To Vladimir Lin in occasion of his 85-th birthday

HOMOMORPHISMS OF COMMUTATOR SUBGROUPS OF BRAID GROUPS WITH SMALL NUMBER OF STRINGS

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Abstract. For any n, we describe all endomorphisms of the braid group B_n and of its commutator subgroup B_n', as well as all homomorphisms $B_n' \to B_n$. These results are new only for small n because endomorphisms of B_n are already described by Castel for $n \geq 6$, and homomorphisms $B_n' \to B_n$ and endomorphisms of B_n' are already described by Kordek and Margalit for $n \geq 7$. We use very different approaches for $n = 4$ and for $n \geq 5$.

Résumé. Pour tout n nous décrivons tous les endomophismes du groupe de tresses B_n et de son sous-groupe dérivé B_n' ainsi que tous les homomorphismes $B_n' \to B_n$. Ces résultats ne sont nouveaux que pour n petits parce que les endomorphismes de B_n sont déjà décrits par Castel pour $n \geq 6$ et les homomorphismes $B_n' \to B_n$ ainsi que les endomorphismes de B_n' sont décrits par Kordek et Margalit pour $n \geq 7$. Nous utilisons des approches très différentes pour $n = 4$ et pour $n \geq 5$.

Introduction

Let $\mathbf{B}_n$ be the braid group with n strings. It is generated by $\sigma_1, \dots, \sigma_{n-1}$ (called *standard* or *Artin* generators) subject to the relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j|>1; \qquad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ for } |i-j|=1.$$

Let $\mathbf{B}_n'$ be the commutator subgroup of $\mathbf{B}_n$.

In this paper we describe all endomorphisms of $\mathbf{B}_n$ and $\mathbf{B}_n'$ and homomorphisms $\mathbf{B}_n' \to \mathbf{B}_n$ for any n. These results are new only for small n because endomorphisms of $\mathbf{B}_n$ are described by Castel in [4] for $n \geq 6$, and homomorphisms $\mathbf{B}_n' \to \mathbf{B}_n$ and endomorphisms of $\mathbf{B}_n'$ are described by Kordek and Margalit in [11] for $n \geq 7$.

The automorphisms of $\mathbf{B}_n$ and $\mathbf{B}_n'$ have been already known for any n: Dyer and Grossman [5] proved that the only non-trivial element of $\mathrm{Out}(\mathbf{B}_n)$ corresponds to the automorphism Λ defined by $\sigma_i \mapsto \sigma_i^{-1}$ for any $i = 1, \dots, n-1$, and in [17] we proved that the restriction map $\mathrm{Aut}(\mathbf{B}_n) \to \mathrm{Aut}(\mathbf{B}_n')$ is an isomorphism for $n \geq 4$ ($\mathbf{B}_3'$ is a free group of rank 2, thus its automorphisms are known as well; see e.g. [15]).

The problem to study homomorphisms between braid groups and, especially, between their commutator subgroups was posed by Vladimir Lin [12–14] because he found its applications to the problem of superpositions of algebraic functions (the initial motivation for Hilbert's 13th problem); see [13] and references therein.

Let us formulate the main results. We start with those about homomorphisms of $\mathbf{B}_n'$ to $\mathbf{B}_n$ and to itself.

I am grateful to the referee for remarks and corrections.

Theorem 1.1. (proven for $n \geq 7$ in [11]). *Let $n \geq 5$. Then every non-trivial homomorphism $\mathbf{B}'_n \to \mathbf{B}_n$ extends to an automorphism of $\mathbf{B}_n$.*

We prove this theorem in §2. Since $\mathbf{B}''_n = \mathbf{B}'_n$ and $\mathrm{Aut}(\mathbf{B}_n) = \mathrm{Aut}(\mathbf{B}'_n)$ for $n \geq 5$, the following two corollaries are, in fact, equivalent versions of Theorem 1.1.

