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CLASSIFICATION OF FLEXIBLE M-CURVES OF DEGREE 8 UP TO ISOTOPY

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1 Introduction

In the first part of his 16-th problem, Hilbert asked how the connected components of a plane real algebraic curve of degree m can be arranged on $\mathbf{RP}^2$ up to isotopy. At that time, the answer was known only for $m \le 5$. Gudkov [GuU] solved this problem for $m = 6$ and Viro [V2,4] did it for $m = 7$. For $m = 8$, the complete answer is still unknown. Following [V3], we shall call the *real scheme of a curve* the arrangement of its connected components on $\mathbf{RP}^2$ up to isotopy.

It is reasonable to start the classification with M-curves (a curve is called an *M-curve* if it has the maximal possible number $(m-1)(m-2)/2+1$ of connected components). After the studies of Fiedler, Viro, Shustin, Korchagin, and Chevallier, there remained only 9 real schemes whose realizability was open (we use the encoding system from [V3])

$$\begin{array}{lll} \langle 4 \sqcup 1\langle 2 \sqcup 1\langle 14\rangle\rangle\rangle\,, & \langle 1 \sqcup 1\langle 1\rangle \sqcup 1\langle 18\rangle\rangle\,, & \langle 1 \sqcup 1\langle 6\rangle \sqcup 1\langle 13\rangle\rangle\,, \\ \langle 7 \sqcup 1\langle 2 \sqcup 1\langle 11\rangle\rangle\rangle\,, & \langle 1 \sqcup 1\langle 3\rangle \sqcup 1\langle 16\rangle\rangle\,, & \langle 1 \sqcup 1\langle 7\rangle \sqcup 1\langle 12\rangle\rangle\,, \\ \langle 14 \sqcup 1\langle 2 \sqcup 1\langle 4\rangle\rangle\rangle\,, & \langle 1 \sqcup 1\langle 4\rangle \sqcup 1\langle 15\rangle\rangle\,, & \langle 1 \sqcup 1\langle 9\rangle \sqcup 1\langle 10\rangle\rangle\,. \end{array} \tag{1}$$

Here we exclude two of them:

Theorem 1.1. *There do not exist real algebraic curves of degree 8 with real schemes*

$$\langle 1 \sqcup 1\langle 3\rangle \sqcup 1\langle 16\rangle\rangle\,, \qquad \langle 1 \sqcup 1\langle 6\rangle \sqcup 1\langle 13\rangle\rangle\,. \tag{2}$$

This is an immediate consequence of Theorem 1.2(a) below. In [O1,4], three of the arrangements (1) were realized as the set of real points of a smooth symplectic surface of degree 8 in $\mathbf{CP}^2$ which is invariant under the complex conjugation $\mathrm{Conj} : \mathbf{CP}^2 \to \mathbf{CP}^2$ (the degree of a surface A is m if $A \sim mL$ where L is a complex line). Such a surface is J-holomorphic for a suitable Conj-invariant tame almost complex structure J. So, we shall call it a *real pseudo-holomorphic curve* (it was called $\mathcal{L}_p$-flexible curve in [O1,4]). Real pseudo-holomorphic curves satisfy all the restrictions which are known for the topology of real algebraic curves of degree 8 (in particular, they are flexible curves in the sense, introduced by Viro in [V3]).

We say that a real scheme in $\mathbf{RP}^2$ *is realized by a real pseudo-holomorphic curve* A if it is isotopic in $\mathbf{RP}^2$ to $\mathbf{R}A = A \cap \mathbf{RP}^2$, the set of real points of A. Here we complete the classification up to isotopy of the arrangements of 22 ovals on $\mathbf{RP}^2$ realizable by real pseudo-holomorphic M-curves of degree 8:

Theorem 1.2. (a) *The two real schemes* (2) *are not realizable by real pseudo-holomorphic curves of degree* 8.

(b) *The other seven real schemes* (1) *are realizable by real pseudo-holomorphic curves of degree* 8.

REMARKS. 1. We do not claim in Theorem 1.2(b) that all the seven schemes are realizable in the same almost complex structure.

2. In Corollaries 4.10 and 4.15 we study possible distribution of ovals between chains (see the definition in section 4.2) for real M-schemes of the form $\langle 1 \sqcup 1\langle\alpha_1\rangle \sqcup 1\langle\alpha_2\rangle\rangle$. We show that only a few of the distributions might be realized by real pseudo-holomorphic curves.

3. In a recent paper [O6], we realize the real scheme $\langle 7 \sqcup 1\langle 2 \sqcup 1\langle 11\rangle\rangle\rangle$ by a real *algebraic* curve in $\mathbf{RP}^2$.

4. Non-trivial examples of arrangements of real curves which are pseudo-holomorphically realizable but algebraically unrealizable can be found in [FiO] (using the techniques of auxiliary pencils of cubics developed in [Fi]) and in [OS] (using Hilbert–Rohn–Gudkov approach developed in [R1,2], [GuU]). However it is still unknown if there exists a smooth real pseudo-holomorphic curve in $\mathbf{CP}^2$ whose real scheme is algebraically unrealizable. This question (in a slightly different context of flexible curves) was asked by Viro [V3].

The proof of the part (a) of Theorem 1.2 is based on the methods proposed by the author in [O1] and used in [O2-5] and [OP]. The new tools are:

- Using one pencil of lines, we prove that some chain of ovals of a real pseudo-holomorphic curve can be degenerated to a singular point of the type A_n, and then we use this information studying the braid coming from another pencil of lines (we already applied this idea in [O4,5]).
- We use a generalised Fox–Milnor theorem for ribbon surfaces of Euler characteristic 1 (see sections 3.3, 4.6, and 4.7).

- We use the periodicity of Tristram signatures to get a contradiction with Murasugi–Tristram inequality for all the braids from a sequence of the form $\{b\,\sigma_i^k\sigma_j^{-k}\}_{k\in\mathbf{Z}}$ (see sections 3.2, 4.8).

In section 2 (resp. section 3), we give necessary definitions and facts about real pseudo-holomorphic curves (resp. links).

In section 4 (resp. section 5), we prove the part (a) (resp. (b)) of Theorem 1.2. As a "side product" of the proof we obtain the results mentioned in Remark 2.

In the Appendix, we present the computer programs (for the system Mathematica [Wo]) which were used for the computations in section 4 (and also in our previous papers [O1-5] and [OP]). It makes this paper (together with the previous ones) self-contained, at least for a reader to whom Mathematica is available. However, the programs are so short that they can be easily translated for any other system of symbolic computations.

A survey of the classification and the complete list. In Table 1, we give the complete list of real schemes realizable by real pseudo-holomorphic M-curves of degree 8 (we present this table in the same form as in [V3]). The 6 schemes whose algebraic realizability is still unknown are marked with an asterisk. Near each real scheme, we indicate the author of its first realization.

The fact that the other real schemes are not realizable follows from:

- Gudkov–Rohlin congruence which implies that $p-n\equiv k^2 \bmod 8$ for any M-curve of an even degree $2k$ where p (resp. n) is the number of ovals contained inside an even (resp. odd) number of other ovals;
- Bezout's theorem for an auxiliary conic which implies that there are ≤ 3 nests and if there is a nest of depth 3 then there are no other nests;
- the result of Viro [V2] which states that if
$$\langle 19-a-b-c \sqcup 1\langle a\rangle \sqcup 1\langle b\rangle \sqcup 1\langle c\rangle\rangle\,, \qquad a,b,c>0\,, \tag{3}$$
is the real scheme of a curve of degree 8 then each of the numbers a, b, c is odd (it generalises an earlier result of Fiedler [F]);
- the result of Viro [V3, (4.12)] which excludes those schemes (3) with odd a,b, and c which are not listed in Table 1 (the proof is not published but later the idea of this proof was used in [KoS], [S3] and one can find a description of the method in [KoS, Sections (5.1) and (5.2)]);
- the result of Shustin [S3] which excludes $\langle 1\langle 20-a\sqcup 1\langle a\rangle\rangle\rangle$ with $a>0$;
- Theorem 1.2(a) of this paper.

Table 1. Real schemes of pseudo-holomorphic M-curves

p=19, n=3	p=15, n=7	p=11, n=11	p=7, n=15	p=3, n=19
$\langle 18\sqcup 1\langle 3\rangle\rangle^{\mathrm{Ha}}$	$\langle 14\sqcup 1\langle 7\rangle\rangle^{\mathrm{G}}$	$\langle 10\sqcup 1\langle 11\rangle\rangle^{\mathrm{V}}$	$\langle 6\sqcup 1\langle 15\rangle\rangle^{\mathrm{V}}$	$\langle 2\sqcup 1\langle 19\rangle\rangle^{\mathrm{V}}$
$\langle 17\sqcup 1\langle 1\rangle\sqcup 1\langle 2\rangle\rangle^{\mathrm{Ha}}$	$\langle 13\sqcup 1\langle 1\rangle\sqcup 1\langle 6\rangle\rangle^{\mathrm{V}}$	$\langle 9\sqcup 1\langle 1\rangle\sqcup 1\langle 10\rangle\rangle^{\mathrm{K}}$	$\langle 5\sqcup 1\langle 1\rangle\sqcup 1\langle 14\rangle\rangle^{\mathrm{V}}$	$\langle 1\sqcup 1\langle 1\rangle\sqcup 1\langle 18\rangle\rangle^{\mathrm{O}\,*}$
	$\langle 13\sqcup 1\langle 2\rangle\sqcup 1\langle 5\rangle\rangle^{\mathrm{G}}$	$\langle 9\sqcup 1\langle 2\rangle\sqcup 1\langle 9\rangle\rangle^{\mathrm{V}}$	$\langle 5\sqcup 1\langle 2\rangle\sqcup 1\langle 13\rangle\rangle^{\mathrm{V}}$	$\langle 1\sqcup 1\langle 2\rangle\sqcup 1\langle 17\rangle\rangle^{\mathrm{V}}$
	$\langle 13\sqcup 1\langle 3\rangle\sqcup 1\langle 4\rangle\rangle^{\mathrm{V}}$	$\langle 9\sqcup 1\langle 3\rangle\sqcup 1\langle 8\rangle\rangle^{\mathrm{V}}$	$\langle 5\sqcup 1\langle 3\rangle\sqcup 1\langle 12\rangle\rangle^{\mathrm{V}}$	
		$\langle 9\sqcup 1\langle 4\rangle\sqcup 1\langle 7\rangle\rangle^{\mathrm{V}}$	$\langle 5\sqcup 1\langle 4\rangle\sqcup 1\langle 11\rangle\rangle^{\mathrm{V}}$	$\langle 1\sqcup 1\langle 4\rangle\sqcup 1\langle 15\rangle\rangle^{*}$
		$\langle 9\sqcup 1\langle 5\rangle\sqcup 1\langle 6\rangle\rangle^{\mathrm{V}}$	$\langle 5\sqcup 1\langle 5\rangle\sqcup 1\langle 10\rangle\rangle^{\mathrm{V}}$	$\langle 1\sqcup 1\langle 5\rangle\sqcup 1\langle 14\rangle\rangle^{\mathrm{V}}$
			$\langle 5\sqcup 1\langle 6\rangle\sqcup 1\langle 9\rangle\rangle^{\mathrm{V}}$	
			$\langle 5\sqcup 1\langle 7\rangle\sqcup 1\langle 8\rangle\rangle^{\mathrm{V}}$	$\langle 1\sqcup 1\langle 7\rangle\sqcup 1\langle 12\rangle\rangle^{*}$
				$\langle 1\sqcup 1\langle 8\rangle\sqcup 1\langle 11\rangle\rangle^{\mathrm{V}}$
				$\langle 1\sqcup 1\langle 9\rangle\sqcup 1\langle 10\rangle\rangle^{*}$
$\langle 17\sqcup 3\langle 1\rangle\rangle^{\mathrm{W}}$	$\langle 12\sqcup 2\langle 1\rangle\sqcup 1\langle 5\rangle\rangle^{\mathrm{V}}$	$\langle 8\sqcup 1\langle 1\rangle\sqcup 1\langle 1\rangle\sqcup 1\langle 9\rangle\rangle^{\mathrm{V}}$	$\langle 4\sqcup 1\langle 1\rangle\sqcup 1\langle 1\rangle\sqcup 1\langle 13\rangle\rangle^{\mathrm{S}}$	$\langle 1\langle 1\rangle\sqcup 1\langle 1\rangle\sqcup 1\langle 17\rangle\rangle^{\mathrm{S}}$
	$\langle 12\sqcup 1\langle 1\rangle\sqcup 2\langle 3\rangle\rangle^{\mathrm{V}}$	$\langle 8\sqcup 1\langle 1\rangle\sqcup 1\langle 3\rangle\sqcup 1\langle 7\rangle\rangle^{\mathrm{V}}$	$\langle 4\sqcup 1\langle 1\rangle\sqcup 1\langle 3\rangle\sqcup 1\langle 11\rangle\rangle^{\mathrm{S}}$	$\langle 1\langle 1\rangle\sqcup 1\langle 7\rangle\sqcup 1\langle 11\rangle\rangle^{\mathrm{S}}$
		$\langle 8\sqcup 1\langle 1\rangle\sqcup 1\langle 5\rangle\sqcup 1\langle 5\rangle\rangle^{\mathrm{V}}$	$\langle 4\sqcup 1\langle 1\rangle\sqcup 1\langle 5\rangle\sqcup 1\langle 9\rangle\rangle^{\mathrm{V}}$	$\langle 1\langle 5\rangle\sqcup 1\langle 7\rangle\sqcup 1\langle 7\rangle\rangle^{\mathrm{S}}$
		$\langle 8\sqcup 1\langle 3\rangle\sqcup 1\langle 3\rangle\sqcup 1\langle 5\rangle\rangle^{\mathrm{V}}$	$\langle 4\sqcup 1\langle 1\rangle\sqcup 1\langle 7\rangle\sqcup 1\langle 7\rangle\rangle^{\mathrm{S}}$	
			$\langle 4\sqcup 1\langle 3\rangle\sqcup 1\langle 5\rangle\sqcup 1\langle 7\rangle\rangle^{\mathrm{V}}$	
			$\langle 4\sqcup 1\langle 5\rangle\sqcup 1\langle 5\rangle\sqcup 1\langle 5\rangle\rangle^{\mathrm{S}}$	
$\langle 1\sqcup 1\langle 2\sqcup 1\langle 17\rangle\rangle\rangle^{\mathrm{Hi}}$	$\langle 1\sqcup 1\langle 6\sqcup 1\langle 13\rangle\rangle\rangle^{\mathrm{V}}$	$\langle 1\sqcup 1\langle 10\sqcup 1\langle 9\rangle\rangle\rangle^{\mathrm{V}}$	$\langle 1\sqcup 1\langle 14\sqcup 1\langle 5\rangle\rangle\rangle^{\mathrm{Hi}}$	$\langle 1\sqcup 1\langle 18\sqcup 1\langle 1\rangle\rangle\rangle^{\mathrm{V}}$
$\langle 2\sqcup 1\langle 2\sqcup 1\langle 16\rangle\rangle\rangle^{\mathrm{C}}$	$\langle 2\sqcup 1\langle 6\sqcup 1\langle 12\rangle\rangle\rangle^{\mathrm{K}}$	$\langle 2\sqcup 1\langle 10\sqcup 1\langle 8\rangle\rangle\rangle^{\mathrm{K}}$	$\langle 2\sqcup 1\langle 14\sqcup 1\langle 4\rangle\rangle\rangle^{\mathrm{K}}$	
$\langle 3\sqcup 1\langle 2\sqcup 1\langle 15\rangle\rangle\rangle^{\mathrm{K}}$	$\langle 3\sqcup 1\langle 6\sqcup 1\langle 11\rangle\rangle\rangle^{\mathrm{K}}$	$\langle 3\sqcup 1\langle 10\sqcup 1\langle 7\rangle\rangle\rangle^{\mathrm{V}}$	$\langle 3\sqcup 1\langle 14\sqcup 1\langle 3\rangle\rangle\rangle^{\mathrm{V}}$	
$\langle 4\sqcup 1\langle 2\sqcup 1\langle 14\rangle\rangle\rangle^{\mathrm{O}\,*}$	$\langle 4\sqcup 1\langle 6\sqcup 1\langle 10\rangle\rangle\rangle^{\mathrm{K}}$	$\langle 4\sqcup 1\langle 10\sqcup 1\langle 6\rangle\rangle\rangle^{\mathrm{K}}$	$\langle 4\sqcup 1\langle 14\sqcup 1\langle 2\rangle\rangle\rangle^{\mathrm{K}}$	
$\langle 5\sqcup 1\langle 2\sqcup 1\langle 13\rangle\rangle\rangle^{\mathrm{C}}$	$\langle 5\sqcup 1\langle 6\sqcup 1\langle 9\rangle\rangle\rangle^{\mathrm{V}}$	$\langle 5\sqcup 1\langle 10\sqcup 1\langle 5\rangle\rangle\rangle^{\mathrm{V}}$	$\langle 5\sqcup 1\langle 14\sqcup 1\langle 1\rangle\rangle\rangle^{\mathrm{Hi}}$	
$\langle 6\sqcup 1\langle 2\sqcup 1\langle 12\rangle\rangle\rangle^{\mathrm{K}}$	$\langle 6\sqcup 1\langle 6\sqcup 1\langle 8\rangle\rangle\rangle^{\mathrm{K}}$	$\langle 6\sqcup 1\langle 10\sqcup 1\langle 4\rangle\rangle\rangle^{\mathrm{K}}$		
$\langle 7\sqcup 1\langle 2\sqcup 1\langle 11\rangle\rangle\rangle^{\mathrm{O}}$	$\langle 7\sqcup 1\langle 6\sqcup 1\langle 7\rangle\rangle\rangle^{\mathrm{V}}$	$\langle 7\sqcup 1\langle 10\sqcup 1\langle 3\rangle\rangle\rangle^{\mathrm{V}}$		
$\langle 8\sqcup 1\langle 2\sqcup 1\langle 10\rangle\rangle\rangle^{\mathrm{K}}$	$\langle 8\sqcup 1\langle 6\sqcup 1\langle 6\rangle\rangle\rangle^{\mathrm{K}}$	$\langle 8\sqcup 1\langle 10\sqcup 1\langle 2\rangle\rangle\rangle^{\mathrm{K}}$		
$\langle 9\sqcup 1\langle 2\sqcup 1\langle 9\rangle\rangle\rangle^{\mathrm{V}}$	$\langle 9\sqcup 1\langle 6\sqcup 1\langle 5\rangle\rangle\rangle^{\mathrm{V}}$	$\langle 9\sqcup 1\langle 10\sqcup 1\langle 1\rangle\rangle\rangle^{\mathrm{V}}$		
$\langle 10\sqcup 1\langle 2\sqcup 1\langle 8\rangle\rangle\rangle^{\mathrm{K}}$	$\langle 10\sqcup 1\langle 6\sqcup 1\langle 4\rangle\rangle\rangle^{\mathrm{K}}$			
$\langle 11\sqcup 1\langle 2\sqcup 1\langle 7\rangle\rangle\rangle^{\mathrm{V}}$	$\langle 11\sqcup 1\langle 6\sqcup 1\langle 3\rangle\rangle\rangle^{\mathrm{V}}$			
$\langle 12\sqcup 1\langle 2\sqcup 1\langle 6\rangle\rangle\rangle^{\mathrm{K}}$	$\langle 12\sqcup 1\langle 6\sqcup 1\langle 2\rangle\rangle\rangle^{\mathrm{K}}$			
$\langle 13\sqcup 1\langle 2\sqcup 1\langle 5\rangle\rangle\rangle^{\mathrm{C}}$	$\langle 13\sqcup 1\langle 6\sqcup 1\langle 1\rangle\rangle\rangle^{\mathrm{V}}$			
$\langle 14\sqcup 1\langle 2\sqcup 1\langle 4\rangle\rangle\rangle^{\mathrm{O}\,*}$				
$\langle 15\sqcup 1\langle 2\sqcup 1\langle 3\rangle\rangle\rangle^{\mathrm{K}}$				
$\langle 16\sqcup 1\langle 2\sqcup 1\langle 2\rangle\rangle\rangle^{\mathrm{C}}$				
$\langle 17\sqcup 1\langle 2\sqcup 1\langle 1\rangle\rangle\rangle^{\mathrm{V}}$				

