ON THE HYPERBOLICITY LOCUS OF A REAL CURVE

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Abstract. Given a real algebraic curve in the projective 3-space, its hyperbolicity locus is the set of lines with respect to which the curve is hyperbolic. We give an example of a smooth irreducible curve whose hyperbolicity locus is disconnected but the connected components are not distinguished by the linking numbers with the connected components of the curve.

1. Introduction

Let C be a real algebraic curve in $\mathbb{RP}^n$. We say that it is *hyperbolic* with respect to a codimension 2 projective subspace L, if C is disjoint from L and each hyperplane passing throught L has only real intersections with (the complexification of) C. Such L is called in [5] a *witness to the hyperbolicity of* C.

Shamovich and Vinnikov asked (see the last part of [5, Question 3.13]) whether the set of all witnesses to the hyperbolicity of an irreducible curve C is connected (following [2], we call this set the *hyperbolicity locus* of C and denote it by $\mathcal{H}(C)$). This is evidently true for $n = 2$. However, Kummer and Shaw [2] have shown that the answer is negative for $n = 3$. They gave an example of a sextic genus one smooth real curve C in $\mathbb{RP}^3$ consisting of two topological circles A and B such that C is hyperbolic with respect to each of some two lines L, L' but the linking numbers of A and B with L and L' are different: $\big(\mathrm{lk}(A,L),\mathrm{lk}(B,L)\big) \neq \big(\mathrm{lk}(A,L'),\mathrm{lk}(B,L')\big)$.

Note that in [3, Theorem 3 and Lemma 3.12] we gave (without stating it explicitly) an infinite series of examples with any number of connected components of $\mathcal{H}(C)$. These are the curves $W_g(a_0,\dots,a_g)$ in the notation of [3], in particular, the curve constructed in [2] is our $W_1(2,2)$. As in [2], in all these examples any two components of $\mathcal{H}(C)$ are distinguished by the linking numbers. This fact follows from [3, Proposition 3.13].

So, a new question naturally arises ([2, Question 2]): is it possible that $\mathcal{H}(C)$ is disconnected but the elements of different components have the same linking numbers with all components of C? Here we give an affirmative answer to this question for $n = 3$.

We construct a rational curve C in $\mathbb{RP}^3$ of degree 8 and two lines $L, L' \in \mathcal{H}(C)$ which belong to different connected components of $\mathcal{H}(C)$ because the links $C \cup L$ and $C \cup L'$ are not isotopic in $\mathbb{RP}^3$. The curve C has only one connected component, thus the linking numbers cannot distinguish between the components of $\mathcal{H}(C)$.

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2. The example

2.1. An auxiliary line arrangement. Let (x, y, z) be coordinates in an affine chart of $\mathbb{RP}^3$. Let L be the z-axis and L' be the common line at infinity of the planes $z = \text{const}$. Let R be the rotation by 90° around L. We set: $p_0 = (3, -1, -1)$, $q_0 = (3, 1, 1)$, and $p_k = R^k(p_0)$, $q_k = R^k(q_0)$, $\ell_k = (p_k q_k)$, $\ell'_k = (p_k, q_{k+1})$; see Figure 1 where the black and grey parts of the lines ℓ_k and ℓ'_k represent the sign of the z-coordinate on them.

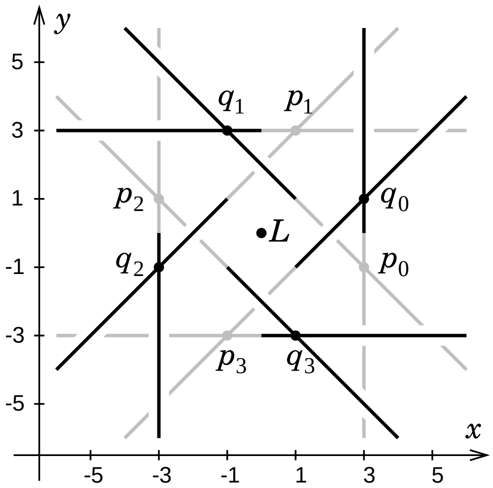

Figure 1. Projection to Oxy

In Figure 2 we show the same line arrangement in the Oxz projection. Here the color represents the sign of the y-coordinate. Note that since Figure 2 is obtained from Figure 1 by a rotation around the axis Ox, the direction of the y-axes in Figure 2 is opposite to the direction of the z-axes in Figure 1.

