LINK THEORY AND NEW RESTRICTIONS FOR M-CURVES OF DEGREE 9

S.Yu. Orevkov

All known restrictions on the topology of real algebraic curves in $\mathbf{RP}^2$ of odd degrees ≤ 9 are obtained combining Bézout theorem, Rokhlin's formula [1] for complex orientations and Fiedler theorem [2] on interchanging of orientations. In this way Viro [3] completed the isotopic classification of real curves of degree 7. Korchagin [4] systematically applied this method for M-curves of degree 9 but it happened not to be enough. An example of a new prohibition for 9th degree curves given by the author in [5; (1.5.4)], was obtained essencially also by the same method but using a new formula for complex orientations.

However, there are rather powerful methods to obtain restrictions for curves of even degrees based on construction of 2-cycles in the double covering of $\mathbf{CP}^2$ branched along the complexification of the curve (homological method). They can not be applied directly for odd degrees by the simple reason that the double covering does not exist. To avoid this problem, one can consider coverings of higher degree. In this way the inequality of Viro-Zvonilov was obtained [6]. Another way, proposed by the author in [5], is to consider the Seifert form of certain link instead of the intersection form on the double covering. Then signature estimates take form of Murasugi-Tristram inaquality.

The aim of this note is to show that this method works for projective curves of odd degrees. We did not apply systematically this method to all the M-schemes of degree 9 whose realizability is still unknown (there are several hundreds of them). The prohibition proven here is just the first example where we found that the homological method works while other known methods do not.

Theorem 1. *There does not exist M-curves of degree 9 in $\mathbf{RP}^2$ with real schemes*[1]

$$\langle J \sqcup \beta \sqcup 1\langle 1\langle \alpha\rangle\rangle\rangle \qquad \text{where } \alpha+\beta = 26, \tag{1}$$

$$\beta = 0,1,2,3,4,5,6,8,10. \tag{2}$$

The following two lemmas are not difficult and we omit their proofs.

Lemma 1. *If a complex scheme corresponding to (1), (2) satisfies [5; (1.4A)] then $|\alpha_+ - \alpha_-| \ge 2$ where α_+ (resp. α_-) is the number of positive (resp. negative) ovals among the α the most inner ovals.*

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[1]See [7] for the definition of real and complex schemes.

Lemma 2. *Let P be a convex n-gon in $\mathbf{RP}^2$ whose vertices lye on a real cubic. If $n > 6$ then for all sides $[ab]$ of P, except maybe two, the points a and b are connected by a segment of the cubic not containing the other vertices of P.*

Proof of Theorem 1. It follows from Lemma 1 and the orientation interchanging rule that there exists a line through two outer empty[2] ovals o_1, o_2 separating some two of the inner empty ovals o_3, o_4. Any conic through $o_1, ..., o_4$ meets the 2 non-empty ovals in ≥ 8 points, hence, if it meets one more empty oval, it can not meet the odd component J. Therefore, all empty ovals lye around vertices of some convex 26-gon Q non-intersecting J (one vertex in each empty oval). Denote by Q_{inn} (resp. Q_{out}) the set of vertices of Q lying in inner (resp. outer) empty ovals. If the number of sides of Q connecting Q_{inn} with Q_{out} were > 6 then there would exist a convex octagon P whose odd vertices belong to Q_{inn} and even ones belong to Q_{out}. Then, by Lemma 2, a cubic through vertices of P and one more vertex of Q would cut the curve in ≥ 28 points. Thus, if one choose a point $p \in Q_{inn}$ connected to Q_{out} by an edge of Q then, in the notation of [5, Sect. 3.5], the scheme of the arrangement of the curve with respect to the pencil of lines through p, has form

$$[\supset_7 o_7^{\alpha_1 - 2} o_5^{\beta_1} o_7^{\alpha_2} o_5^{\beta_2} o_7^{\alpha_3} o_5^{\beta_3} \subset_2], \qquad \sum \alpha_i = \alpha, \ \sum \beta_i = \beta. \tag{3}$$

Denote by $\sigma_1, \dots, \sigma_{m-1}$ the standard generators of the braid group B_m. Put $\pi_{k,l} = \sigma_k \dots \sigma_l$, $\tau_{k,l} = \pi_{l,k-1}^{-1} \pi_{k,l+1}$ $(k > l)$, and $\Delta_m = \pi_{1,m-1} \pi_{1,m-2} \dots \pi_{1,1}$. Applying to (3) the procedure described in [5; Sect. 3.5], we obtain the braid

$$b = b_{\alpha_1,\beta_1,\dots,\alpha_3,\beta_3} = \sigma_7 \Big(\prod_{i=1}^{3} \sigma_7^{-\alpha_i} \tau_{7,5} \sigma_5^{-\beta_i} \tau_{7,5}^{-1} \Big) \tau_{7,2} \Delta_9 \in B_9. \tag{4}$$