Ha=Harnack [H]
Hi=Hilbert [Hi]
W=Wiman [Wi]
G=Gudkov [Gu]
K=Korchagin [Ko1,2]
V=Viro [V1,4]
S=Shustin [S1,2]
C=Chevallier [C]
O=Orevkov [O1,4,6]

2 Pseudo-holomorphic Curves and Quasipositive Braids

2.1 Real pseudo-holomorphic curves. Let X be a smooth 4-manifold and ω a symplectic (i.e. closed and nowhere vanishing) 2-form on X. Recall that an *almost complex structure* J on X is a smooth family of linear operators $\{J_p : T_pX \to T_pX\}_{p\in X}$ such that $J_p \circ J_p = -1$ for each $p \in X$. Following Gromov [Gr], we say that an almost complex structure J on X is *tamed by* ω if $\omega_p(v, Jv) > 0$ for any $p \in X$ and $v \in T_pX$. A *J-holomorphic* (or *pseudo-holomorphic* if J is not specified) curve in X is a smooth surface $A \subset X$ such that $Jv \in T_pA$ for any $p \in A$ and $v \in T_pA$.

Let ω be Fubini–Studi symplectic form on $\mathbf{CP}^2$. It is anti-invariant under the complex conjugation, i.e. $\mathrm{Conj}^*\omega = -\omega$ where $\mathrm{Conj} : \mathbf{CP}^2 \to \mathbf{CP}^2$, $(z_0 : z_1 : z_2) \mapsto (\bar z_0 : \bar z_1 : \bar z_2)$. We say that an almost complex structure J on $\mathbf{CP}^2$ is *real* if $J_p \circ \mathrm{Conj}_* = \mathrm{Conj}_* \circ J_p^{-1}$ for each $p \in \mathbf{CP}^2$. A *real pseudo-holomorphic curve* is a Conj-invariant surface A in $\mathbf{CP}^2$ which is a J-holomorphic curve with respect to some real almost complex structure J tamed by ω. When speaking of J-holomorphic curves with singularities, we shall always suppose that near singular points the almost complex structure is standard and hence, the curve near singularities is real analytic.

The *degree* of a pseudo-holomorphic curve A in $\mathbf{CP}^2$ is its homological degree, i.e. the number m such that $A \sim mL$ in $H_2(\mathbf{CP}^2)$ where L is a complex line. In particular, a *pseudo-holomorphic line* (resp. *conic*) is a pseudo-holomorphic curve of degree 1 (resp. 2). Given a tame almost complex structure J on $\mathbf{CP}^2$, it follows from Gromov's results [Gr] that there exists a unique J-holomorphic line (resp. conic) through any two (resp. five) given points. Moreover, if J and the given two (five) points are real then the line (conic) through them is also real because otherwise it would have too many intersections with its conjugate. Therefore, given a real point p, one can consider the pencil of J-holomorphic lines through p (we denote it by $\mathcal{L}_p$). The behaviour of a J-holomorphic curve with respect to such a pencil is very similar to the behaviour of a real algebraic curve with respect to a pencil of true lines.

2.2 Encoding of an arrangement of a curve with respect to a pencil of lines. Suppose that A is a real pseudo-holomorphic curve of degree m all whose singularities are real and of the type A_n. Let $\mathcal{L}_p$ be the pencil of pseudo-holomorphic lines through a generic point p. Suppose that there exists a real line $l_\infty \in \mathcal{L}_p$ meeting A transversally at m distinct real points.

We shall encode the arrangement of $\mathbf{R}A$ with respect to $\mathcal{L}_p$ (the $\mathcal{L}_p$-*scheme of* $\mathbf{R}A$) the same way as in [O1,5], [OP], but we also cover the case of singular points A_n. Namely, let us choose smooth coordinates (x, y) in $\mathbf{R}^2 = \mathbf{RP}^2 \setminus \mathbf{R}l_\infty$ so that the lines of $\mathcal{L}_p$ are vertical lines $x = \mathrm{const}$. Let $(x_1, y_1), \dots, (x_s, y_s)$ be all the real points where $\mathbf{R}A$ is not transversal to the vertical lines. Since p is generic, we may suppose that $x_1 < \cdots < x_s$ and that at some neighbourhood of each point (x_j, y_j), the curve A is given by

$$(y - y_j)^2 \pm (x - x_j)^{n_j+1} = 0\,, \quad n_j \geq 0\,. \tag{4}$$

We encode the arrangement of $\mathbf{R}A$ with respect to $\mathcal{L}_p$ by the word

$[A^{\pm}_{n_1}]_{k_1}\dots[A^{\pm}_{n_s}]_{k_s}$ where the sign of A_{n_j} is taken from (4) and k_j-1 is the number of intersections of $\mathbf{R}A$ with the open half-line $\{x=x_j, y<y_j\}$. We also abbreviate:

$$[A_0^+]_k \longrightarrow \supset_k\,,\quad [A_0^-]_k \longrightarrow \subset_k\,,\quad [A_1^-]_k \longrightarrow \times_k\,,\quad \subset_k\supset_k \longrightarrow o_k$$

(so, o_k denotes an empty oval in the k-th horizontal band).

2.3 The braid associated to a real curve and a pencil of lines. If l_t, $t\in[0,1]$, is a generic closed path in $\mathcal{L}_p\setminus\{l_\infty\}$, it defines a braid with m strings $b\in B_m$. Indeed, the m points $l_t\cap A$ travel on the complex plane (we may identify all l_t using the complexification of the chosen affine coordinates). Let b be the braid corresponding to a simple closed path γ surrounding all the lines from the upper half-plane of $\mathcal{L}_p$ which are not tangent to A (when we say of the upper half-plane, we identify $\mathcal{L}_p\setminus\{l_\infty\}$ with $\mathbf{C}$).

Let $\mathrm{pr}_p:\mathbf{CP}^2\setminus\{p\}\to\mathbf{CP}^1=\mathbf{C}\cup\{\infty\}$ be the projection along the lines of $\mathcal{L}_p$ such that $\mathrm{pr}_p(l_\infty)=\infty$. Since p is a generic point, all the ramifications of $\mathrm{pr}_p|_A$ inside γ are simple. The fact that A and the lines from $\mathcal{L}_p$ are J-holomorphic for the same almost complex structure J implies that all the ramifications are positive, i.e. the braid associated to a small loop around any branch point is a standard generator σ_j, $j=1,\dots,m-1$ of the braid group. Hence, the obtained braid is *quasipositive*, i.e. is a product of braids conjugate to the standard generators of the braid group. The term "quasipositive braid" was introduced by Rudolph [Ru]. A presentation of a quasipositive braid as the product of conjugates of the standard generators is commonly referred to as a *braid monodromy decomposition*.

We shall suppose that any real line $l\in\mathcal{L}_p$ meets A at least at $m-2$ real points (counting the multiplicities). Under this assumption, the braid b is determined by the $\mathcal{L}_p$-scheme of $\mathbf{R}A$ and it can be easily computed as follows. Put $\pi_{k,l}=\sigma_k\dots\sigma_l$ and $\Delta_m=\pi_{1,m-1}\pi_{1,m-2}\dots\pi_{1,1}$. Then $b=b_{\mathbf{R}}\Delta_m$ where $b_{\mathbf{R}}$ is obtained from the encoding word by the following algorithm:

ALGORITHM 2.1. (i) *Replace each* $[A^+_{2n}]_k$ *with* $\times_k^n\supset_k$; *each* $[A^-_{2n}]_k$ *with* $\subset_k\times_k^n$; *each* $[A^+_{2n+1}]_k$ *with* $\subset_k\times_k^n\supset_k$; *each* $[A^-_{2n+1}]_k$ *with* $\times_k^{n+1}$.

(ii) *Replace each subword* $\supset_k\times_{i_1}\dots\times_{i_n}\subset_l$ *with* $\sigma_k^{-1}\delta_1\dots\delta_n\tau_{k,l}$ *where*

$$\delta_j=\begin{cases}\sigma_{i_j}^{-1}\,, & i_j<k-1\,,\\ \sigma_{i_j+2}^{-1}\,, & i_j>k-1\,,\\ \tau_{k,k+1}\sigma_{k+1}^{-1}\tau_{k+1,k}\,, & i_j=k-1\,;\end{cases}\qquad \tau_{k,l}=\begin{cases}\pi_{l,k+1}^{-1}\pi_{k,l-1}\,, & k<l\,,\\ \pi_{l,k-1}^{-1}\pi_{k,l+1}\,, & k>l\,,\\ 1\,, & k=l\,.\end{cases}$$

(iii) *Replace each* $\times_k$ (*which was not replaced in the step* (ii)) *with* σ_k^{-1}.

If any real line $l \in \mathcal{L}_p$ meets A at least at $m-4$ real points (counting the multiplicities) then the braid b is determined by the $\mathcal{L}_p$-scheme of $\mathbf{R}A$ up to some unknown integers $e_1, \dots, e_h$, where h is the number of intervals of the pencil with $m-4$ real intersections with A. In this case the family of braids $\{b_{\vec{e}}\}_{\vec{e}\in\mathbf{Z}^h}$ can be explicitly written down in terms of the $\mathcal{L}_p$-scheme using the algorithm described in [O1, Sect. 3.4] preceded by the step (i) of Algorithm 2.1.

