive destabilization):

$$\bar{1}2\bar{3} = \bar{1}\bar{3}32\bar{3} = \bar{3}\bar{1}32\bar{3} = \bar{3}\bar{1}\bar{2}32 \xrightarrow{conj} \bar{1}\bar{2}32\bar{3} = \bar{1}\bar{2}\bar{2}32 \xrightarrow{conj} 2\bar{1}\bar{2}\bar{2}3 \xrightarrow{M_+^{-1}} 2\bar{1}\bar{2}\bar{2} \xrightarrow{conj} \bar{1}\bar{2}.$$

Appendix. Markov's Theorem from the Point View of Contact Topology.

Here we discuss some "classical" and recent results on contact isotopy of Legendrian and transversal knots in $\mathbb{R}^3$ and deduce "topological" Markov's theorem from its contact version.

We start with a brief description of related notions and constructions, referring to the articles [9] and [10] for more details. Notice that the contact structure in $\mathbb{R}^3$ use there is given by the form $\alpha_{jet} := dz - ydx$ and originates from the identification of $\mathbb{R}^3$ with the space $J^1\mathbb{R}$ of 1-jets of functions on the real axis $\mathbb{R}$. The substitution $(x, y, z) \mapsto (x, 2y, z + xy)$ transforms α_{jet} into the rotation invariant form $\alpha_{rot} := dz + xdy - ydx = dz + r^2 d\theta$ used in the main part. Thus both forms define the same contact structure. The advantage of the form α_{jet} is that it provides a possibility to control over Legendrian and transversal knots by their projections on xy- and xz-planes.

A link L in $\mathbb{R}^3$ is *Legendrian* (*transversal*) w.r.t. a contact form α if the restriction $\alpha|_L$ vanishes identically (never vanishes). The *contact* orientation of a transversal link L is induced by the restriction $\alpha|_L$. A link isotopy $\{L_t\}$ is Legendrian (resp., transversal) if every L_t has this property. We always assume that Legendrian, transversal, and usual ("topological") isotopies preserve the orientation of the link.

Every link in $\mathbb{R}^3$ admits both Legendrian and transversal representation. Legendrian and transversal links have additional $\mathbb{Z}$-valued invariants constraining the existence of a contact isotopy: these are the Maslov and Thurston-Bennequin indices in the Legendrian case and the Thurston-Bennequin index in the transversal case, denoted by $\mu(L)$ and $tb(L)$ respectively.

Assume that L_1 and L_2 are disjoint links, both Legendrian or transversal. Let $lk(L_1, L_2)$ be their linking number. Then $\mu(L_1 \sqcup L_2) = \mu(L_1) + \mu(L_2)$ (linear behavior) and $tb(L_1 \sqcup L_2) = tb(L_1) + 2lk(L_1, L_2) + tb(L_2)$ (quadratic behavior). This reduces the computation of the indices to the case of knots.

The Thurston-Bennequin index of a *knot* is independent of its orientation, while the Maslov index changes the sign if we reverse the orientation. The Thurston-Bennequin index of a transversal link $L(b)$ represented by an algebraic braid b with n strands equals $tb(L(b)) = \deg(b) - n$ where $\deg(b)$ is the algebraic degree of b.

Every oriented Legendrian link L can be smoothly approximated by a transversal link L^+ whose contact orientation coincides with that induced from L. Moreover, such a link L^+ is unique up to transversal isotopy. Similarly, there exists a unique transversal isotopy class of links L^- which approximate L with the reversed orientation. The indices of $L^\pm$ are related to those of L as $tb(L^\pm) = tb(L_0) \pm \mu(L_0)$.

There exist several constrains on possible values of Maslov and Thurston-Bennequin indices of Legendrian and transversal links in $\mathbb{R}^3$. The first one is that $tb(L)$ (resp., $tb(L) \pm \mu(L)$) has the same parity as the number of components of the transversal (Legendrian) link L. In particular, $tb(L)$ is odd for every transversal knot. Another constrain is the Bennequin inequality $tb(L) \le -\chi(F)$ for every transversal link L and its Seifert surface F. Unlike the first constrain, this one is highly non-trivial and reflects the fact that the standard contact structure in $\mathbb{R}^3$ is *tight* (see [7] for more details). For a Legendrian link L this inequality reads $tb(L) + |\mu(L)| \le -\chi(F)$. Some further inequalities are listed in [10].