It is shown in [5] that a necessary condition for the realisability of (3) is Murasugi-Tristram inequality which can be rewritten in the form

$$n(\hat{b}) \geq |\sigma(\hat{b})| + m - e(b) \tag{5}$$

where $m = 9$ is the degree, $e(b) = 11$ is the sum of exponents of b, and $\sigma(\hat{b})$, $n(\hat{b})$ are respectively the signature and the nullity of the link $\hat{b}$ which is the closure of the btaid b. According to [5; Sect. 2.6], we have $\sigma(\hat{b}_{\alpha_1 \dots \beta_3}) = \sum \alpha_i + \sum \beta_i + \sigma(W)$ and $\det \hat{b} = \det W$ where W is a matrix which can be explicitely written from the presentation (4), all whose entries are constants independent of α_i, β_i except 6 diagonal entries, equal to $\alpha_1, \dots, \beta_3$. A computation shows that $\det W = 16(\Sigma\alpha_i + \Sigma\beta_i)$ and $\sigma(\hat{b}_{26,0,\dots,0}) = -5$. Since $\det W \neq 0$ when the point $(\alpha_1, ..., \beta_3)$ belongs to the hyperplane $\sum \alpha_i + \sum \beta_i = 26$, the signature of the matrix W is constant along this hyperplane. Hence, $\sigma(\hat{b}) = -5$ for all b. The fact $\det W \neq 0$ means $n(\hat{b}) = 1$. This contradicts (5). The Theorem 1 is proven.

Theorem 2. *There does not exist M-curves of degree 9 in $\mathbf{RP}^2$ with the real schemes (1) for even β.*

Remarks. **1.** The scheme (1) with $\alpha = 0, 1$ is realizable; non-realizability of (1) for $\beta = 0, 1$ was already known (see [4]).

[2] An oval is called *empty* if there are no other ovals inside it.

2. The complex schemes

$$\langle J \sqcup (\tfrac{\beta+7}{2})_+ \sqcup (\tfrac{\beta-7}{2})_- \sqcup 1_\pm\langle 1_\mp\langle (\tfrac{\alpha-1}{2})_+ \sqcup (\tfrac{\alpha+1}{2})_-\rangle\rangle\rangle,$$

$$\langle J \sqcup (\tfrac{\beta+12}{2})_+ \sqcup (\tfrac{\beta-12}{2})_- \sqcup 1_-\langle 1_-\langle (\tfrac{\alpha}{2})_+ \sqcup (\tfrac{\alpha}{2})_-\rangle\rangle\rangle \tag{6}$$

show that Lemma 1 is not true for values of β not listed in Theorem 1.
3. The only known to the author restriction for odd degrees obtained by methods not mentioned in the introduction is the prohibition of M-schemes $\langle J \sqcup 1\langle\alpha\rangle\rangle$ proven by V.Kharlamov (unpublished) using real θ-characteristics related to conic bundles.

Sketch of the proof of Theorem 2. The complex schemes (6) are the only ones for which the proof of Theorem 1 fails. Choose again a pencil of lines through a point in one of the ovals $\langle\alpha\rangle$. Using Bézout theorem for auxilary conics one can show that all other ovals of $\langle\alpha\rangle$ are not separated by lines of this pencil passing through the ovals $\langle\beta\rangle$. Thus, the schemes of the arrangement with respect to the pencil have form $[\supset_7 o_7^{\alpha-2} w_1 \dots w_h \subset_2]$ where $w_i = o_5^{\beta_i} o_4^{\beta'_i}$, $\sum(\beta_i + \beta'_i) = \beta$ and $\beta_i, \beta'_i > 0$. The corresponding braids are

$$b = \sigma_7^{1-\alpha}\tau_{7,5}\Big(\prod_{i=1}^{h}\sigma_5^{-\beta_i}\tau_{5,4}\sigma_4^{-\beta'_i}\tau_{4,5}\Big)\tau_{5,2}\Delta_9\,.$$

Obviousely, $2h \le \beta - 2$, and it follows from (6) that $h \ge 12$. For any given h one can compute $\det\hat b$ as a polynomial in α, β_i, β'_i. If all the variables are ≥ 1 then it does not vanish. Therefore, like in the proof of Theorem 1, one can show that $\sigma(\hat b)$ is constant for each h. Calculating $\sigma(\hat b)$ for $\beta_i = \beta'_i = 1$, we see that it equals $2h - 7$. Since $n(\hat b) = 1$ this contradicts (5).

The straight-forward computation of $\det\hat b$ for $h > 20$ is so huge that it hardly can be performed even with a computer. However, using additional topological arguments, one can compute the signature in a simpler and more natural way. The details will be published in another paper.

Theorem 3. *There does not exist an M-curve of degree 7 with complex scheme* $\langle J \sqcup 10_+ \sqcup 3_- \sqcup 1_-\langle 1_-\rangle\rangle$.

Proof. The arrangement of the curve with respect to a pencile of lines whose center is in the nest $1\langle 1\rangle$ has form $[\supset_3 o_3^{\beta_0-1} w_1 \dots w_h \subset_4]$ where $w_i = o_4^{\beta_{2i-1}} o_3^{\beta_{2i}}$, $\sum\beta_i = 13$. It follows from the orientations interchanging rule that $h \ge 3$. Computations show that $e(b) = 8$, $\sigma(\hat b) = -2h$, $n(\hat b) = 1$. This contradicts (5) for $m = 7$.

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Steklov Math. Inst., Russ. Acad. Sci. Gubkina 8, Moscow, 117966, Russia