As we already mentioned, the quasipositivity of the braid b is a necessary condition for the realizability of a given $\mathcal{L}_p$-scheme as the set of real points of some real pseudo-holomorphic curve. It can be easily seen (see [FiO, Section 4] for details) that this condition is sufficient as well. Namely,

PROPOSITION 2.2. *Let $C \subset \mathbf{RP}^2$ be a $\mathcal{C}^\infty$-smooth curve which has a finite number of singular points of the type A_n. Suppose that C is real analytic in some neighbourhoods of these points. Suppose also that there exists a generic point in $\mathbf{RP}^2$ such that almost any real line through p meets C at least at $m-4$ points. Denote by $\mathbf{R}\mathcal{L}_p$ the pencil of real lines through p. Let $\mathcal{B} = \{b_{\vec{e}}\}_{\vec{e}\in\mathbf{Z}^h} \subset B_m$ be the family of braids constructed from $(C, \mathbf{R}\mathcal{L}_p)$ by the above procedure. Then $\mathcal{B}$ contains a quasipositive braid if and only if there exist a tame* Conj*-invariant almost complex structure J in $\mathbf{CP}^2$ and a J-holomorphic* Conj*-invariant curve $A \subset \mathbf{CP}^2$ such that $C = A \cap \mathbf{RP}^2$ and each line from $\mathbf{R}\mathcal{L}_p$ is the set of real points of some real J-holomorphic line.*

COROLLARY 2.3. *Let $C, C' \subset \mathbf{RP}^2$ be unions of immersed real circles with transversal intersections. Let $p \in \mathbf{RP}^2$ be a generic point.*

a) *Suppose that the $\mathcal{L}_p$-scheme of C' is obtained from the $\mathcal{L}_p$-scheme of C either by one of the replacements from Step (i) of Algorithm 2.1 or their inverses, or by one of the following substitutions:*

$$\subset_j\supset_{j\pm1} \to \varnothing \qquad\qquad \subset_j\supset_k \to \supset_k\subset_j \tag{5}$$

$$\times_j\supset_{j\pm1} \longleftrightarrow \times_{j\pm1}\supset_j \quad \subset_{j\pm1}\times_j \longleftrightarrow \subset_j\times_{j\pm1} \quad \times_j[A_n^\pm]_k \longleftrightarrow [A_n^\pm]_k\times_j \tag{6}$$

where $|k-j|>1$.

If C is realizable by a real pseudo-holomorphic curve then so is C'.

b) *Suppose that the $\mathcal{L}_p$-scheme of C' is obtained from the $\mathcal{L}_p$-scheme of C by the substitution*

$$\supset_j\subset_j \to \times_j\,. \tag{7}$$

Let $[\ell_1\ell_2]$ be the segment of $\mathbf{R}\mathcal{L}_p$ where the modification (7) is performed. Suppose that ℓ_1 meets C at m real points.

If C is realizable by a real pseudo-holomorphic curve then so is C'.

REMARK. We do not claim in Corollary 2.3 that the curves C and C' are pseudo-holomorphic with respect to *the same* tame real almost complex structure.

In section 5 we shall need a local version of Proposition 2.2.

3 Some Facts on Links

3.1 Murasugi–Tristram inequality. Let L be an oriented link in the 3-sphere S^3. Let V be its Seifert matrix corresponding to a Seifert surface F. For a complex number ζ such that $|\zeta| = 1$, $\zeta \neq 1$, we set $V_\zeta = V_\zeta(L) = (1-\zeta)V + (1-\bar{\zeta})V^T$. The *Tristram signature* $\operatorname{Sign}_\zeta(L)$ and *nullity* $\operatorname{Null}_\zeta(L)$ are defined as

$$\operatorname{Sign}_\zeta(L) = \text{the signature of } V_\zeta(L)\,,\ \operatorname{Null}_\zeta(L) = \mu(F) + \text{the nullity of } V_\zeta(L)$$

where $\mu(F)$ is the number of connected components of F. The *Alexander polynomial* of L is $\Delta_L(t) = \det(V - tV^T)$ considered up to a factor $\pm t^k$. The *determinant* of L is $\det L = |\det(V+V^T)| = |\Delta_L(-1)|$.

It is known that $\operatorname{Sign}_\zeta(L)$, $\operatorname{Null}_\zeta(L)$, $\Delta_L(t)$, and $\det L$ are link invariants (do not depend on the choice of F). If b is a braid then we denote $\operatorname{Sign}_\zeta(b) = \operatorname{Sign}_\zeta(L)$, $\operatorname{Null}_\zeta(b) = \operatorname{Null}_\zeta(L)$, and $\Delta_b(t) = \Delta_L(t)$ where L is the closure of b in S^3.

PROPOSITION 3.1 (Murasugi–Tristram inequality). *Let F be a smooth oriented surface in the 4-ball whose boundary is L (taking into account the orientations). Then for any ζ such that $|\zeta| = 1$, $\zeta \neq 1$, we have*

$$\operatorname{Null}_\zeta(L) \geq |\operatorname{Sign}_\zeta(L)| + \chi(F)$$

where χ denotes the Euler characteristic.

For a braid $b = \prod_j \sigma_{i_j}^{k_j}$, let us set $e(b) = \sum_j k_j$ (the exponent sum). Recall (see 2.3) that a braid b is called *quasipositive* if $b = \prod_j a_j \sigma_1 a_j^{-1}$ for some braids a_j (note that all the generators σ_i are conjugate to each other).

COROLLARY 3.2. *Let b be a quasipositive braid with m strings. Then for any ζ such that $|\zeta| = 1$, $\zeta \neq 1$, we have*

$$\operatorname{Null}_\zeta(b) \geq |\operatorname{Sign}_\zeta(b)| + m - e(b)\,.$$

REMARK. In fact, Corollary 3.2 is much easier than Proposition 3.1 and it can be immediately proved as follows. Let $b = \prod_{j=1}^n a_j\sigma_1 a_j^{-1}$, $n = e(b)$, be a quasipositive presentation of b. Let b_0 be the trivial braid and $b_k = \prod_{j=1}^k a_j \sigma_1 a_j^{-1}$ for $k = 1, \dots, n$. Set $s_k = \operatorname{Null}_\zeta(b_k) - |\operatorname{Sign}_\zeta(b_k)|$. Since $a_k^{-1} b_k a_k = (a_k^{-1} b_{k-1} a_k) \cdot \sigma_1$, there exists a Seifert matrix of b_k obtained from that of b_{k-1} by adding one row and one column (see, e.g. [O1]). Hence, $|s_k - s_{k-1}| = 1$, and we have $m = s_0 \le s_1 + 1 \le s_2 + 2 \le \cdots \le s_n + n = \mathrm{Null}_\zeta(b) - |\,\mathrm{Sign}_\zeta(b)| + e(b)$. □

3.2 Periodicity of Tristram signatures.

PROPOSITION 3.3. *Let L_k, $k \in \mathbf{Z}$, be the closure in S^3 of the braid $\sigma_\nu^k b$ where b is a fixed braid. Let $t = \exp(2\pi i\, p/q)$ be a primitive root of the unity of degree q for $q > 2$. Then*

$$\mathrm{Sign}_{-t}(L_{k+q}) = \mathrm{Sign}_{-t}(L_k) - |2p - q|\,, \qquad \mathrm{Null}_{-t}(L_{k+q}) = \mathrm{Null}_{-t}(L_k)\,.$$

Proof. Changing the numbering, we may assume that $k > 0$. Then the Seifert matrices of L_k and L_{k+q} computed in [O1] are $n \times n$- and $(n+q) \times (n+q)$-matrices

$$V(L_k) = \begin{pmatrix} a & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \text{ and } V(L_{k+q}) = \begin{pmatrix} -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & & \ddots & -1 & 1 & 0 \\ 0 & \dots & \dots & 0 & a & V_{12} \\ 0 & \dots & \dots & 0 & V_{21} & V_{22} \end{pmatrix}.$$

for some integer a, $(n-1) \times 1$-matrix V_{12}, $1 \times (n-1)$-matrix V_{21}, and $(n-1) \times (n-1)$-matrix V_{22}. Consider a $(n+q) \times (n+q)$-matrix $U = \|u_{ij}\|$ where

$$u_{ij} = \begin{cases} 1 & \text{for } i = j\ , \\ t^{j-i}(t^i - 1)/(t^j - 1) & \text{for } i \le j < q\ , \\ 0 & \text{otherwise}\,. \end{cases}$$

Then $U^* \cdot V_{-t}(L_{k+q}) \cdot U = \mathrm{diag}(d_1, \dots, d_{q-2}) \oplus ((1+t)W + (1+\bar t)W^T)$ where

$$d_j = -\,\frac{(t+1)(t^{j+1}-1)}{t\,(t^j - 1)} \qquad \text{and} \quad W = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & a & V_{12} \\ 0 & 0 & V_{21} & V_{22} \end{pmatrix}.$$

It is easy to see that $(1+t)W + (1+\bar t)W^T$ is congruent to $\left(\begin{smallmatrix} 0 & 1+t \\ 1+\bar t & 0 \end{smallmatrix}\right) \oplus V_{-t}(L_k)$, hence,

$$\mathrm{Null}_{-t}(L_{k+q}) = \mathrm{Null}_{-t}(L_k) \text{ and } \mathrm{Sign}_{-t}(L_{k+q}) = \mathrm{Sign}_{-t}(L_k) + \sum_{i=1}^{q-2} \mathrm{sign}\, d_i\,.$$

Note, that $\{d_j \mid 1 \le j \le q-2\} = \{c_j \mid 1 \le j \le q-1,\, j \ne q-p\}$ where

$$c_j = -\,\frac{(t+1)(t\omega^{2j}-1)}{t(\omega^{2j}-1)} = -\,\frac{8\,\mathrm{Re}\,\omega^p \cdot \mathrm{Im}\,\omega^j \cdot \mathrm{Im}\,\omega^{j+p}}{|\omega^{2j}-1|^2}\,, \qquad \omega = e^{\pi i/q}\,.$$

Since $\operatorname{Sign}_\zeta(L) = \operatorname{Sign}_{\bar\zeta}(L)$, it is enough to consider the case $p < q/2$. Then $\operatorname{Re}\omega^p > 0$, $\operatorname{Im}\omega^j > 0$, $\sum \operatorname{sign}\operatorname{Im}\omega^{j+p} = (q-p-1)\cdot 1 + (p-1)\cdot(-1)$. □

REMARK. In essential, our proof of Proposition 3.3 is nothing more than a diagonalisation of a Seifert matrix. It would be interesting to find a more conceptual proof (maybe, based on the Novikov–Wall additivity) which would work for generalised Tristram signatures studied in [Fl].

COROLLARY 3.4. *Let L_k, $k \in \mathbf{Z}$, be the closure in S^3 of the braid $b_1\sigma_{i_1}^k b_2 \sigma_{i_2}^{-k} b_3$ where b_1, b_2, and b_3 are some fixed braids.*

Let $\zeta = \exp(2\pi i p/q)$ for an integer p and an even $q > 2$. Then

$$\operatorname{Sign}_\zeta(L_{k+q}) = \operatorname{Sign}_\zeta(L_k)\,, \qquad \operatorname{Null}_\zeta(L_{k+q}) = \operatorname{Null}_\zeta(L_k)\,. \qquad \square$$

COROLLARY 3.5. *Let $\{L_k\}_{k\in\mathbf{Z}}$ be as in Corollary 3.4. Suppose that an arc $A = \{e^{2\pi i\theta} \mid \theta_1 < \theta < \theta_2\}$ $(0 \le \theta_1 < \theta_2 \le 1)$ does not contain roots of $\Delta_{L_0}(t)$. Let $s_0 = \operatorname{Sign}_\zeta(L_0)$ for $\zeta \in A$. Then for any $q \in 2\mathbf{Z} \setminus [-\alpha,\alpha]$ where $\alpha = 1/(\theta_2-\theta_1)$, there exists ζ_q $(|\zeta_q| = 1,\ \zeta_q \ne 1)$ such that*

$$\operatorname{Sign}_{\zeta_q}(L_q) = s_0 \qquad \text{and} \qquad \operatorname{Null}_{\zeta_q}(L_q) = 1\,.$$

Proof. For $q \in 2\mathbf{Z} \setminus [-\alpha,\alpha]$, set $\zeta_q = e^{2\pi i\theta}$ where $\theta = ([q\theta_2 - 1] + 1)/q$. Then $\theta_1 < \theta < \theta_2$, hence, $\operatorname{Sign}_{\zeta_q}(L_0) = s_0$. By Corollary 3.4, we have $\operatorname{Sign}_{\zeta_q}(L_q) = \operatorname{Sign}_{\zeta_q}(L_0)$. □

Combining Corollary 3.2 and Corollary 3.5, sometimes it is possible to prove that none of the braids $b_1\sigma_{i_1}^k b_2 \sigma_{i_2}^{-k} b_3$ is quasipositive.

3.3 Fox–Milnor theorem for links of ribbon Euler characteristic one. Recall that a *ribbon surface* in the 3-sphere S^3 is an immersion $r : F \to S^3$ of a smooth oriented surface F without closed components which satisfies the following property. There exist disjoint embedded segments $I_1, \dots, I_n$ and $J_1, \dots, J_n$ in F such that

1. $I_k \cap \partial F = \partial I_k$ and I_k is transversal to ∂F for $k = 1, \dots, n$;
2. $J_k \cap \partial F = \varnothing$ for $k = 1, \dots, n$;
3. for each $k = 1, \dots, n$, one has: $r(I_k) = r(J_k)$, the restrictions $r|_{I_k}$ and $r|_{J_k}$ are embeddings, and the images of some neighbourhoods of I_k and J_k are transversal to each other;
4. r is injective outside $I_1 \cup \dots \cup I_n \cup J_1 \cup \dots \cup J_n$.

PROPOSITION 3.6. *Let L be an oriented link in the 3-sphere S^3. Suppose, there exists a ribbon surface $r : F \to S^3$ such that $r(\partial F) = L$ (taking into account the orientations) and $\chi(F) = 1$. Then there exists a polynomial $f(t) \in \mathbf{Z}[t]$ such that $\Delta_L(t) = f(t)\cdot f(t^{-1})$.*

If L is a knot (and hence, F is a disk), this was proved by Fox and Milnor [FoM] even when F is slice (which is a priori stronger than ribbon).

Proof. Repeat word-by-word the second proof of Fox–Milnor theorem given in [K, Example on pp. 212–213]. □

REMARK. A weaker form of Proposition 3.6, namely, with $f(t) \in \mathbf{R}[t]$, is a direct consequence from the symmetricity of Δ_L and Murasugi–Tristram inequality (Proposition 3.1). Indeed, the signature is constantly zero along the unit circle, hence all the roots of the Alexander polynomial on the unit circle are double.