It is always possible to decrease the Thurston-Bennequin index of a Legendrian or transversal knot L. More precisely, there exists transformations ζ_+ and ζ_- (resp., a transformation ρ) of isotopy classes of oriented Legendrian (resp., transversal) knots with the following properties:

(1) The transformations $\zeta_\pm$ and ρ can be realized by adding an appropriate unknotted loop in any given neighborhood of any given point on L; in particular, they can be represented by an appropriate Legendrian knot $\zeta_\pm L$ (resp., a transversal knot ρL) in the topological isotopy class of L.
(2) The operations ζ_+ and ζ_- commute, i.e., there exists a Legendrian isotopy between $\zeta_+(\zeta_- L)$ and $\zeta_-(\zeta_+ L)$.
(3) If a braid b represents a transversal knot L, then the braid $M^- b$ obtained from b by negative stabilization represents ρL.
(4) $tb(\zeta_\pm L) = tb(L) - 1$ and $\mu(\zeta_\pm L) = \mu(L) \pm 1$ in the Legendrian case; $tb(\rho L) = tb(L) - 2$ in the transversal case.
(5) For an oriented Legendrian knot L, the knot $(\zeta_+ L)^+$ (see above) is transversally isotopic to L^+ and the knot $(\zeta_+ L)^-$ to $\rho(L^-)$; similarly, the knot $(\zeta_- L)^-$ is transversally isotopic to L^- and the knot $(\zeta_- L)^+$ to $\rho(L^-)$.
(6) The transformations $\zeta_\pm$ and ρ naturally extend to links; one should only indicate to which component of the link the operation is applied.

We refer to [10] for the definition of the transformations $\zeta_\pm$ and the proof of the properties (1–5). However, it should be noticed that these transformations are known well enough as a part of the contact topology folklore, so looking for references would be an ungrateful task. The property (3) can be used as the definition of the operation ρ. The property (5) means that, informally speaking, after the "positive (negative) transversalization" $L \mapsto L^+$ (resp., $L \mapsto L^-$) the operation $\zeta_\pm$ descends to the stabilization $M^\pm$ (resp., $M^\mp$) of the same (resp., opposite) sign.

Proposition A. *Let L_1 and L_2 be two oriented Legendrian (resp., transversal) links which are topologically isotopic; then one can obtain Legendrian (resp., transversal) isotopic links L_1' and L_2' applying the operations $\zeta_\pm$ (resp., ρ) to each component of L_1 and L_2 sufficiently many times.*

Proposition B. *Let L_1 and L_2 be two oriented Legendrian links; then the links L_1^+ and L_2^+ are transversally isotopic if and only if one can transform L_1 into L_2 applying Legendrian isotopies, the operation ζ_+, and its inverse.*

In the case of knots Proposition A was proved in [10] and Proposition B in [9]. However, since the condition of being connected is not used in the both proofs, the general case follows as well. In view of the property (3), our Theorem and Proposition A imply Markov's theorem for knots in the refined form stated in Introduction. As one can easily see the refined form remains valid in the case of links after an appropriate generalization of negative (de)stabilizations. Such a generalization should represent the operation ρ applied to any prescribed component of the link. For example, one can take operations

$$M_k^- : b \in B_n \ \mapsto \ \sigma_{n-1} \cdots \sigma_k \, b \, \sigma_k^{-1} \cdots \sigma_{n-1}^{-1} \sigma_n^{-1} \in B_{n+1}$$

which are compositions of the conjugation in B_n by $\sigma_{n-1} \cdots \sigma_k$ with the negative stabilization M^-.

In view of Proposition A, the authors of [10] have expressed the conjecture that the transversal (Legendrian) isotopy class of a knot is completely determined by its topological isotopy class and its Thurston-Bennequin (and Maslov) index. This conjecture has been disproved by Yuriĭ Chekanov who has constructed [6] new invariants of Legendrian knots and has given an example of two Legendrian knots which are topologically isotopic and have equal Thurston-Bennequin and Maslov indices but different Chekanov's invariants. Some examples of even finer type have been found in [9]. Namely, there exist Legendrian knots L_1 and L_2 which have equal Thurston-Bennequin and Maslov classes and transversally isotopic "transversalizations" L_1^+ and L_2^+, but nevertheless L_1 and L_2 are not Legendrian isotopic. On the other hand, the Legendrian isotopy class of the unknot is completely determined by its Thurston-Bennequin and Maslov indices, see [7] and [8].

