NEW M-CURVE OF DEGREE 8

S.Yu. Orevkov

[1]

A real algebraic curve in $\mathbf{RP}^2$ is called *M-curve* if it has the maximal possible number $(m-1)(m-2)/2+1$ of connected components where m is the degree. *In this note we construct an M-curve of degree* 8 *which is arranged up to isotopy as in Figure 1* where $\langle n\rangle$ means that there are n ovals outside each other in the corresponding region. Realizability of this arrangement was not previousely known. After this result (together with the recent results of [1, 3] and earlier results of Fiedler, Viro, Korchagin, Shustin, and others, see the surveys [2, 3]) it remains 6 arrangements of 22 ovals whose realizability by M-curves is still unknown.

We shall denote $\mathbf{RP}^1\times\mathbf{RP}^1$ (resp. $\mathbf{RP}^2$ blown-up at one point) by $\mathbf{R}\Sigma_0$ (resp. by $\mathbf{R}\Sigma_1$). We shall depict $\mathbf{R}\Sigma_n$ as a rectangle whose opposite sides are identified according the arrows. If $n=0$ then the sides are $x\times\mathbf{RP}^1$ and $\mathbf{RP}^1\times y$; if $n=1$ then the horizontal sides represent the exceptional curve and the vertical ones represent a fiber (the proper transform of a line passing through the center of the blow-up).

Lemma 1. *There exists a real algebraic curve C_0 on $\mathbf{R}\Sigma_0$ of bidegree $(2,4)$ which is arranged with respect to horizontal lines L_1, L_2, L_3 and vertical lines $L_4,\dots,L_7$ as in Figure 2, (in particular, $(C_0\cdot L_1)_{p_3}=3$) and which has an ordinary double point at p_0 with non-real tangents and two cusps at p_1 and p_5.*

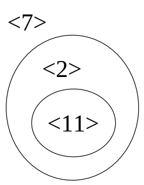

Fig. 1

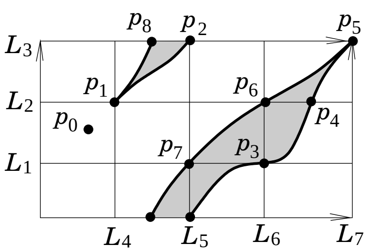

Fig. 2

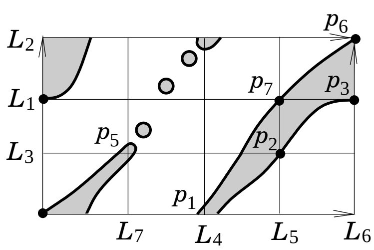

Fig. 3

Proof. The genus formula implies that C_0 must be rational. So, we shall define it by a parametrization $f:\mathbf{P}^1\to\Sigma_0$, $f(t:1)=\big((x:u),(y:z)\big)$ where

$$x=t^2,\qquad u=1,\qquad y=t^2(t+1)(t-t_3),\qquad z=(t-t_1)(t-t_2),$$
$$t_1=\tfrac{5-\sqrt{17}}{4}\approx 0.219\,,\qquad t_2=\tfrac{2-\sqrt{17}}{13}\approx -0.163\,,\qquad t_3=\tfrac{11+\sqrt{17}}{4}\approx 3.781\,.$$

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Let us introduce also the affine coordinates $X = x/u$, $Y = y/z$. One has

$$y(t) + \alpha z(t) = (t+t_1)(t-1)^3, \quad Y(t_0) = Y(-t_0) = -\tfrac{\alpha}{3}, \quad Y(1) = Y(-t_1) = -\alpha$$
$$\text{where} \qquad t_0 = i\sqrt{\tfrac{-4+\sqrt{17}}{3}} \approx 0.203\, i, \qquad \alpha = 2+\sqrt{17} \approx 6.123\,.$$

Let us define the lines $L_1 = \{y + \alpha z = 0\}$, $L_2 = \{y = 0\}$, $L_3 = \{z = 0\}$, $L_4 = \{x = 0\}$, $L_5 = \{x = t_1^2 u\}$, $L_6 = \{x = u\}$, $L_7 = \{u = 0\}$ and the points (in the affine coordinates) $p_0 = (t_0^2, -\frac{\alpha}{3})$, $p_1 = (0,0)$, $p_2 = (t_1^2, \infty)$, $p_3 = (1, -\alpha)$, $p_4 = (t_3^2, 0)$ $p_5 = (\infty, \infty)$, $p_6 = (1, 0)$, $p_7 = (t_1^2, -\alpha)$, $p_8 = (t_2^2, \infty)$. The above formulas imply $(\pm t_0, 0, t_1, 1, t_3, \infty, -1, -t_1, t_2) \mapsto (p_0, \dots, p_8)$. Since $L_j \cdot C_0 = 4$ or 2, we have $C_0 \cap \big(\bigcup L_j\big) = \{p_1, \dots, p_8\}$. Moreover, by the genus formula, C_0 has no self-crossings except p_0. Thus, one can trace C_0 in a unique way. $\square$