LEMMA 3.7. *Let b be a quasipositive braid with m strings. Then there exists a ribbon surface of Euler characteristic $m - e(b)$ bounded by the closure of b in S^3.*

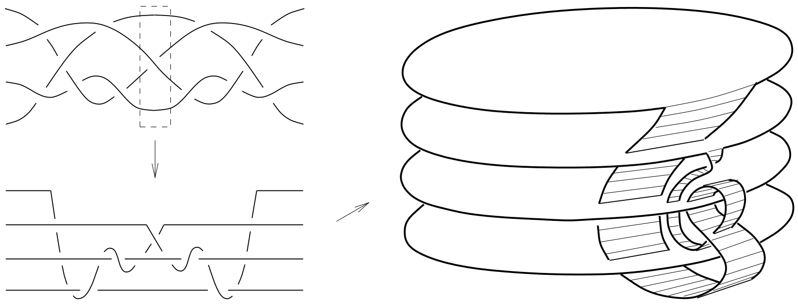

Figure 1

Proof. Let $b = \prod_{j=1}^{e(b)} a_j \sigma_{i_j} a_i^{-1}$. We attach $e(b)$ ribbons to m parallel discs as in Figure 1 (each ribbon corresponds to a factor $a_j \sigma_{i_j} a_i^{-1}$). □

COROLLARY 3.8. *Let b be a quasipositive braid with m strings. If $e(b) = m - 1$ then there exists a polynomial $f(t) \in \mathbf{Z}[t]$ such that $\Delta_b(t) = f(t) \cdot f(t^{-1})$. In particular,* $\det b$ *is a complete square.*

Proof. Combine Proposition 3.6 and Lemma 3.7. □

CONJECTURE. *Under the hypothesis of Proposition 3.6, there exists a polynomial $f(t_1, \dots, t_n) \in \mathbf{Z}[t_1, \dots, t_n]$ such that the Alexander polynomial in several variables $\Delta_L(t_1, \dots, t_n)$ is equal to $f(t_1, \dots, t_n) \cdot f(t_1^{-1}, \dots, t_n^{-1})$.*

REMARK. Applied to real algebraic curves, Corollary 3.8 is a generalisation of Viro's method (used in [V3, Sect. (4.12)], [KoS, Sect. (5.1) and (5.2)],

[S3]) which is based on the observation that the determinant of the intersection matrix of some set of 2-cycles on the double covering is a complete square.

4 Restrictions for the Curves $\langle 1 \sqcup 1\langle \alpha_1 \rangle \sqcup 1\langle \alpha_2 \rangle \rangle$

4.1 Notation. Let A be a real pseudo-holomorphic curve of degree 8 whose set of real points $\mathbf{R}A$ has the following real scheme

$$\langle 1 \sqcup 1\langle \alpha_1 \rangle \sqcup 1\langle \alpha_2 \rangle \rangle\,,\ \alpha_1, \alpha_2 \ge 2\,,\ \alpha_1 + \alpha_2 = 19\,,\ \alpha_1 \text{ is odd}\,,\ \alpha_2 \text{ is even}\,. \quad (8)$$

It has one exterior empty oval v_0 and two non-empty ovals V_1 and V_2. (An oval is called *empty* if it has no other ovals inside; it is not $\varnothing$.) The oval V_k, $(k = 1, 2)$, surrounds interior ovals $v_j^{(k)}$, $j = 1, \dots, \alpha_k$.

4.2 Restrictions coming from Bezout's theorem. As usual, we say that two ovals u_1, u_2 surrounded by an oval V are *separated* by a line ℓ if they lie in the different connected components of $(\operatorname{Int} V) \setminus \ell$.

LEMMA 4.1. *Suppose a curve of degree 8 has five ovals $1, \dots, 5$ arranged as in Figure 2 up to isotopy. Let p_1, p_2, p_3 be some points inside the ovals $1, 2, 3$ respectively. Let H_1 and H_2 be the connected components of the complement of the lines $(p_1 p_2)$ and $(p_1 p_3)$. Then (up to swapping H_1 and H_2) one has:*

1. *if $p \in H_1$ and the line $(p_1 p)$ does not cut the ovals 2 and 3, then $(p_1 p)$ separates the ovals 2 and 3;*
2. *if $p \in H_2$ then $(p_1 p)$ does not separate the ovals 2 and 3.* □

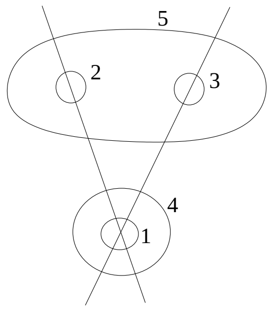

Figure 2

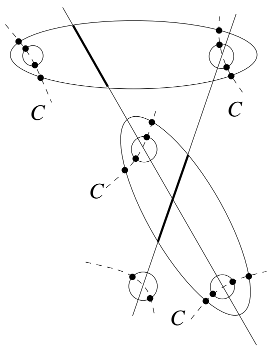

Figure 3

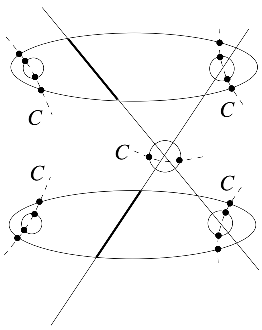

Figure 4

Let the notation be as in sect. 4.1.

LEMMA 4.2. *Let $k = 1$ or 2.*

(a) *Suppose that two ovals $v_k^{(i_1)}$ and $v_k^{(i_2)}$ are separated by a line ℓ passing through $v_{3-k}^{(j_1)}$ and $v_{3-k}^{(j_2)}$. Then, for any $j_3 \in \{1,\dots,\alpha_{3-k}\}$, $j_3 \neq j_1$, the ovals $v_k^{(i_1)}$ and $v_k^{(i_2)}$ are separated by any line ℓ' passing through $v_{3-k}^{(j_1)}$ and $v_{3-k}^{(j_3)}$.*

(b) *Suppose that two ovals $v_k^{(i_1)}$ and $v_k^{(i_2)}$ are separated by a line ℓ passing through $v_{3-k}^{(j_1)}$ and v_0. Then, for any $j_2 \in \{1,\dots,\alpha_{3-k}\}$, the ovals $v_k^{(i_1)}$ and $v_k^{(i_2)}$ are separated by any line ℓ' passing through $v_{3-k}^{(j_2)}$ and v_0.*

Proof. Let us choose points p_0, p_1, p_2, q_1, q_2, q_3 inside the ovals v_0, $v_k^{(i_1)}$, $v_k^{(i_2)}$, $v_{3-k}^{(j_1)}$, $v_{3-k}^{(j_2)}$, and $v_{3-k}^{(j_3)}$ respectively. We assume that the lines ℓ and ℓ' pass through the corresponding points.

(a) Let C be the conic through p_1, p_2, q_1, q_2, and q_3. It cannot cut $\ell \cap \operatorname{Int} V_k$ because it cuts already ℓ at q_1 and q_2. Hence, C cuts V_k at least at 4 points. But C cuts the 5 empty ovals at least at 10 points and the total number of real intersections is ≤ 16, hence, C cuts V_{3-k} only at 2 points. Let us denote the arc $C \cap \operatorname{Int} V_{3-k}$ by γ and the connected components of $\mathbf{RP}^2 \setminus ((q_1p_1) \cup (q_1p_2))$ by H_1 and H_2. Then $\gamma \cap ((q_1p_1) \cup (q_1p_2)) = \{q_1\}$. Hence, for some $\nu = 1$ or 2, we have $\{q_2, q_3\} \subset \gamma \setminus \{q_1\} \subset H_\nu \cap \operatorname{Int} V_{k-3}$ and the result follows from Lemma 4.1.

(b) Let C be the conic through p_0, p_1, p_2, q_1, and q_2. As in part (a), C cannot cut $\ell \cap \operatorname{Int} V_k$, hence, $C \cap \operatorname{Int} V_{3-k}$ is connected. Let q_t, $t \in [1,2]$, be a parametrisation of the arc q_1q_2 of C lying inside V_{3-k}. Then none of the lines (p_0q_t) can pass through p_1 or p_2 because (p_0q_t) already cuts C at p_0 and q_t. Hence, all of these lines separate p_1 and p_2. □

Let us say that two ovals $v_k^{(i_1)}$ and $v_k^{(i_2)}$, $k = 1$ or 2, *are separated* if they are separated either by a line passing through some two ovals $v_{3-k}^{(j_1)}$ and $v_{3-k}^{(j_2)}$ or by a line passing through v_0 and some oval $v_{3-k}^{(j)}$. In the former case we say that *$v_k^{(i_1)}$ and $v_k^{(i_2)}$ are separated by a pair of interior ovals*; in the latter case we say that *$v_k^{(i_1)}$ and $v_k^{(i_2)}$ are separated by the oval v_0*. Lemma 4.2(a) implies:

COROLLARY 4.3. *If two ovals $v_k^{(i_1)}$ and $v_k^{(i_2)}$ are separated by a pair of interior ovals then they are separated by a line passing through any two ovals $v_{3-k}^{(j_1)}$ and $v_{3-k}^{(j_2)}$.* □

LEMMA 4.4. *Let $k = 1$ or 2. Suppose that some two ovals $v_k^{(i_1)}$ and $v_k^{(i_2)}$ are separated. Then any two ovals $v_{3-k}^{(j_1)}$ and $v_{3-k}^{(j_2)}$ are not separated.*

Proof. Suppose that the ovals $v_{3-k}^{(j_1)}$ and $v_{3-k}^{(j_2)}$ are also separated.

Case 1. *The ovals $v_k^{(i_1)}$ and $v_k^{(i_2)}$ are separated by a pair of interior ovals; the ovals $v_{3-k}^{(j_1)}$ and $v_{3-k}^{(j_2)}$ are also separated by a pair of interior ovals.* By Corollary 4.3, this would imply that $v_k^{(i_1)}$ and $v_k^{(i_2)}$ are separated by a line through $v_{3-k}^{(j_1)}$ and $v_{3-k}^{(j_2)}$ and vice versa which is impossible.

Case 2. *The ovals $v_k^{(i_1)}$ and $v_k^{(i_2)}$ are separated by a pair of interior ovals; the ovals $v_{3-k}^{(j_1)}$ and $v_{3-k}^{(j_2)}$ are separated by v_0.* By Corollary 4.3, in this case $v_k^{(i_1)}$ and $v_k^{(i_2)}$ are separated by a line ℓ passing through $v_{3-k}^{(j_1)}$ and $v_{3-k}^{(j_2)}$. By Lemma 4.2(b), $v_{3-k}^{(j_1)}$ and $v_{3-k}^{(j_2)}$ are separated by a line ℓ' passing through $v_k^{(i_1)}$ and v_0. Let C be the conic through all the five ovals. It cannot cut $\ell \cap \operatorname{Int} V_k$ and $\ell' \cap \operatorname{Int} V_{3-k}$, hence, it cuts each of V_k and V_{3-k} at ≥ 4 points (see Figure 3). Contradiction.

Case 3. *The ovals $v_k^{(i_1)}$ and $v_k^{(i_2)}$ are separated by v_0; the ovals $v_{3-k}^{(j_1)}$ and $v_{3-k}^{(j_2)}$ are also separated by v_0.* By Lemma 4.2(b) one has: $v_k^{(i_1)}$ and $v_k^{(i_2)}$ are separated by a line ℓ passing through $v_{3-k}^{(j_1)}$ and v_0; $v_{3-k}^{(j_1)}$ and $v_{3-k}^{(j_2)}$ are separated by a line ℓ' passing through $v_k^{(i_1)}$ and v_0. The end of the proof is the same as in Case 2 (see Figure 4). □

4.3 Complex orientations and chains of ovals. Let the notation be as in sect. 4.1. Let us fix one of the complex orientations on the curve A. For $k = 1$ or 2, let α_k^+ (resp. α_k^-) be the number of ovals which form positive (resp. negative) injective pair with the oval V_k. According to Welschinger's result [W, Corollary 2.9], we have

$$(\alpha_1^+ - \alpha_1^-) = 1\,, \qquad (\alpha_2^+ - \alpha_2^-) = 2\,. \tag{9}$$

Let us write $v_k^{(i_1)} \sim v_k^{(i_2)}$ when $v_k^{(i_1)}$ and $v_k^{(i_2)}$ are not separated. It is clear that this is an equivalence relation. The equivalence classes are called the *chains of ovals*.

Lemma 4.5. *(a) The ovals $v_1^{(1)}, \dots, v_1^{(\alpha_1)}$ form one chain (recall that α_1 is odd).*

(b) A line through $v_1^{(i)}$ and v_0 divides $\operatorname{Int} V_2$ into two halves each of which contains an odd number of empty ovals. The empty ovals lying in one of the halves form a chain; the empty ovals from the other half form either one chain or two chains separated by any pair of ovals $v_1^{(i)}$, $v_1^{(j)}$.

Proof. By Lemma 4.4, for some k where $k=1$ or 2, the ovals $v_k^{(1)}, \dots, v_k^{(\alpha_k)}$ form a single chain. Then, by Fiedler's theorem [F] applied to a pencil of lines through a point in the oval $v_{3-k}^{(1)}$, we have $|\alpha_k^+ - \alpha_k^-| \le 1$. Thus, $k=1$ by (9). It remains to apply the Fiedler's theorem to a pencil of lines through a point in the oval $v_1^{(1)}$. □

4.4 A pseudo-holomorphic degeneration. Let A be as in sect. 4.1. By Corollary 2.3 and Lemma 4.5(a), there exists a real pseudo-holomorphic curve A' (a priori, with respect to another tame real almost complex structure J') such that $\mathbf{R}A'$ is obtained from $\mathbf{R}A$ by replacing the chain of ovals $v_1^{(1)}, \dots, v_1^{(\alpha_1)}$ with a connected curve v_1' which has a singular point q of the type A_n for $n = 2\alpha_1 - 3$ and which is smooth outside it (v_1' is isotopic to the curve $y^2 = x^{n+1}(x^2-1)$). Let us consider the pencil of J'-holomorphic lines $\mathcal{L}_p$ where p is a generic point inside v_1'.