Corollary 1.2. *If $n \geq 5$, then any non-trivial endomorphism of $\mathbf{B}'_n$ is bijective.*

Corollary 1.3. *If $n \geq 5$, then any non-trivial homomorphism $\mathbf{B}'_n \to \mathbf{B}_n$ is an automorphism of $\mathbf{B}'_n$ composed with the inclusion map.*

Let R be the homomorphism

$$R : \mathbf{B}_4 \to \mathbf{B}_3, \qquad \sigma_1, \sigma_3 \mapsto \sigma_1, \quad \sigma_2 \mapsto \sigma_2 \tag{1}$$

(we denote it by R because, if we interpret $\mathbf{B}_n$ as $\pi_1(X_n)$ where X_n is the space of monic squarefree polynomials of degree n, then R is induced by the mapping which takes a degree 4 polynomial to its cubic resolvent).

For a group G, we denote its commutator subgroup, center, and abelianization by G', $Z(G)$, and $G^{\mathfrak{ab}}$ respectively. We also denote the inner automorphism $y \mapsto xyx^{-1}$ by $\tilde{x}$, the commutator $xyx^{-1}y^{-1}$ by $[x, y]$, and the centralizer of an element x (resp. of a subgroup H) in G by $Z(x; G)$ (resp. by $Z(H; G)$).

Given two group homomorphisms $f : G_1 \to G_2$ and $\tau : G_1^{\mathfrak{ab}} \to Z(\mathrm{im}\, f; G_2)$, we define the *transvection* of f by τ as the homomorphism $f_{[\tau]} : G_1 \to G_2$ given by $x \mapsto f(x)\tau(\bar{x})$ where $\bar{x}$ is the image of x in $G_1^{\mathfrak{ab}}$. To simplify notation, we will not distinguish between τ and its composition with the canonical projection $G_1 \to G_1^{\mathfrak{ab}}$. So, we shall often speak of a transvection by $\tau : G_1 \to Z(\mathrm{im}\, f; G_2)$.

We say that two homomorphisms $f, g : G_1 \to G_2$ are *equivalent* if there exists $h \in \mathrm{Aut}(G_2)$ such that $f = hg$. If, moreover, $h \in \mathrm{Inn}(G_2)$, we say that f and g are *conjugate.*

Theorem 1.4. *Any homomorphism $\varphi : \mathbf{B}'_4 \to \mathbf{B}_4$ either is equivalent to a transvection of the inclusion map, or $\varphi = fR$ for a homomorphism $f : \mathbf{B}'_3 \to \mathbf{B}_4$ (since $\mathbf{B}'_3$ is free [9], it has plenty of homomorphisms to any group).*

We prove this theorem in §3.

Corollary 1.5. *Any endomorphism of $\mathbf{B}'_4$ is either an automorphism or a composition of R with a homomorphism $\mathbf{B}'_3 \to \mathbf{B}'_4$.*

As we already mentioned, $\mathbf{B}'_3$ is free, thus its homomorphisms are evident. Now let us describe endomorphisms of $\mathbf{B}_n$. We say that a homomorphism is *cyclic* if its image is a cyclic group (probably, infinite cyclic).

Theorem 1.6. (proven for $n \geq 6$ in [4]). *If $n \geq 5$, then any non-cyclic endomorphism of $\mathbf{B}_n$ is a transvection of an automorphism.*

For $n \geq 7$, this result is derived in [11] from Theorem 1.1. The same proof works without any change for any $n \geq 5$.

Theorem 1.7. *Any endomorphism of $\mathbf{B}_4$ is either a transvection of an automorphism, or it is of the form fR for some $f : \mathbf{B}_3 \to \mathbf{B}_4$ (see Proposition 1.9 for a general form of such f).*

This theorem also can be derived from Theorem 1.4 in the same way as it is done in [11] for $n \geq 7$.