A similar counterexample for transversal knots has been constructed in [4]. It is shown that the braids

$$\sigma_1^{2p+1}\sigma_2^{2q}\sigma_1^{2r}\sigma_2^{-1} \quad \text{and} \quad \sigma_1^{2p+1}\sigma_2^{-1}\sigma_1^{2r}\sigma_2^{2q} \quad \text{with } p,q,r>1 \text{ and } q\neq r$$

represent the knots K_1 and K_2 which are topologically isotopic and have equal Thurston-Bennequin indices but which are not isotopic transversally. On the other hand, there are several types of knots and links for which the transversal isotopy class is completely determined by its topological isotopy class and Thurston-Bennequin indices of the components, see [5]. For example, those are unlinks and iterated torus knots.

The discussions made so far lead to the following problems:

P1 *Does there exist two Legendrian knots L_1 and L_2 which are not Legendrian isotopic, but the "transversalizations" of both signs L_1^+ and L_2^+ (resp., L_1^- and L_2^-) are transversally isotopic?*

The negative answer to this question is conjectured (indirectly) in [9].

P2 *Find an analogue of Alexander's and Markov's theorems for Legendrian links.*

We finish the paper with a description of a natural construction of closed Legendrian braids. It can be considered as the first step toward the solution of Problem P2. First, we describe possible Legendrian isotopy classes of unknots. Let $\bar{L}_{0,0}$ be the curve in the xz-plane given by the equation $z^2=\cos^3(x)$ with $|x|\le\pi/2$ and $|z|\le 1$, see Figure 10.

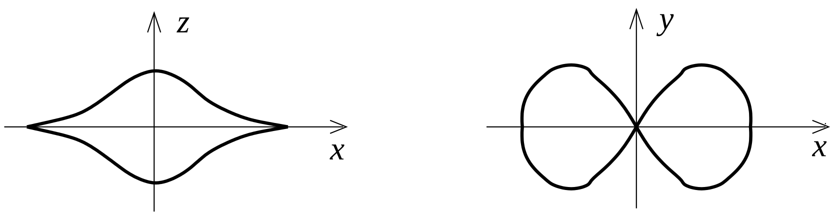

FIGURE 10. THE xz- AND xy-PROJECTIONS OF $L_{0,0}$.

This curve lifts uniquely to a smooth Legendrian curve $L_{0,0}$ in $\mathbb{R}^3$ with the standard contact structure given by $dz-ydx$. Namely, the lift of each branch given $z(x)=\pm\cos^{3/2}(x)$ is parameterized by $(x,z'(x),z(x))$ with $z'(x):=\frac{dz(x)}{dx}=\mp\frac{3}{2}\cos^{1/2}(x)\sin(x)$. Observe that in a neighborhood of each cusp-point $(\pm\frac{\pi}{2},0,0)$ the curve $L_{0,0}$ admits the parameterization

$$x(t)=\pm\arccos(t^2),\quad y(t)=\mp\tfrac{3}{2}t\sqrt{1-t^4},\quad z(t)=t^3$$

with t close to 0. This shows that $L_{0,0}$ is a smooth Legendrian unknot. Direct computation gives $\mu(L_{0,0}) = 0$ and $tb(L_{0,0}) = -1$, see [10], §3.4.

Set $L(p,q) := \zeta_+^p \circ \zeta_-^q(L_{0,0})$. Then $\mu(L_{p,q}) = p-q$ and $tb(L_{p,q}) = -1-p-q$. By the results of Bennequin and Eliashberg that every Legendrian unknot L is Legendrian isotopic to $L(p,q)$ with $p = (\mu(L) - tb(L) - 1)/2$ and $q = (-\mu(L) - tb(L) - 1)/2$.