LEMMA 4.6. *Up to a choice of the line l_∞, a choice of the orientation of the pencil, and a choice of the half of $\operatorname{Int} v_1'$ containing p, the $\mathcal{L}_p$-scheme of $\mathbf{R}A'$ can be reduced by (5) to one of the following:*

$$\supset_4\supset_3\subset_5[A_n^-]_4\supset_4\subset_3 o_4^{\beta_1} o_\nu o_4^{\beta_2-1}\subset_4\,,\ \ \beta_1,\beta_2 \text{ are odd},\ \beta_1+\beta_2=\alpha_2\,,\ \ \beta_1\le\beta_2\,, \tag{10}$$

$$\supset_4\supset_3\subset_3 o_4^{\beta_1}\subset_7[A_n^-]_6\supset_6 o_4^{\beta_2} o_\nu o_4^{\beta_3-1}\subset_4\,,\ \ \beta_3 \text{ is odd},\ \beta_1>0\,,\ \ \beta_1+\beta_2+\beta_3=\alpha_2\,, \tag{11}$$

where $\nu = 3$ or 5 (o_ν represents the oval v_0). □

COROLLARY 4.7. *There exist $e_1, e_2 \in \mathbf{Z}$, $\nu \in \{3,5\}$, and β_j's as in* (10), (11) *such that one of the following braids* (12) *or* (13) *is quasipositive:*

$$b_1\sigma_5^{1-e_1}\sigma_3^{e_1}b_2\sigma_5^{-e_2}\sigma_3^{1+e_2}\sigma_4^{-\beta_1}b_3\sigma_4^{1-\beta_2}\Delta\,, \tag{12}$$

$$b_1\sigma_5^{1-e_1}\sigma_3^{1+e_1}\sigma_4^{-\beta_1}\sigma_5^{-1}b_2\sigma_5\sigma_4^{-\beta_2}b_3\sigma_4^{1-\beta_3}\Delta\,, \tag{13}$$

where (12) *and* (13) *correspond to* (10) *and* (11) *respectively, and*

$$b_1 = \sigma_4^{-1}\sigma_3^{-1}\sigma_5^{-1}\sigma_4^{-1},\ \ b_2 = \sigma_6^{-1}\sigma_7^{-1}\sigma_4\sigma_5\sigma_6^{1-\alpha_1}\sigma_5^{-1}\sigma_4^{-1}\sigma_6,$$
$$b_3 = \sigma_\nu^{-1}\sigma_4\sigma_\nu^{-1}\sigma_4^{-1}\sigma_\nu\,.$$

If b is any of the braids (12), (13) *then*

$$e(b) = 7\,. \tag{14}$$

In Fig. 5, we depicted the $\mathcal{L}_p$-scheme (10) by the dashed line and the braid (12) by the solid line for $\nu=5$. A box $\boxed{n}$ means σ_k^n for the braid and $\times_k^{|n|}$ for the $\mathcal{L}_p$-scheme if the dashed lines enter into the box (k is the height of the box).

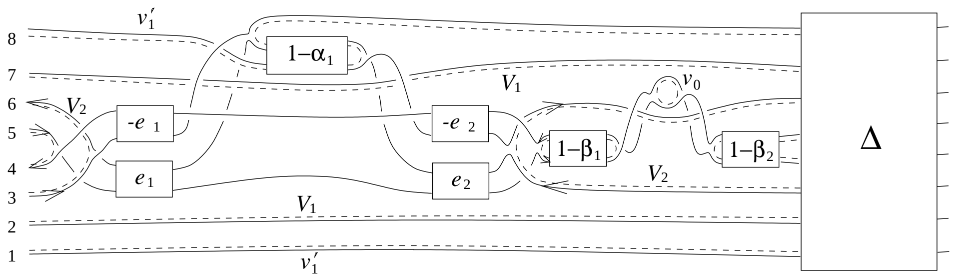

Figure 5

Remark. If we do not degenerate the chain $v_1^{(1)},\dots,v_1^{(\alpha_1)}$ into a singularity A_n then we get an interval of $\{l\in\mathbf{R}\mathcal{L}_p \mid \operatorname{Card}\mathbf{R}l\cap\mathbf{R}A=4\}$ between any two consecutive ovals $v_1^{(j)}$. This would give us a braid depending of many unknown integer parameters (see sect. 2.3) instead of the braid (12) which depends only on e_1 and e_2 (in Lemma 4.8 we express e_1 via e_2). In this case, it would be very difficult to prove that the braid is not quasipositive for all possible values of the parameters. The usage of pseudo-holomorphic curves is essential here because staying in the real algebraic context, it is very difficult (if possible at all) to prove the existence of such a degeneration.

4.5 Linking numbers. Let the notation be as in sect. 4.1–4.4. Now, using linking numbers of sublinks of the braids (12) and (13) as it was done in [O1, Sect. 8.2], we shall compute all possible values for e_1 in the case (13) and express e_1 in terms of e_2 in the case (12).

Lemma 4.8. *(a) Suppose, the $\mathcal{L}_p$-scheme of $\mathbf{R}A'$ is* (10). *Then the braid* (12) *is quasipositive at least in one of the six cases listed in Table 2a.*

(b) Suppose, the $\mathcal{L}_p$-scheme of $\mathbf{R}A'$ is (11). *Then the braid* (13) *is quasipositive at least in one of the six cases listed in Table 2b.*

Table 2a

Case	ν	e_2	e_1
1	3	even	$12-\alpha_2-e_2$
2	3	even	$4-\alpha_2-e_2$
3	3	odd	$-6-e_2$
4	5	even	$-4+\alpha_2-e_2$
5	5	odd	$14-e_2$
6	5	odd	$6-e_2$

Table 2b

Case	ν	β_1	e_1
1	3	odd	-14
2	3	odd	-6
3	5	odd	6
4	3	even	-6
5	5	even	6
6	5	even	14

Proof. Let pr_p and γ be as in sect. 2.3. Denote by H the domain bounded by γ and let $N = \mathrm{pr}_p^{-1}(H) \cap A$ and $L = \partial N$ (it is the link representing the braid b). Since A is an M-curve, $A \setminus \mathbf{R}A$ is a disjoint union $A_+ \sqcup A_-$ and $\mathrm{Conj}(A_\pm) = A_\mp$. Let us denote $N_\pm = \mathrm{pr}_p^{-1}(H) \cap A_\pm$ and $L_\pm = \partial N_\pm$. Let us denote the number of components of N and L by μ_N and μ_L respectively. Since $\mathrm{pr}_p|_N$ has $e(b) = 7$ simple branch points, we have $\chi(N) = 1$. The genus of $A_+ \cup A_-$ is zero because A is an M-curve. Since $N \subset A_+ \cup A_-$, the genus of N is also zero, and hence,

$$1 = \chi(N) = 2\mu_N - \mu_L\,. \tag{15}$$

(a) The complex orientations imply (see arrows in Fig. 5) that $e_1 + e_2$ is even. Consider the standard homomorphism φ of B_n to the symmetric group defined by $\sigma_i \mapsto (i, i+1)$. Then (see Fig. 5)

$$\varphi(b) = \begin{cases} (183)(27)(4)(5)(6) & \text{if } e_1 \text{ and } e_2 \text{ are even,} \\ (184)(27)(3)(5)(6) & \text{if } e_1 \text{ and } e_2 \text{ are odd.} \end{cases} \tag{16}$$

In the both cases we have $\mu_L = 5$, and hence, $\mu_N = 3$ by (15). Let us denote the components of L by $L_1, \dots, L_5$ in the order of the cycles in (16).

Consider the case when e_1 and e_2 are even. Let us denote the linking number of L_i and L_j by l_{ij}. Then we have $l_{12} = 3$, $l_{15} = (3 - \alpha_1)/2$, $l_{23} = l_{24} = l_{25} = 1$, $l_{34} = -\alpha_2/2$ and

$$l_{13} = 2 - (e_1 + e_2)/2, l_{14} = 1, l_{35} = -1, l_{45} = 2 + (e_1 + e_2)/2 \qquad \text{for } \nu = 3;$$

$$l_{13} = 3 - (e_1 + e_2)/2, l_{14} = 0, l_{35} = 0, \quad l_{45} = 1 + (e_1 + e_2)/2 \qquad \text{for } \nu = 5.$$

We see from the complex orientations that $L_1 \cup L_4 \subset L_+$ and $L_2 \cup L_3 \cup L_5 \subset L_-$ (up to swapping L_+ and L_-). L_2 cannot bound a connected component of N because if $L_2 = \partial N_0$ then $N_0 \cup \mathrm{Conj}(N_0)$ would be an isolated component of A. Thus, all possible distributions of L_j between the components of N are those listed in the first column of Table 3 where N_1, N_2, N_3 are connected components of N and $L_{ij\dots}$ denotes $L_i \cup L_j \cup \dots$ The linking numbers $\partial N_i \cdot \partial N_j$ must be zero for $i \neq j$. In the second and the third column of Table 3, we put the linking numbers which lead either to a contradiction or to a relation between e_1 and e_2.

Table 3. The braid (12) for even e_1 and e_2

∂N_1, ∂N_2, ∂N_3	$\nu = 3$	$\nu = 5$
L_1, L_4, L_{235}	$L_1 \cdot L_4 = 1$	$L_4 L_{235} = 2 + (e_1 + e_2 - \alpha_2)/2$
L_{14}, L_{23}, L_5	$L_{14} L_5 = -6 + (e_1 + e_2 + \alpha_2)/2$	$L_{23} \cdot L_5 = 1$
L_{14}, L_{25}, L_3	$L_{14} L_3 = 2 - (e_1 + e_2 + \alpha_2)/2$	$L_{25} \cdot L_3 = 1$

The case when e_1 and e_2 are odd is similar. The linking numbers are $l_{12}=3$, $l_{14}=-8$, $l_{23}=l_{24}=l_{25}=1$, $l_{35}=0$ and

$$l_{13}=2-(e_1+e_2)/2,\ l_{15}=0,\ l_{34}=0,\quad l_{45}=2+(e_1+e_2)/2 \quad \text{for } \nu=3;$$
$$l_{13}=3-(e_1+e_2)/2,\ l_{15}=1,\ l_{34}=-1,\ l_{45}=1+(e_1+e_2)/2 \quad \text{for } \nu=5.$$

We have $L_+=L_1\cup L_5$ and $L_-=L_2\cup L_3\cup L_4$. Possible distributions of L_i's between ∂N_j's and the corresponding linking numbers are presented in Table 4.

Table 4. The braid (12) for odd e_1 and e_2

$\partial N_1,\partial N_2,\partial N_3$	$\nu=3$	$\nu=5$
L_1, L_5, L_{234}	$L_5\cdot L_{234}=3+(e_1+e_2)/2$	$L_1\cdot L_5=1$
L_{15}, L_{23}, L_4	$L_{23}\cdot L_4=1$	$L_{15}\cdot L_4=-7+(e_1+e_2)/2$
L_{15}, L_{24}, L_3	$L_{24}\cdot L_3=1$	$L_{15}\cdot L_3=3-(e_1+e_2)/2$

(b) The proof follows the same scheme.

Suppose that β_1 is odd. Then we have $\varphi(b)=(158)(27)(3)(4)(6)$; $L_+=L_2\cup L_4\cup L_5$ and $L_-=L_1\cup L_3$; the linking numbers are $l_{12}=3$, $l_{14}=-8$, $l_{23}=l_{24}=l_{25}=1$, $l_{35}=0$, and

$$l_{13}=1,\quad l_{15}=3+e_1/2,\quad l_{34}=1-e_1/2,\quad l_{45}=-1 \qquad \text{for } \nu=3\,;$$
$$l_{13}=0,\quad l_{15}=2+e_1/2,\quad l_{34}=2-e_1/2,\quad l_{45}=0 \qquad \text{for } \nu=5\,.$$

The linking numbers of ∂N_i's are in Table 5.

Table 5. The braid (13) for odd β_1

$\partial N_1,\partial N_2,\partial N_3$	$\nu=3$	$\nu=5$
L_1, L_3, L_{245}	$L_1\cdot L_3=1$	$L_3\cdot L_{245}=3-e_1/2$
L_{13}, L_{24}, L_5	$L_{13}\cdot L_5=3+e_1/2$	$L_{24}\cdot L_5=1$
L_{13}, L_{25}, L_4	$L_{13}\cdot L_4=-7-e_1/2$	$L_{25}\cdot L_4=1$

Suppose that β_1 is even. Then we have $\varphi(b)=(148)(27)(3)(5)(6)$; $L_+=L_1\cup L_5$ and $L_-=L_2\cup L_3\cup L_4$; the linking numbers are $l_{12}=3$, $l_{14}=-8$, $l_{23}=l_{24}=l_{25}=1$, $l_{35}=0$, and

$$l_{13}=2-e_1/2,\quad l_{15}=0,\quad l_{34}=0,\qquad l_{45}=2+e_1/2 \qquad \text{for } \nu=3\,;$$
$$l_{13}=3-e_1/2,\quad l_{15}=1,\quad l_{34}=-1,\quad l_{45}=1+e_1/2 \qquad \text{for } \nu=5\,.$$

The linking numbers of ∂N_i's are in Table 6.

4.6 Restrictions provided by the generalised Fox–Milnor theorem. Here we prove some restrictions applying Corollary 3.8 to braids which do not depend on unknown parameters, namely, to the braids b corresponding to Lemma 4.8(b). For each of these 612 braids we compute

Table 6. The braid (13) for even β_1

∂N_1, ∂N_2, ∂N_3	$\nu = 3$	$\nu = 5$
L_1, L_5, L_{234}	$L_5 \cdot L_{234} = 3 + e_1/2$	$L_1 \cdot L_5 = 1$
L_{15}, L_{23}, L_4	$L_{23} \cdot L_4 = 1$	$L_{15} \cdot L_4 = -7 + e_1/2$
L_{15}, L_{24}, L_3	$L_{24} \cdot L_3 = 1$	$L_{15} \cdot L_3 = 3 - e_1/2$

its determinant (using the computer program given in the Appendix) and check if its absolute value is a complete square or not. For most of the braids, it is not. Hence, they are not quasipositive by Corollary 3.8. In the remaining cases (there are 28 of them) we compute the Alexander polynomial (see Example 3 in the Appendix). The only six cases when it is of the form $(t-1)^4 f(t) f(t^{-1})$, are those listed in Table 7 where

$$p_1(t) = t^{10} - t^9 + 2t^8 - t^7 + 3t^6 - 2t^5 + 3t^4 - 2t^3 + 2t^2 - t + 1\,,$$

$$p_2(t) = t^{14} - t^{13} + 2t^{12} - 2t^{11} + 3t^{10} - 2t^9 + 4t^8 - 3t^7 + 3t^6 - 2t^5 + 2t^4 - t^3 + t^2 - t + 1\,,$$

$$p_3(t) = t^{12} - t^{11} + 2t^{10} - 2t^9 + 4t^8 - 3t^7 + 4t^6 - 3t^5 + 3t^4 - 2t^3 + 2t^2 - t + 1\,,$$

and $\Phi_n(t)$ is the n-th cyclotomic polynomial. This proves the following lemma.