Let $\Delta = \Delta_n = \prod_{i=1}^{n-1}\prod_{j=1}^{n-i}\sigma_j$ (the Garside's half-twist), $\delta = \delta_n = \sigma_{n-1}\dots\sigma_2\sigma_1$, and $\gamma = \gamma_n = \sigma_1\delta_n$. One has $\delta^n = \gamma^{n-1} = \Delta^2$, and it is known that $Z(\mathbf{B}_n)$ is generated by Δ^2, and each periodic braid (i.e. a root of a central element) is conjugate to δ^k or γ^k for some $k \in \mathbb{Z}$.

It is well-known that $\mathbf{B}_3$ admits a presentation $\langle \Delta, \delta \mid \Delta^2 = \delta^3\rangle$. By combining this fact with basic properties of canonical reduction systems, it is easy to prove the following descriptions of homomorphisms from $\mathbf{B}_3$ to $\mathbf{B}_n$ for $n = 3$ or 4.

Proposition 1.8. *Any non-cyclic endomorphism of $\mathbf{B}_3$ is equivalent to a transvection by τ of a homomorphism of the form $\Delta \mapsto \Delta$, $\delta \mapsto X\delta X^{-1}$ for some $X \in \mathbf{B}_3$ and $\tau : \mathbf{B}_3^{\mathfrak{ab}} \to Z(\mathbf{B}_3) = \langle \Delta^2\rangle$.*

Proposition 1.9. *For any non-cyclic homomorphism $\varphi : \mathbf{B}_3 \to \mathbf{B}_4$, one of the following two possibilities holds:*

(a) *φ is equivalent to a transvection by τ of a homomorphism of the form $\Delta_3 \mapsto \Delta_4$, $\delta_3 \mapsto X\gamma_4 X^{-1}$ for some $X \in \mathbf{B}_4$ and $\tau : \mathbf{B}_3^{\mathfrak{ab}} \to Z(\mathbf{B}_4) = \langle \Delta_4^2\rangle$;*

(b) *φ is equivalent to $(\iota\psi)_{[\tau]}$ where ψ is a non-cyclic endomorphism of $\mathbf{B}_3$, $\iota : \mathbf{B}_3 \to \mathbf{B}_4$ is the standard embedding, and τ is a homomorphism $\mathbf{B}_3^{\mathfrak{ab}} \to Z(\mathbf{B}_4) = \langle \Delta_4^2\rangle$.*

Remark 1.10. Since $\mathbf{B}_n^{\mathfrak{ab}} \cong Z(\mathbf{B}_n) \cong \mathbb{Z}$, the transvection in Theorem 1.6 (and in the non-degenerate case in Theorem 1.7) is uniquely determined by a single integer number. In contrast, $(\mathbf{B}_4')^{\mathfrak{ab}} \cong \mathbb{Z}^2$, thus the transvection in Theorem 1.4 depends on two integers (here $Z(\mathrm{im}(\mathbf{B}_4' \hookrightarrow \mathbf{B}_4); \mathbf{B}_4) = Z(\mathbf{B}_4'; \mathbf{B}_4) = Z(\mathbf{B}_4) \cong \mathbb{Z}$). Notice also that two transvections are involved in the case (b) of Proposition 1.9, thus the general form of φ in this case is

$$\Delta_3 \mapsto f\big(\iota(\Delta_3)^{6k+1}\Delta_4^{6l}\big), \qquad \delta_3 \mapsto f\big(\iota(X\delta_3X^{-1}\Delta_3^{4k})\Delta_4^{4l}\big)$$

with $k, l \in \mathbb{Z}$, $X \in \mathbf{B}_3$, $f \in \mathrm{Aut}(\mathbf{B}_4)$.

2. The case $n \geq 5$

In this section we prove Theorem 1.1 which describes homomorphisms $\mathbf{B}_n' \to \mathbf{B}_n$ for $n \geq 5$. The proof is very similar to the proof of the case $n \geq 5$ of the main theorem of [17] which describes $\mathrm{Aut}\,\mathbf{B}_n'$. As we already mentioned, Theorem 1.1 for $n \geq 7$ is proven by Kordek and Margalit in [11]. Some elements of their proof are valid for $n \geq 5$ (see Proposition 2.4 below) which allowed us to omit a big part of our original proof based on [17].