Lemma 2. *There exists a real algebraic curve C_1 on $\mathbf{R}\Sigma_0$ of bidegree $(4,4)$ which is arranged with respect to horizontal lines L_1, E_2 and vertical lines L_3, E_7 as in Figure 8. Moreover, $(C_1 \cdot L_1)_{q_6} = 4$ and C_1 has the singularity of the type X_{10} (ordinary tangency of three smooth branches) at q_5.*

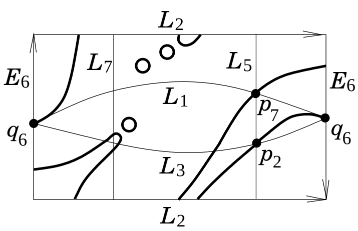

FIG. 4

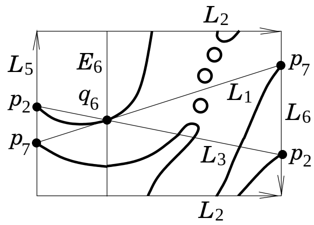

FIG. 5

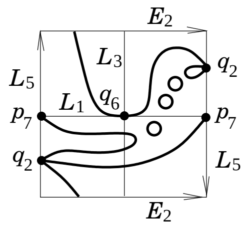

FIG. 6

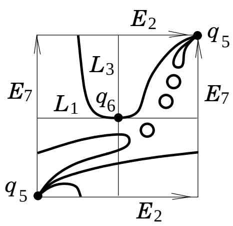

FIG. 7

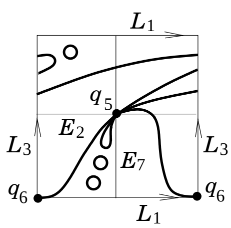

FIG. 8

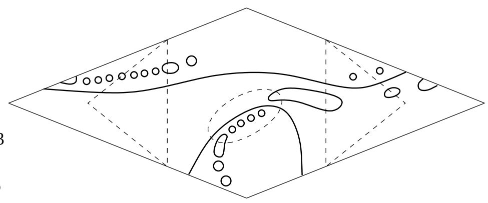

FIG. 9

Proof. The blow-up (resp. blow-down) which transforms a point p into a line E (resp. vice versa) will be denoted below by $p \uparrow E$ (resp. $E \downarrow p$).

Step 1. Smooth out C_0 as in Figure 3.

Step 2. $p_6 \uparrow E_6$, then $L_6 \downarrow q_6$ (we obtain $\mathbf{R}\Sigma_1$ shown in Figures 4 and 5).

Step 3. $p_2 \uparrow E_2$, then $L_2 \downarrow q_2$ (we obtain $\mathbf{R}\Sigma_1$ shown in Figure 6).

Step 4. $p_7 \uparrow E_7$, then $L_5 \downarrow q_5$ (we obtain $\mathbf{R}\Sigma_0$ shown in Figures 7 and 8). $\square$

Smoothing out the singularity X_{10} in Figure 8 (possibility of such a smoothing follows from [4; Lemma 8]) we can obtain a curve defined by a polynomial whose Newton polygon is $[(0,0),(4,0),(4,3),(0,4)]$ and whose chart in the sense of [5] is shown in the middle hexagon in Figure 9. Gluing it together with charts of the triangles $[(4,0),(10,0),(4,3)]$ and $[(10,0),(16,0),(4,3)]$, we obtain the chart of the triangle $[(0,0),(16,0),(0,4)]$ depicted in Figure 9. The charts in the small triangles are taken from [5, 6]; the both of them are birational transforms of the same smoothing of X_{10} (the one used above).

To complete the construction of the curve in Figure 1, we proceed as in [1]. Namely, we start with four real conics in $\mathbf{RP}^2$ which maximally touch each other at the same point and we glue the chart in Figure 9 into the point of the tangency, see Figure 10.

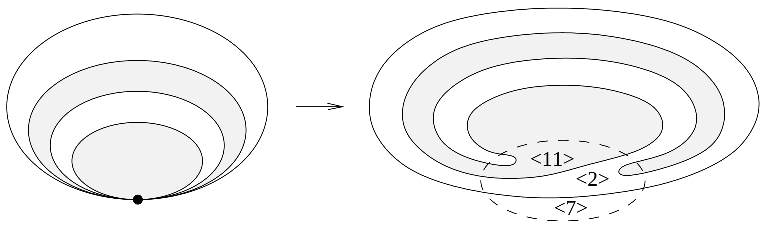

Fig. 10

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Laboratoire E. Picard, UFR MIG, Université Paul Sabatier, 118 route de Narbonne, 31062, Toulouse, France
E-mail address: orevkov@picard.ups-tlse.fr

Steklov Mathematical Institut of Russian Academy of Sciences, ul. Gubkina 6, GSP-1, Moscow, 117966, Russia
E-mail address: orevkov@mi.ras.ru