Table 7. The braids (13) satisfying Corollary 3.8

α_1	α_2	β_1	β_2	β_3	ν	e_1	$f(t)$
7	12	9	2	1	5	+6	$p_1(t)$
7	12	10	1	1	3	−6	$p_2(t)$
7	12	10	1	1	5	+6	$p_3(t)$
9	10	7	2	1	5	+6	$p_1(t)$
15	4	1	2	1	3	−6	$\Phi_5(t)\Phi_{10}(t)$
15	4	1	2	1	5	+6	$\Phi_5(t)\Phi_{10}(t)$

LEMMA 4.9. *If a braid* (13) *with* $\nu \in \{3, 5\}$, *and* α_i*'s and* β_j*'s from* (8) *and* (11) *is quasipositive then the values of the parameters are as in Table 7.* □

COROLLARY 4.10. *Suppose that a real pseudo-holomorphic* M*-curve* A *with real scheme* (8) *contains six ovals arranged with respect to some line as in Fig. 6 up to isotopy. Then the* $\mathcal{L}_p$*-scheme of* $\mathbf{R}A$ *with respect to some point* p *is* (11) *with* $(\beta_1, \beta_2, \beta_3; \nu)$ *as in Table 7, in particular,* $(\alpha_1, \alpha_2) \in \{\,(7,12),\ (9,10),\ (15,4)\,\}$.

4.7 Fox–Milnor theorem applied to infinite families of braids. Here we prove restrictions applying Corollary 3.8 to braids depending on

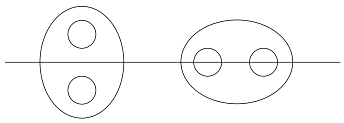

Figure 6

unknown parameters, namely, to the braids corresponding to Lemma 4.8(a). For odd positive β_1, β_2 satisfying $\beta_1+\beta_2\le 16$ and for $k\in\{1,\dots,6\}$, let $b^{(k)}_{\beta_1,\beta_2}(e_2)$ denote the braid (12) with $\alpha_1=19-\beta_1-\beta_2$ and with (ν,e_1) from the k-th line of Table 2a. Let $d^{(k)}_{\beta_1,\beta_2}(e_2)=\det b^{(k)}_{\beta_1,\beta_2}(e_2)/16$. Using the algorithm from [O1, Sect. 2.6] (see Appendix) or using Göritz matrices [GL] (as it was done in [O4], [OS]), one can compute

$$\begin{aligned}
d^{(1)}_{\beta_1,\beta_2}(e_2) &= c_1c_2e_2^2+c_1(c_2+\gamma)(\alpha_2-10)e_2+3\alpha_2^3\beta_1\beta_2\\
&\quad-9/4\,\beta_1^4-131\beta_1^3\beta_2-513/2\,\beta_1^2\beta_2^2-129\beta_1\beta_2^3-5/4\,\beta_2^4\\
&\quad+90\beta_1^3+1868\beta_1^2\beta_2+1830\beta_1\beta_2^2+52\beta_2^3\\
&\quad-1169\beta_1^2-9006\beta_1\beta_2-709\beta_2^2+5126\beta_1+3326\beta_2-1609,\\
d^{(2)}_{\beta_1,\beta_2}(e_2) &= c_1c_2e_2^2+c_1(c_2+\gamma)(\alpha_2-2)e_2+3\alpha_2^3\beta_1\beta_2\\
&\quad-9/4\,\beta_1^4-83\beta_1^3\beta_2-321/2\,\beta_1^2\beta_2^2-81\beta_1\beta_2^3-5/4\,\beta_2^4\\
&\quad+54\beta_1^3+544\beta_1^2\beta_2+522\beta_1\beta_2^2+32\beta_2^3\\
&\quad-257\beta_1^2-1102\beta_1\beta_2-181\beta_2^2+414\beta_1+342\beta_2-113,\\
d^{(3)}_{\beta_1,\beta_2}(e_2) &= c_1c_2e_2^2+8c_1(c_2+\gamma)e_2+136\alpha_2\beta_1\beta_2\\
&\quad-123\beta_1^2-2666\beta_1\beta_2-59\beta_2^2+2289\beta_1+1137\beta_2-763,\\
d^{(4)}_{\beta_1,\beta_2}(e_2) &= c_1c_2e_2^2-c_1\big((c_2+\gamma)(\alpha_2-6)+4\gamma\big)e_2+3\alpha_2^3\beta_1\beta_2\\
&\quad-9/4\,\beta_1^4-91\beta_1^3\beta_2-353/2\,\beta_1^2\beta_2^2-89\beta_1\beta_2^3-5/4\,\beta_2^4\\
&\quad+60\beta_1^3+734\beta_1^2\beta_2+708\beta_1\beta_2^2+34\beta_2^3\\
&\quad-387\beta_1^2-1858\beta_1\beta_2-231\beta_2^2+818\beta_1+602\beta_2-257,\\
d^{(5)}_{\beta_1,\beta_2}(e_2) &= c_1c_2e_2^2-4c_1(3c_2+4\gamma)e_2+536\alpha_2\beta_1\beta_2\\
&\quad-443\beta_1^2-10538\beta_1\beta_2-251\beta_2^2+8285\beta_1+4829\beta_2-2795,\\
d^{(6)}_{\beta_1,\beta_2}(e_2) &= c_1c_2e_2^2-4c_1(c_2+2\gamma)e_2+88\alpha_2\beta_1\beta_2\\
&\quad-83\beta_1^2-1754\beta_1\beta_2-51\beta_2^2+1557\beta_1+981\beta_2-547
\end{aligned}$$

where $c_1 = \alpha_2 - 18$, $c_2 = (2\beta_1 - 1)(2\beta_2 - 1)$, and $\gamma = \beta_2 - \beta_1$ (recall that $\alpha_2 = \beta_1 + \beta_2$).

LEMMA 4.11. *$|d^{(k)}_{\beta_1,\beta_2}(e_2)|$ cannot be a complete square for the values of (β_1, β_2, k) listed in Table 8 and for any $e_2 \in \mathbf{Z}$.*

Table 8. The braids (12) whose determinants are never complete squares

β_1	β_2	k	p	β_1	β_2	k	p	β_1	β_2	k	p
3	13	1,...,6	5	5	9	5	25	3	5	1,...,6	5
5	11	1	9	3	7	1	5	1	5	1,...,6	25
7	9	1,...,6	25	5	5	2,3,4,6	25	3	3	1,...,6	5

Proof. It is easy to see that if we fix integers β_1 and β_2 of the same parity then each $d^{(k)}_{\beta_1,\beta_2}$ is a polynomial in e_2 with integral coefficients. Computing all its values $\bmod p$ for p given in Table 8, we obtain the result (see Example 5 in Appendix). □

COROLLARY 4.12. *The real scheme $\langle 1 \sqcup 1\langle 13\rangle \sqcup 1\langle 6\rangle\rangle$ is not realizable as the set of real points of a real pseudo-holomorphic curve of degree* 8 *in* $\mathbf{CP}^2$.

4.8 Restrictions provided by the periodicity of Tristram signatures.

LEMMA 4.13. *Let A be a non-degenerate Hermitian $n \times n$-matrix. Then*

$$\operatorname{Sign} A \equiv n + 2a \mod 4 \qquad \textit{where} \quad \operatorname{sign}\det A = (-1)^a. \qquad \square$$

PROPOSITION 4.14. *Let $b = b^{(k)}_{\beta_1,\beta_2}(e_2)$ (see section 4.7). If b is quasi-positive then it is one of*

$b^{(1)}_{1,13}(-1),\ b^{(1)}_{7,7}(-2),\ b^{(4)}_{7,7}(4);$

$b^{(1)}_{1,11}(-6),\ b^{(2)}_{1,11}(-10),\ b^{(3)}_{1,11}(-7),\ b^{(4)}_{1,11}(8),\ b^{(5)}_{1,11}(9),\ b^{(6)}_{1,11}(5);$

$b^{(1)}_{1,9}(0),\ b^{(5)}_{1,9}(5);$

$b^{(1)}_{1,7}(4),\ b^{(2)}_{1,7}(0),\ b^{(3)}_{1,7}(-5),\ b^{(4)}_{1,7}(-2),\ b^{(5)}_{1,7}(7),\ b^{(6)}_{1,7}(3);$

$b^{(1)}_{1,3}(6), b^{(6)}_{1,3}(5);$

$b^{(1)}_{1,1}(4), b^{(2)}_{1,1}(0), b^{(3)}_{1,1}(-11), b^{(3)}_{1,1}(3), b^{(4)}_{1,1}(-2), b^{(5)}_{1,1}(-1), b^{(5)}_{1,1}(13), b^{(6)}_{1,1}(-5), b^{(6)}_{1,1}(9).$

COROLLARY 4.15. *Suppose that a real pseudo-holomorphic M-curve A with real scheme* (8) *does not contain six ovals arranged with respect to* *some line as in Fig. 6 up to isotopy. Then the $\mathcal{L}_p$-scheme of* $\mathbf{R}A$ *with respect to some point p is* (10) *with*

$$(\beta_1,\beta_2)\in\big\{\,(1,13),\ (7,7),\ (1,11),\ (1,9),\ (1,7),\ (1,3),\ (1,1)\big\}$$

and if $(\beta_1,\beta_2)=(1,13)$ *then* $\nu=3$. *In particular,* $(\alpha_1,\alpha_2)\notin\{(3,16),(6,13)\}$.

Combining Corollaries 4.10 and 4.15, we obtain immediately Theorem 1.2(a) and hence, Theorem 1.1.

Proof of Proposition 4.14. For the triples (β_1,β_2,k) listed in Table 8 the result follows from Lemma 4.11. For each of the other triples (there are 84 of them; they are listed in the first columns of Tables 9a–g we proceed as follows.

Step 1. Choice of $q_{\max}$. Fix a value $e_2=e_2^0$ of the parity indicated in Table 2a. In all the cases we choose $e_2^0=0$ or 1 except the three cases $(\beta_1,\beta_2,k)=(1,9,2)$, $(1,7,2)$, and $(1,9,1)$ where we choose $e_2^0=2$. Let $b_0=b^{(k)}_{\beta_1,\beta_2}(e_2^0)$ and $s=\mathrm{Sign}_{-1}(b_0)$. A computation shows that $s=2$ for $(\beta_1,\beta_2,k)\in\{(1,3,4),\ (1,1,5),\ (1,1,6)\}$ and $s=0$ otherwise. Then we compute the Alexander polynomial $\Delta_{b_0}(t)$. The change of variable $t=\varphi(u)=(u+i)/(u-i)$ transforms the circle $|t|=1$ into the real line $\mathrm{Im}\,u=0$ and we have $\varphi(0)=-1$, $\varphi(\pm1)=\pm i$, $\varphi(\infty)=1$. Since $\Delta_{b_0}(t)$ is symmetric, we have $\Delta_{b_0}(\varphi(u))=Q(u^2)/(u-i)^n$ where $n=\deg\Delta_{b_0}(t)$ and $Q(z)$ is a polynomial with integral coefficients. The roots of $\Delta_{b_0}(t)$ on the unit circle are exactly $\varphi\big(\pm\sqrt{z_1}\big),\dots,\varphi\big(\pm\sqrt{z_\ell}\big)$ where $z_1,\dots,z_\ell$ are non-negative real roots of $Q(z)$. Then we find an interval $I=[z',z'']$ on the positive half-line such that $\operatorname{sign}Q(z)=-(-1)^{s/2}\operatorname{sign}Q(0)$ when $z'<z<z''$. Let $q_{\max}$ be the maximal even number such that $1/q_{\max}>|\theta'-\theta''|$ where $e^{2\pi i\theta'}=\varphi\big(\sqrt{z'}\big)$, $e^{2\pi i\theta''}=\varphi\big(\sqrt{z''}\big)$, and $0<\theta''<\theta'\le1/2$. The intervals I are listed in Tables 9a–g.

Step 2. Let us show that the braid $b=b^{(k)}_{\beta_1,\beta_2}(e_2)$ is not quasipositive when $|e_2-e_2^0|>q_{\max}$. Indeed, if b were quasipositive then by Corollary 3.2 and (14) one would have $\mathrm{Sign}_\zeta(b)=0$ for a generic ζ on the unit circle. We used here that $\Delta_b(t)$ is not identically zero (this follows from the fact that each of the $d^{(k)}_{\beta_1,\beta_2}$ has no integer roots; see sect. 4.7). By Lemma 4.13, we have $\mathrm{Sign}_\zeta(b_0)\equiv2\mod4$ for $\zeta=\varphi(\sqrt{z})$, $z\in\mathrm{Int}\,I$. Thus, if $q=|e_2-e_2^0|>q_{\max}$ then, by Corollary 3.5, there exists ζ_q such that $\mathrm{Sign}_{\zeta_q}(b)=\mathrm{Sign}_{\zeta_q}(b_0)\ne0$ which implies that b is not quasipositive.

Step 3. In the last columns of Tables 9a–g (entitled "sq."), we give the set of the values of e_2 such that $|e_2-e_2^0|\le q_{\max}$ and $d^{(k)}_{\beta_1,\beta_2}(e_2)$ is a complete square. Computing the Alexander polynomial for each of these values of e_2, one can check that it has the form $f(t)f(t^{-1})$ only in the cases marked by an asterisk. In the cases marked by $'$ one can avoid the computation of the Alexander polynomial excluding them by the arguments similar to those in Step 2. In each of these cases we find p such that $\operatorname{sign} Q(u^2) = -(-1)^{s/2} \operatorname{sign} Q(0)$ where $\varphi(u) = e^{2\pi i p/q}$ and $q = |e_2 - e_2^0|$. This reduces the computations considerably: for instance, $\deg(\Delta_b) = 277$ for $b = b_{1,11}^{(2)}(60)$. □

Remark. To do the computations, I used the program "Mathematica 3.0 for Silicon Graphics. Copyright 1988-96 Wolfram Research. Inc." (see [Wo]). First, to find the intervals where $\operatorname{sign} Q(z)$ is constant, I used the command `NRoots[Q==0,z]` hoping that it finds approximations of the roots with the declared precision. However, I found that this command gives absolutely wrong results when the degree of the polynomial is large. Therefore, I justified the intervals presented in Tables 9a–g using Sturm's method.