Let $\mathbf{S}_n$ be the symmetric group. Let $e : \mathbf{B}_n \to \mathbb{Z}$ and $\mu : \mathbf{B}_n \to \mathbf{S}_n$ be the homomorphisms defined on the generators by $e(\sigma_i) = 1$ and $\mu(\sigma_i) = (i, i+1)$ for $i = 1, \dots, n-1$. So, $e(X)$ is the exponent sum (signed word length) of X. Let $\mathbf{P}_n = \ker\mu$ be the pure braid group. Following [12], we denote $\mathbf{P}_n \cap \mathbf{B}_n'$ by $\mathbf{J}_n$, and $\mu|_{\mathbf{B}_n'}$ by μ', thus $\mathbf{J}_n = \ker\mu'$.

For a pure braid X, we denote the linking number between the i-th and the j-th strings of X by $\mathrm{lk}_{ij}(X)$. It can be defined as $\frac{1}{2}e(X_{ij})$ where X_{ij} is the 2-braid obtained from X by removal of all strings except the i-th and the j-th ones. For $1 \leq i < j \leq n$, we set $\sigma_{ij} = (\sigma_{j-1}\dots\sigma_{i+1})\sigma_i(\sigma_{j-1}\dots\sigma_{i+1})^{-1}$ (here $\sigma_{i,i+1} = \sigma_i$). Then $\mathbf{P}_n$ is generated by $\{\sigma_{ij}^2\}_{1\leq i<j\leq n}$ (see [1]) and we denote the image of σ_{ij}^2 in $\mathbf{P}_n^{\mathfrak{ab}}$ by A_{ij}. We use the additive notation for $\mathbf{P}_n^{\mathfrak{ab}}$ and $\mathbf{J}_n^{\mathfrak{ab}}$.

Lemma 2.1. ([17, Lemma 2.3]). $\mathbf{P}_n^{\mathfrak{ab}}$ *(for any n) is free abelian group with basis $(A_{ij})_{1\le i<j\le n}$. The natural projection $\mathbf{P}_n\to\mathbf{P}_n^{\mathfrak{ab}}$ is given by $X\mapsto\sum_{i<j}\operatorname{lk}_{ij}(X)A_{ij}$.*

If $n\ge 5$, then the homomorphism $\mathbf{J}_n^{\mathfrak{ab}}\to\mathbf{P}_n^{\mathfrak{ab}}$ induced by the inclusion map defines an isomorphism of $\mathbf{J}_n^{\mathfrak{ab}}$ with $\{\sum x_{ij}A_{ij}\mid\sum x_{ij}=0\}$ (notice that this statement is wrong for $n=3$ or 4; see [17, Proposition 2.4]).

From now on, till the end of this section, we assume that $n\ge 5$ and $\varphi:\mathbf{B}_n'\to\mathbf{B}_n$ is a non-cyclic homomorphism. Since any group homomorphism $G_1\to G_2$ maps G_1' to G_2', we have $\varphi(\mathbf{B}_n'')\subset\mathbf{B}_n'$. By [9] (see also [17, Remark 2.2]), we have $\mathbf{B}_n''=\mathbf{B}_n'$, thus

$$\varphi(\mathbf{B}_n')\subset\mathbf{B}_n'.$$

Then [12, Theorem D] implies that

$$\varphi(\mathbf{J}_n)\subset\mathbf{J}_n.$$

Thus we may consider the endomorphism φ_* of $\mathbf{J}_n^{\mathfrak{ab}}$ induced by $\varphi|_{\mathbf{J}_n}$. We shall not distinguish between $\mathbf{J}_n^{\mathfrak{ab}}$ and its isomorphic image in $\mathbf{P}_n^{\mathfrak{ab}}$ (see Lemma 2.1).