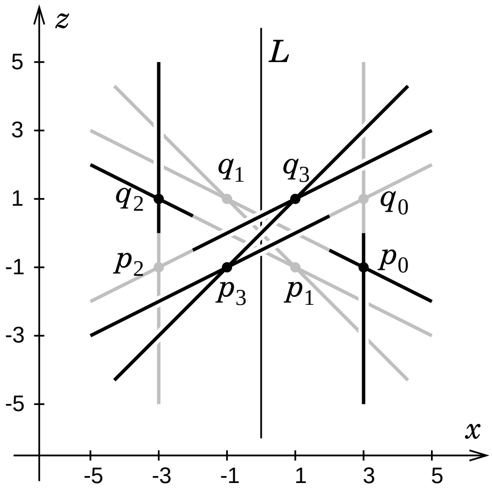

Figure 2. Projection to Oxy

We orient the lines ℓ_k and ℓ'_k so that $dz > 0$ on them. Note that in this case we have $d\theta > 0$ on them as well where $(x, y) = (r\cos\theta, r\sin\theta)$. This means that (under a suitable choice of the orientations of L and L') both L and L' are positively linked with any of the lines ℓ_k, ℓ'_k.

2.2. Construction of the curve C. We perturb the union of 8 lines constructed in §2.1 so that the double point q_0 transforms as in Figure 3(left) and all the other double points (q_1, q_2, q_3, and $p_0, \dots, p_3$) as in Figure 3(right). Such a perturbation is possible due to [1, Theorem 2.4]: one should add the lines one by one. It is easy to see that $L, L' \in \mathcal{H}(C)$.

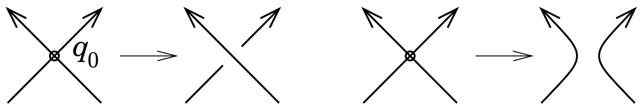

FIGURE 3. Pertubations at q_0 (left) and at other points (right)

2.3. Proof that L and L' belong to different components of $\mathcal{H}(C)$. It is enough to show that the oriented links $C \cup L$ and $C \cup L'$ are not isotopic in $\mathbb{RP}^3$. Indeed, their lifts on the 3-sphere (we denote them Λ and Λ') are distinguished by the *link determinant*, i.e., the determinant of the symmetrized Seifert matrix (the value at -1 of the Alexander polynomial). Namely, we have $\det(\Lambda) = 64$ and $\det(\Lambda') = 0$.

For the computations we used the program from [4, Appendix] available at `http://picard.ups-tlse.fr/~orevkov/sm.mat`. It takes the input in the form of braid. So, for the reader's convenience we give here the braids β and β' whose braid closures are Λ and Λ' respectively.

To find β', we rotate a line by 360° around the origin of the plane in Figure 1 and write down the contributions of all crossings consecutively scanned by this line (including the crossings at infinity). So, we have $\beta' = \beta'_{1/2}\tau_9(\beta'_{1/2})$ where $\beta'_{1/2}$ is the contribution of the rotation by 180° starting at a horizontal position, and $\tau_n : B_n \to B_n$ is the braid group automorphism given by $\sigma_i \mapsto \sigma_{n-i}$. We need to apply τ_9 on the second half-turn because the orientation of the line reverses (see also [3, §§4.3–4.5]). We have

$$
\begin{aligned}
\beta'_{1/2} = {} & \sigma_1\Delta_{45}\sigma_8 && \pm(2,0),\ (\infty,0)\\
& \times \sigma_2^{-1} && (3,1)\quad \text{(no contribution of } (-3,-1))\\
& \times \sigma_3\sigma_6 && \pm(5,3)\\
& \times \sigma_2\Delta_{45}\sigma_7 && \pm(3,3),\ (\infty,\infty)\\
& \times \sigma_3\sigma_6 && \pm(3,5)\\
& \sigma_1\Delta_{45}\sigma_8 && \pm(0,2),\ (0,\infty)\\
& \times \sigma_3\sigma_6 && \pm(-3,5)\\
& \times \sigma_2\Delta_{45}\sigma_7 && \pm(-3,3),\ (-\infty,\infty)\\
& \times \sigma_3\sigma_6 && \pm(-5,3)
\end{aligned}
$$

where $\Delta_{45} = \sigma_4\sigma_5\sigma_4$ is the contribution of each triple crossing at infinity (in the comments we refer to the coordinates of the contributing crossings in Figure 1).

Similarly, $\beta = \beta_{1/2}\tau_9(\beta_{1/2})$ with

$$
\begin{aligned}
\beta_{1/2} = {} & (\sigma_1\sigma_4\sigma_7)\sigma_2^{-1}(\sigma_3\sigma_5)(\sigma_2\sigma_4\sigma_6)(\sigma_3\sigma_5)\\
& \times (\sigma_1\sigma_4\sigma_7)(\sigma_3\sigma_5)(\sigma_2\sigma_4\sigma_6)(\sigma_3\sigma_5)\\
& \times (\sigma_8\sigma_7\dots\sigma_1)
\end{aligned}
$$

where the first two lines represent C and the third line represents L'.

Remark. If we replace the negative crossing in Figure 3(left) by a positive one (which corresponds to replacing σ_2^{-1} by σ_2 in our braids), then $\mathcal{H}(C)$ will be connected by [3, Proposition 3.13] because the obtained 8-th degree curve will be maximally writhed in this case.

References

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