Table 9a. Braids (12) corresponding to $\langle 1 \sqcup \langle 3 \rangle \sqcup \langle 16 \rangle \rangle$

b	(e_2, s)	I	$q_{\max}$	sq.	b	(e_2, s)	I	$q_{\max}$	sq.
$b_{1,15}^{(1)}$	(0,0)	$[0.49, 0.77]$	28	$\{-6\}$	$b_{5,11}^{(1)}$		see §4.7		
$b_{1,15}^{(2)}$	(0,0)	$[0.35, 0.55]$	30	$\{-4\}$	$b_{5,11}^{(2)}$	(0,0)	$[3.37, 6.05]$	26	$\{-20', -10\}$
$b_{1,15}^{(3)}$	(1,0)	$[1.07, 3.00]$	12	$\{-3\}$	$b_{5,11}^{(3)}$	(1,0)	$[1.08, 2.38]$	16	$\varnothing$
$b_{1,15}^{(4)}$	(0,0)	$[1.19, 3.00]$	14	$\{2\}$	$b_{5,11}^{(4)}$	(0,0)	$[1.19, 5.23]$	8	$\{8'\}$
$b_{1,15}^{(5)}$	(1,0)	$[1.06, 7.63]$	6	$\varnothing$	$b_{5,11}^{(5)}$	(1,0)	$[1.06, 2.65]$	14	$\{13\}$
$b_{1,15}^{(6)}$	(1,0)	$[1.11, 4.12]$	10	$\{11'\}$	$b_{5,11}^{(6)}$	(1,0)	$[1.11, 7.44]$	6	$\varnothing$

5 Constructions

The real scheme $\langle 1 \sqcup 1\langle 1 \rangle \sqcup 1\langle 18 \rangle \rangle$ is realized by a real pseudo-holomorphic curve in [O1]. The real schemes $\langle 4 \sqcup 1\langle 2 \sqcup 1\langle 14 \rangle \rangle \rangle$ and $\langle 14 \sqcup 1\langle 2 \sqcup 1\langle 4 \rangle \rangle \rangle$ are realized by real pseudo-holomorphic curves in [O4].

5.1 Construction of $\langle 7 \sqcup 1\langle 2 \sqcup 1\langle 11 \rangle \rangle \rangle$. Let C_0 be the union of four ellipses on $\mathbf{RP}^2$ which pairwise touch each other with the tangency of order four at the same point q. Let p be a generic point on $\mathbf{RP}^2$. We shall use a local analogue of Proposition 2.2. Namely, a given $\mathcal{L}_p$-scheme is realizable

Table 9b. Braids (12) corresponding to $\langle 1 \sqcup \langle 5\rangle \sqcup \langle 14\rangle\rangle$

b	(e_2,s)	I	$q_{\max}$	sq.	b	(e_2,s)	I	$q_{\max}$	sq.
$b^{(1)}_{1,13}$	$(0,0)$	$[22.37, 2341.43]$	16	$\{-6^*\}$	$b^{(1)}_{5,9}$	$(0,0)$	$[14.78, 1449.51]$	12	$\{-10\}$
$b^{(2)}_{1,13}$	$(0,0)$	$[1.27, 2.12]$	24	$\varnothing$	$b^{(2)}_{5,9}$	$(0,0)$	$[3.23, 5.48]$	30	$\{6\}$
$b^{(3)}_{1,12}$	$(1,0)$	$[3.36, 7.01]$	22	$\varnothing$	$b^{(3)}_{5,9}$	$(1,0)$	$[1.85, 3.00]$	28	$\varnothing$
$b^{(4)}_{1,13}$	$(0,0)$	$[1.33, 5.55]$	10	$\varnothing$	$b^{(4)}_{5,9}$	$(0,0)$	$[1.33, 1.74]$	46	$\{-8\}$
$b^{(5)}_{1,13}$	$(1,0)$	$[4.37, 7.33]$	34	$\{11\}$	$b^{(5)}_{5,9}$	$(1,0)$	$[2.58, 3.71]$	40	$\varnothing$
$b^{(6)}_{1,13}$	$(1,0)$	$[3.00, 7.57]$	18	$\varnothing$	$b^{(6)}_{5,9}$	$(1,0)$	$[1.36, 2.38]$	22	$\varnothing$
$b^{(1)}_{3,11}$	$(0,0)$	$[15.94, 1947.89]$	14	$\varnothing$	$b^{(1)}_{7,7}$	$(0,0)$	$[34.46, 838.76]$	22	$\{-2^*\}$
$b^{(2)}_{3,11}$	$(0,0)$	$[3.61, 4.91]$	50	$\{14'\}$	$b^{(2)}_{7,7}$	$(0,0)$	$[10.81, 26.8]$	30	$\{-6\}$
$b^{(3)}_{3,11}$	$(1,0)$	$[0.11, 0.21]$	28	$\varnothing$	$b^{(3)}_{7,7}$	$(1,0)$	$[3.00, 7.28]$	18	$\varnothing$
$b^{(4)}_{3,11}$	$(0,0)$	$[3.79, 5.80]$	38	$\{-16\}$	$b^{(4)}_{7,7}$	$(0,0)$	$[1.31, 5.21]$	10	$\{4^*\}$
$b^{(5)}_{3,11}$	$(1,0)$	$[0.65, 0.82]$	54	$\varnothing$	$b^{(5)}_{7,7}$	$(1,0)$	$[4.40, 7.48]$	32	$\varnothing$
$b^{(6)}_{3,11}$	$(1,0)$	$[1.41, 3.00]$	16	$\varnothing$	$b^{(6)}_{7,7}$	$(1,0)$	$[1.37, 8.96]$	8	$\varnothing$

by a pseudo-holomorphic smoothing of C_0 if and only if the braid $b = b_{\mathbf{R}}\Delta^4$ is quasipositive where $b_{\mathbf{R}}$ is constructed according to Algorithm 2.1.

In Figure 7, we depicted a curve by a dashed line and the corresponding braid by a solid line. As in Figure 5, a box $\boxed{n}$ means σ_k^n for the braid and $\times_k^{|n|}$ for the $\mathcal{L}_p$-scheme. So, we have

$$b = \sigma_2^{-1}\tau_{2,1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-6}\tau_{1,3}\sigma_3^{-8}\sigma_2^{-1}\sigma_3^{-2}\tau_{3,2}\sigma_2^{-1}\tau_{2,3}\sigma_3^{-1}\tau_{3,2}\,\Delta^4$$

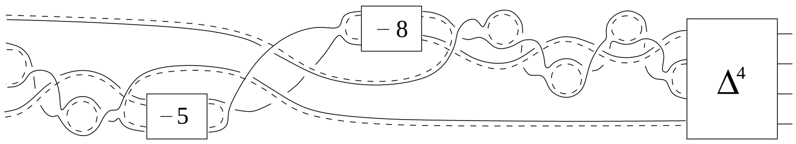

Figure 7

One can check that

$$b = (a_1^{-1}\sigma_3 a_1)(a_2^{-1}\sigma_2 a_2) \qquad \text{where} \quad a_1 = \sigma_2^2\sigma_1\sigma_2, \quad a_2 = \sigma_3^2\sigma_2\sigma_1^2\sigma_2\sigma_3^2\sigma_2\sigma_1\,.$$

5.2 Construction of $\langle \mathbf{1} \sqcup \mathbf{1}\langle \mathbf{7}\rangle \sqcup \mathbf{1}\langle \mathbf{12}\rangle\rangle$. We proceed as in section 5.1 but using the smoothing depicted in Figure 8.

$$b(e_1,e_2) = \sigma_2^{-11}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{1+e_1}\sigma_3^{1-e_1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-6}\cdot$$
$$\cdot\,\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{1+e_2}\sigma_3^{1-e_2}\sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_3^{-1}\sigma_2^{-1}\sigma_3\,\Delta^4.$$

A corresponding pseudo-holomorphic curve exists iff there exist integers

Table 9c. Braids (12) corresponding to $\langle 1 \sqcup \langle 7\rangle \sqcup \langle 12\rangle\rangle$

b	(e_2,s)	I	$q_{\max}$	sq.	b	(e_2,s)	I	$q_{\max}$	sq.
$b^{(1)}_{1,11}$	$(0,0)$	$[0.47,0.69]$	34	$\{-18,-6^*,-4\}$	$b^{(4)}_{3,9}$	$(0,0)$	$[0.50,0.92]$	20	$\varnothing$
$b^{(2)}_{1,11}$	$(0,0)$	$[0.38,0.46]$	72	$\{-10^*,-6,60'\}$	$b^{(5)}_{3,9}$	$(1,0)$	$[0.42,0.78]$	20	$\varnothing$
$b^{(3)}_{1,11}$	$(1,0)$	$[1.09,2.29]$	16	$\{-11',-7^*,3\}$	$b^{(6)}_{3,9}$	$(1,0)$	$[1.35,2.59]$	20	$\{5,15'\}$
$b^{(4)}_{1,11}$	$(0,0)$	$[0.47,0.93]$	18	$\{4,8^*\}$	$b^{(1)}_{5,7}$	$(0,0)$	$[7.26,21.26]$	22	$\{-2,2\}$
$b^{(5)}_{1,11}$	$(1,0)$	$[0.45,0.78]$	22	$\{-3,7,9^*,21'\}$	$b^{(2)}_{5,7}$	$(0,0)$	$[6.58,12.71]$	30	$\{-12\}$
$b^{(6)}_{1,11}$	$(1,0)$	$[1.13,5.00]$	8	$\{-5',5^*,9\}$	$b^{(3)}_{5,7}$	$(1,0)$	$[1.09,2.30]$	16	$\varnothing$
$b^{(1)}_{3,9}$	$(0,0)$	$[0.44,0.77]$	22	$\{-4,-2\}$	$b^{(4)}_{5,7}$	$(0,0)$	$[0.45,0.93]$	16	$\{10'\}$
$b^{(2)}_{3,9}$	$(0,0)$	$[2.86,4.07]$	42	$\varnothing$	$b^{(5)}_{5,7}$	$(1,0)$	$[0.46,0.78]$	24	$\varnothing$
$b^{(3)}_{3,9}$	$(1,0)$	$[0.25,0.69]$	12	$\{-7\}$	$b^{(6)}_{5,7}$	$(1,0)$	$[3.86,9.08]$	20	$\{-3\}$

Table 9d. Braids (12) corresponding to $\langle 1 \sqcup \langle 9\rangle \sqcup \langle 10\rangle\rangle$

b	(e_2,s)	I	$q_{\max}$	sq.	b	(e_2,s)	I	$q_{\max}$	sq.
$b^{(1)}_{1,9}$	(2,0)	$[1.25,3.65]$	12	$\{0^*\}$	$b^{(1)}_{3,7}$		see §4.7		
$b^{(2)}_{1,9}$	(0,0)	$[0.37,0.69]$	20	$\{12\}$	$b^{(2)}_{3,7}$	(0,0)	$[1.83,3.00]$	26	$\{-2\}$
$b^{(3)}_{1,9}$	(1,0)	$[1.23,3.00]$	14	$\{-7\}$	$b^{(3)}_{3,7}$	(1,0)	$[1.19,2.41]$	18	$\{-15',-5,5,9'\}$
$b^{(4)}_{1,9}$	(0,0)	$[1.155,3.0]$	12	$\varnothing$	$b^{(4)}_{3,7}$	(0,0)	$[1.22,8.41]$	6	$\varnothing$
$b^{(5)}_{1,9}$	(1,0)	$[1.23,4.19]$	10	$\{-7\}$	$b^{(5)}_{3,7}$	(1,0)	$[1.20,2.68]$	16	$\{16'\}$
$b^{(6)}_{1,9}$	(1,0)	$[1.24,4.30]$	10	$\{5^*\}$	$b^{(6)}_{3,7}$	(1,0)	$[1.25,3.00]$	14	$\{-11',-7',3',13'\}$
$b^{(1)}_{5,5}$	(0,0)	$[2.28,8.38]$	12	$\varnothing$	$b^{(5)}_{5,5}$	(1,0)	$[1.22,2.63]$	17	$\varnothing$

e_1 and e_2 such that the braid $b(e_1,e_2)$ is quasipositive. One can check that

$$b(3,7)=\sigma_3^2\sigma_2\cdot\sigma_1\cdot\sigma_2^{-1}\sigma_3^{-2}.$$

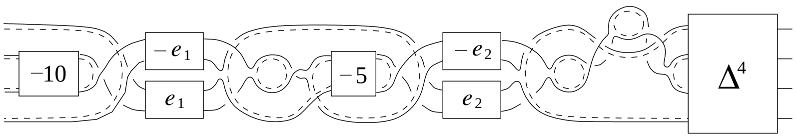

Figure 8

5.3 Construction of $\langle 1 \sqcup 1\langle 9\rangle \sqcup 1\langle 10\rangle\rangle$. As we have seen in sections 4.6 and 4.8, this real scheme is realizable as the set of real points of a real pseudo-holomorphic curve if and only if one of the following three braids is quasipositive: either the braid (13) with $(\beta_1,\beta_2,\beta_3;\nu,e_1)=(7,2,1;5,6)$,

Table 9e. Braids (12) corresponding to $\langle 1 \sqcup \langle 11\rangle \sqcup \langle 8\rangle\rangle$

b	(e_2,s)	I	$q_{\max}$	sq.	b	(e_2,s)	I	$q_{\max}$	sq.
$b_{1,7}^{(1)}$	(0,0)	$[3.78, 5.16]$	52	$\{-48', -26, 2, 4^*, 12\}$	$b_{1,7}^{(2)}$	(2,0)	$[1.49, 7.33]$	8	$\{-6', 0^*\}$
$b_{1,7}^{(3)}$	(1,0)	$[3.00, 7.68]$	16	$\{-15', -5^*\}$	$b_{1,7}^{(4)}$	(0,0)	$[2.37, 7.30]$	14	$\{-2^*, 4\}$
$b_{1,7}^{(5)}$	(1,0)	$[4.10, 7.57]$	28	$\{-21', -7', 7^*, 15, 21\}$	$b_{1,7}^{(6)}$	(1,0)	$[3.00, 9.47]$	14	$\{3^*, 13\}$

Table 9f. Braids (12) corresponding to $\langle 1 \sqcup \langle 15\rangle \sqcup \langle 4\rangle\rangle$

b	(e_2,s)	I	$q_{\max}$	sq.	b	(e_2,s)	I	$q_{\max}$	sq.
$b_{1,3}^{(1)}$	(0,0)	$[0.39, 0.56]$	36	$\{-18, 6^*, 12'\}$	$b_{1,3}^{(4)}$	(0,2)	$[0.00, 0.03]$	18	$\{-8, -2, 2, 4\}$
$b_{1,3}^{(2)}$	(0,0)	$[1.71, 2.60]$	32	$\{-6, -4, 0', 6, 26'\}$	$b_{1,3}^{(5)}$	(1,0)	$[2.32, 4.03]$	26	$\{-13, 11, 17'\}$
$b_{1,3}^{(3)}$	(1,0)	$[1.90, 3.00]$	30	$\{-13, -7, -3, -1\}$	$b_{1,3}^{(6)}$	(1,0)	$[1.26, 4.68]$	10	$\{-1, 1, 5^*, 11'\}$

or $b_{1,9}^{(1)}(0)$, or $b_{1,9}^{(6)}(5)$. One can check that $b_{1,9}^{(1)}(0)$ is equal to

$$(b_7^{-1}\sigma_5 b_7)(a_1^{-1}\sigma_6 a_1)(a_2^{-1}\sigma_7 a_2)(a_3^{-1}\sigma_3 a_3)(a_4^{-1}\sigma_6 a_4)(a_5^{-1}\sigma_1 a_5)(a_6^{-1}\sigma_2 a_6)$$

where (recall that $\pi_{j,k}$ denotes $\sigma_j \ldots \sigma_k$; see section 2.3)

$$a_1 = \sigma_5^{-1}\sigma_6\, a_2, \quad a_2 = \sigma_6^6\sigma_5\sigma_4\sigma_5^{-1}\, c, \quad a_3 = \sigma_4\pi_{4,7}\sigma_5\, c, \quad a_4 = \sigma_7^{-2}\pi_{6,3}\, a_6,$$
$$a_5 = \sigma_2^{-1}\sigma_3\, a_6, \quad a_6 = \pi_{3,7}^2(\sigma_3\sigma_5\sigma_4)^2, \quad b_n = \sigma_6^n\sigma_7\sigma_5^{-2}, \quad c = \pi_{4,7}\sigma_5\sigma_4.$$