Now assume that L_b is a Legendrian braid with the "axis" L_a, which is also a Legendrian knot. Then there exists a tubular neighborhood $U \cong \Delta \times L_a$ of L_a and coordinates $(v, w; \theta)$ in U such that

1. θ is the coordinate along $L_a \cong S^1$;
2. $(v, w) \in \Delta$ are normal coordinates to L_a in U;
3. the contact structure in U is given by the form $dv - w d\theta$;
4. the projection $\pi_{v,\theta} : L_b \to [0,1] \times L_a$ of L_b onto the strip $[0,1] \times L_a$ with coordinates (v, θ) has only simple transversal crossings.

Observe that the projection of L_b onto (v,θ)-strip determines L_b completely. Indeed, if $(v,w) = (f_i(\theta), g_i(\theta))$ is a local parameterization of a strand of L_b, then $g_i(\theta)$ is the derivative of $f_i(\theta)$, $g_i(\theta) = f_i'(\theta)$. It follows then that the projection has only *positive* crossings.

Vice versa, given a Legendrian knot L_a and a positive braid b, there exists a Legendrian link L_b realized as the closure of b in arbitrary tubular neighborhood U of L_a. Moreover, the Legendrian isotopy class of such a link L_b is well-defined. We shall use the notation $L_a \ltimes b$ to denote such a link L_b.

Lemma 6.

1. *The link $L_{p,q} \ltimes b$ is represented by the braid $\Delta^{-p-q-1} \cdot b$.*
2. *For any Legendrian knot L_a and a positive braid $b \in B_+(n)$,*
   $\mu\big(L_a \ltimes b\big) = \mu(L_a) \cdot n$ *and* $tb\big(L_a \ltimes b\big) = n^2 tb(L_a) + \deg(b)$.

In particular, $\mu\big(L_{p,q} \ltimes b\big) = n(p-q)$ *and* $tb\big(L_{p,q} \ltimes b\big) = \deg(b) - n^2(p+q+1)$.

Observe that every braid $b \in B(n)$ can be decomposed as $b = \Delta^{-k} \cdot b_+$ with appropriate $k \geq 0$ and $b_+ \in B_+(n)$.

Proof. Every Legendrian knot L in $\mathbb{R}^3$ has two natural framings: the Legendrian one given by the contact distribution $\xi := ker(dz - ydx) \subset T\mathbb{R}^3$ and the topological one given by its Seifert surface. In particular, the coordinates (v, w, θ) in a tubular neighborhood of L introduced above define the Legendrian framing. By definition, the Thurston-Bennequin index $tb(L)$ is the linking number between L and the knot L' obtained from L by pushing it slightly in the positive (or negative) normal direction to the contact distribution $\xi = ker(dz - ydx)$. Thus $tb(L)$ is the rotation number of the Legendrian framing with respect to the topological one. The part (1) of the lemma follows.

It follows from definition that the Maslov index of a Legendrian link L in $\mathbb{R}^3$ is the winding number of the projection of L onto xy-plane. Since every strand of $L_a \ltimes b$ is C^1 close to L_a we immediately obtain $\mu\big(L_a \ltimes b\big) = \mu(L_a) \cdot n$.

Now assume that $b_0 \in B_+(n)$ is the trivial braid. Let L_i, $i = 1 \dots n$, be the strands of $L_a \ltimes b_0$. Then every L_i is Legendrian isotopic to L_a and represents the "push in the direction normal to ξ". So the linking number $lk(L_i, L_j) = tb(L_i) = tb(L_a)$. Then $tb(L_a \ltimes b_0) = \sum_i tb(L_i) + \sum_{i<j} 2lk(L_i, L_j) = n^2 tb(L_a)$.

To obtain the general case, we use the algorithm for computing of the Thurston-Bennequin index of a Legendrian link L in $\mathbb{R}^3$ by its projection onto xz-plane, see

[10], §3.4. After a small Legendrian perturbation, the only singularities of such a projection are transversal crossings and cusps. A crossing is called positive (negative) if both strands cross the vertical line in the same (resp., opposite) direction. Then $tb(L)$ is the number of positive crossings minus the number of negative crossings minus half the number of cusps. Now, it remains to observe that for $b \in B_+(n)$ the xz-projections of $L_a \ltimes b$—compared with that of $L_a \ltimes b_0$—has exactly $\deg(b)$ additional positive crossings. $\square$

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