5.4 Construction of $\langle 1 \sqcup 1\langle 15\rangle \sqcup 1\langle 4\rangle\rangle$. This real scheme is realizable by a real pseudo-holomorphic curve if and only if one of the following four braids is quasipositive: either the braid (13) with $(\beta_1, \beta_2, \beta_3; \nu, e_1) = (1, 2, 1; 3, -6)$ or $(1, 2, 1; 5, 6)$, or the braid $b_{1,3}^{(1)}(6)$, or $b_{1,3}^{(6)}(5)$. One can check that $b_{1,3}^{(1)}(6)$ is equal to

$$(b_{13}^{-1}\sigma_5 b_{13})(\tilde a_1^{-1}\sigma_7\tilde a_1)(\tilde a_2^{-1}\sigma_6\tilde a_2)(a_3^{-1}\sigma_3 a_3)(a_4^{-1}\sigma_6 a_4)(a_5^{-1}\sigma_1 a_5)(a_6^{-1}\sigma_2 a_6)$$

where

$$\tilde a_1 = \sigma_6^{-1}\sigma_5^{-1}\sigma_7^4\,\tilde a_2, \qquad \tilde a_2 = \sigma_5\sigma_6^{-1}\sigma_7\sigma_4\sigma_5^{-1}\, c$$

and the braids $a_3, \ldots, a_6$, b_n, and c are the same as in section 5.3.

Appendix. A program for computation of Seifert matrix of a braid.

In this appendix we present two computer programs for Mathematica [Wo] which were used in sections 4.6 – 4.8 (and also in [O1-5], [OP]). The first one, `SeifertMatrix`, computes Seifert matrix of an explicitly given braid. The second one, `ssmW` (**s**ymmetrised **s**eifert **m**atrix W), allows to find the determinant of a braid which contains factors of the form $\sigma_{k_1}^{e_1}, \ldots, \sigma_{k_n}^{e_n}$ with

Table 9g. Braids (12) corresponding to $\langle 1 \sqcup \langle 17\rangle \sqcup \langle 2\rangle\rangle$

b	(e_2,s)	I	$q_{\max}$	sq.	b	(e_2,s)	I	$q_{\max}$	sq.
$b_{1,1}^{(1)}$	(0,0)	$[0.19, 0.31]$	32	$\{-14', 4^*, 22'\}$	$b_{1,1}^{(4)}$	(0,0)	$[1.06, 1.82]$	22	$\{-8, -2^*, 4\}$
$b_{1,1}^{(2)}$	(2,0)	$[1.08, 3.92]$	10	$\{-6, 0^*, 6\}$	$b_{1,1}^{(5)}$	(1,2)	$[1.12, 1.94]$	22	$\{-13', -1^*, 13^*\}$
$b_{1,1}^{(3)}$	(1,0)	$[1.11, 2.15]$	19	$\{-11^*, 3^*\}$	$b_{1,1}^{(6)}$	(1,2)	$[0.00, 0.03]$	18	$\{-5^*, 9^*\}$

known k_j's and unknown e_j's as a polynomial in $e_1, \dots, e_n$ (in fact, it computes the matrix W from [O1, section 2.6]).

A.1 The texts of the programs.

```
SeifertMatrix=Function[{m,brd}, Module[{n,e,V,X,q,c,i,j,h,a,b},
   a={{ 0,1,-1,0},{-1, 0,1,0},{0,0,0,0},{1,-1,0,0}};
   b={{-1,1, 0,0},{ 1,-1,0,0},{0,0,0,0},{0, 0,0,0}};
   n=Length[brd]; V=Table[0,{i,n},{j,n}]; X=Table[{n},{h,m-1}];
   Do[ h=Abs[brd[[q]]]; e=Sign[brd[[q]]];
      c[1]=X[[h,1]]; X[[h]]={c[2]=q};
      c[3]=If[h<m-1,X[[h+1,1]],n]; c[4]=If[h>1,X[[h-1,1]],n];
      Do[Do[ V[[ c[i],c[j] ]] += a[[i,j]]+e*b[[i,j]], {i,4}],{j,4}],
   {q,n}];
   Transpose[Delete[Transpose[Delete[V,X]],X]]/2
]];
```

```
ssmW=Function[{m,brd}, Module[{bq,n,d,e,V,X,p=1,q,r=1,c,i,j,h,a,b},
  a={{0,0,-1,1, 2},{0,0,1,-1,-2},{-1,1,0,0,0},{1,-1,0,0,0},
     {2,-2,0,0,0}}; b={{-2,2,0,0},{2,-2,0,0},{0,0,0,0},{0,0,0,0}};
  d=n=Length[brd]; Do[If[Not[IntegerQ[brd[[q]]]],d++;p++],{q,n}];
  V=Table[0,{i,d},{j,d}]; X=Table[{d},{h,m-1}];
  Do[ bq=brd[[q]];
    If[ IntegerQ[bq], h=Abs[bq]; e=Sign[bq],
      h=Abs[bq[[2]]]; V[[r,r]]=2*bq[[1]]*Sign[bq[[2]]]; c[5]=r++;
    ];
    c[1]=X[[h,1]]; X[[h]]={c[2]=p++};
    c[3]=If[h<m-1,X[[h+1,1]],d]; c[4]=If[h>1,X[[h-1,1]],d];
    If[ IntegerQ[bq],
      Do[Do[ V[[ c[i],c[j] ]] += a[[i,j]]+e*b[[i,j]], {i,4}],{j,4}],
      Do[Do[ V[[ c[i],c[j] ]] += a[[i,j]], {i,5}],{j,5}]
    ],
  {q,n}];
  Transpose[Delete[Transpose[Delete[V,X]],X]]/2
]];
```

A.2 User's guide. Suppose that `m` is a positive integer and `brd` is a list of non-zero integers which range between $\pm(m-1)$. The command

`SeifertMatrix[m,brd]`

computes a Seifert matrix of the braid $b = \sigma_{k_1}^{\varepsilon_1} \dots \sigma_{k_n}^{\varepsilon_n}$ where k_q and ε_q are respectively the absolute value and the sign of the q-th element of the list `brd`.

REMARK. The number of connected components of the corresponding Seifert surface F is equal to $m - \mathrm{Card}\{k_1, \dots, k_n\}$. Thus, F is connected if and only if each of the numbers $1, \dots, m-1$ appears among $k_1, \dots, k_n$.

EXAMPLE 1. Computation of a Seifert matrix of the trefoil (we suppose that the program of section A.1 is written in the file `sm.mat`).

```
In[1]:= <<sm.mat;

In[2]:= V=SeifertMatrix[2,{1,1,1}]

Out[2]= {{-1, 1}, {0, -1}}
```

EXAMPLE 2. Computation of the Alexander polynomial of the torus knot $(3,4)$.

```
In[3]:= V=SeifertMatrix[3,{1,2,1,2,1,2,1,2}];

In[4]:= Factor[Det[ V - t*Transpose[V] ]]

                    2          2     4
Out[4]= (1 - t + t ) (1 - t  + t )
```

EXAMPLE 3. Computation of the Alexander polynomial of the braid (13) with $(\alpha_1; \beta_1, \beta_2, \beta_3; \nu, e_1) = (15; 1, 2, 1; 5, 6)$; see the last line of Table 7 in section 4.6.

```
In[5]:=De={1,2,3,4,5,6,7,1,2,3,4,5,6,1,2,3,4,5,1,2,3,4,1,2,3,1,2,1};

In[6]:=b1={-4,-3,-5,-4}; b2={-6,-7,4,5}; bb2={-5,-4,6};

In[7]:=b3[3]={-3,4,-3,-4,3}; b3[5]={-5,4,-5,-4,5};

In[8]:=b=Join[b1,{-5,-5,-5,-5,-5,3,3,3,3,3,3,3,-4,-5},b2,
      Table[-6,{14}],bb2,{5,-4,-4},b3[5],De]; V=SeifertMatrix[8,b];

In[9]:=Factor[Det[ V - t*Transpose[V] ]]

                4  24          2     3     4 2             2     3     4 2
Out[9]=(-1 + t)  t   (1 - t + t  - t  + t )  (1 + t + t  + t  + t )
```

Now suppose that `brd` is a list whose elements are of two kinds. An element of the first kind is an integer h such that $0 < |h| < m$; an element of the second kind is a list of the form $\{\,iter,\ h\,\}$ where $iter$ is any expression and h is an integer such that $0 < |h| < m$. The command

`ssmW[m,brd]`

computes the matrix W^J (see [O1, Section 2.6]) associated with the braid $b = \sigma_{k_1}^{e_1} \dots \sigma_{k_n}^{e_n}$ where J is the set of indices of the second kind and (k_q, e_q) are defined as follows. If the q-th element of `brd` is an integer h then $k_q = |h|$ and $e_q = \operatorname{sign} h$. If the q-th element of `brd` is $\{\,iter,\ h\,\}$ then $k_q = |h|$ and $e_q = iter \times \operatorname{sign} h$.

Recall the following property of the matrix W^J (a specialisation of [O1, Corollary 2.11] for the case when $|e_q| = 1$ for $q \notin J$):

$$\operatorname{Sign} b = \operatorname{Sign} W^J - \sum_{q\in J}(e_q - \operatorname{sign} e_q)\,, \quad \operatorname{Null} b = \mu(F) + \operatorname{Null} W^J\,,$$
$$\det b = \pm \det W^J$$

where $\mu(F)$ is the number of connected components of the Seifert surface. As above, $\mu(F) = m - \operatorname{Card}\{k_1, \dots, k_n\}$.

Example 4. Computation of the determinant of the torus knot $(2, n)$.

```
In[10]:= ssmW[2,{{n,1}}]

Out[10]= {{n}}

In[11]:= W=ssmW[2,{1,1,{n-2,1}}]

Out[11]= {{-2 + n, 0, 1}, {0, -2, 1}, {1, 1, -1}}

In[12]:= Det[W]

Out[12]= n
```

Example 5. Computation of $d_{3,3}^{(k)}(e_2) \mod 5$ and $d_{1,5}^{(k)}(e_2) \mod 25$ (see the last two lines of Table 8 in section 4.7) for $k = 1$.

```
In[13]:= alpha1=13; alpha2=6;

In[14]:= b=Join[b1,{{1-e1,5},{e1,3}},b2,{{1-alpha1,6}},bb2,
          {{-e2,5},{1+e2,3},{-beta1,4}},b3[3],{{1-beta2,4}},De];

In[15]:= W=ssmW[8,b];

In[16]:= d=Det[W//.{beta1->3,beta2->3,e1->12-alpha2-e2}]/16

                                      2
Out[16]= 6817 - 1200 e2 + 300 e2
```

This is $d = d^{(1)}_{3,3}(e_2)$ as a polynomial in e_2. It is clear that $d \equiv 2 \mod 5$ for any e_2, hence d cannot be $\pm n^2$.

```
In[17]:= d=Det[W//.{beta1->1,beta2->5,e1->12-alpha2-e2}]/16

                                              2
Out[17]= 2833 - 624 e2 + 108 e2

In[18]:= Union[Table[ Mod[d//.e2->n,25], {n,25} ]]

Out[18]= {2, 3, 7, 8, 12, 13, 15, 17, 18, 22, 23}
```

This is the set of all possible values of $d^{(1)}_{1,5}(e_2) \mod 25$. Its intersection with the set of possible values of $\pm n^2 \mod 25$ is empty. Indeed,

```
In[19]:= Union[Table[Mod[n*n,25],{n,25}],Table[Mod[-n*n,25],{n,25}]]

Out[19]= {0, 1, 4, 6, 9, 11, 14, 16, 19, 21, 24}
```

A.3 The implemented algorithms. The algorithm implemented in the program `SeifertMatrix` is a modification of the algorithm described in [O1, Section 2.5]. It is based on the following observation. Let $b = \sigma_{k_1}^{\varepsilon_1} \dots \sigma_{k_n}^{\varepsilon_n}$, $\varepsilon_q = \pm 1$. Let F be the Seifert surface of b constructed in [O1, Section 2.5]. We shall denote the Seifert form by $V(x,y)$, $x,y \in H_1(F)$. Let us choose the base $x_1,\dots,x_s$ of $H_1(F)$ as in [O1]. Its elements correspond to the bounded regions (we shall denote them by $X_1,\dots,X_s$) of the complement of the standard projection of the braid b onto the plane.

It is easy to check that the description of the form V given in [O1] can be reformulated as follows. $V(x_\mu, x_\nu) = \sum_{q=1}^n V_q(x_\mu, x_\nu)$ is the sum of the local contributions over all the crossings. All the non-zero contributions of the q-th crossing (i.e. of the factor $\sigma_{k_q}^{\varepsilon_q}$) are

$$V_q(x_{c_i(q)}, x_{c_j(q)}) = (a_{ij} + \varepsilon_q b_{ij})/2$$

where $X_{c_1(q)},\dots,X_{c_4(q)}$ are the regions which are respectively to the left, right, up, and down from the crossing point and a_{ij}, b_{ij} are the entries of the matrices

$$A = \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The algorithm is as follows. We prepare a zero $n \times n$-matrix `V`. Its rows and columns correspond to the extended set of the regions $X_1,\dots,X_n$ to whom we added those which are unbounded from the right. We numerate the regions according to their left corners. Then we skip the crossings from the left to the right accumulating their contributions in the matrix `V`. Between each two iterations (i.e. between two crossings) the list `X` contains the indices of all the regions intersected by a vertical line which passes between the crossing. At the end, we delete the rows and columns indexed by the list `X` (they correspond to the unbounded regions). Since we know from advance that the last region will be deleted, we use it as the "dummy region" throughout the computation.

The algorithm implemented in the program `ssmW` is that from [O1, Section2.6] modified in a similar way.

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