Following [12], we set

$$c_i=\sigma_1^{-1}\sigma_i\quad(i=3,\dots,n-1)\quad\text{and } c=c_3.$$

Lemma 2.2. *Suppose that $\mu\varphi=\mu'$ and $\varphi(c)=c$. Then $\varphi_*=\mathrm{id}$.*

Proof. The exact sequence $1\to\mathbf{J}_n\to\mathbf{B}_n'\to\mathbf{A}_n\to 1$ defines an action of $\mathbf{A}_n$ on $\mathbf{J}_n^{\mathfrak{ab}}$ by conjugation. Let V be a complex vector space with base $e_1,\dots,e_n$ endowed with the natural action of $\mathbf{S}_n$ induced by the action on the base. We identify $\mathbf{P}_n^{\mathfrak{ab}}$ with its image in the symmetric square $\operatorname{Sym}^2 V$ under the homomorphism $A_{ij}\to e_ie_j$. Then, by Lemma 2.1, we may identify $\mathbf{J}_n^{\mathfrak{ab}}$ with $\big\{\sum x_{ij}e_ie_j\,\big|\,x_{ij}\in\mathbb{Z},\sum x_{ij}=0\big\}$. These identifications are compatible with the action of $\mathbf{A}_n$. Thus $W:=\mathbf{J}_n^{\mathfrak{ab}}\otimes\mathbb{C}$ is a $\mathbb{C}\mathbf{A}_n$-submodule of $\operatorname{Sym}^2 V$.

For an element v of a $\mathbb{C}\mathbf{S}_n$-module, let $\langle v\rangle_{\mathbb{C}\mathbf{S}_n}$ be the $\mathbb{C}\mathbf{S}_n$-submodule generated by v. It is shown in the proof of [17, Lemma 3.1], that $W=W_2\oplus W_3$ where

$$W_2=\langle(e_1-e_2)(e_3+\dots+e_n)\rangle_{\mathbb{C}\mathbf{S}_n},\qquad W_3=\langle(e_1-e_2)(e_3-e_4)\rangle_{\mathbb{C}\mathbf{S}_n},$$

and that W_2 and W_3 are irreducible $\mathbb{C}\mathbf{S}_n$-modules isomorphic to the Specht modules corresponding to the partitions $(n-1,1)$ and $(n-2,2)$ respectively. Since the Young diagrams of these partitions are not symmetric, W_2 and W_3 are also irreducible as $\mathbb{C}\mathbf{A}_n$-modules.

The condition $\mu\varphi=\mu'$ implies that φ_* is $\mathbf{A}_n$-equivariant. Hence, by Schur's lemma, $\varphi_*=a\,\mathrm{id}_{W_2}\oplus b\,\mathrm{id}_{W_3}$. We have the identity

$$(n-2)(e_1-e_2)e_3=(e_1-e_2)(e_3+\dots+e_n)+\sum_{i\ge 4}(e_1-e_2)(e_3-e_i)$$

whence, denoting $e_5+\dots+e_n$ by e,

$$(n-2)\varphi_*((e_1-e_3)e_2)=(e_1-e_3)\big(a(e_2+e_4+e)+b((n-3)e_2-e_4-e)\big),$$
$$(n-2)\varphi_*((e_2-e_4)e_3)=(e_2-e_4)\big(a(e_1+e_3+e)+b((n-3)e_3-e_1-e)\big).$$

The condition $\varphi(c) = c$ implies the φ-invariance of $c^2 \in \mathbf{J}_n$. Since the image of c^{-2} in $\mathbf{J}_n^{\mathfrak{ab}}$ is $A_{12} - A_{34}$, we obtain that $e_1e_2 - e_3e_4$ is φ_*-invariant. Hence

$$\begin{aligned}(n-2)(e_1e_2 - e_3e_4) &= (n-2)\varphi_*(e_1e_2 - e_3e_4)\\ &= (n-2)\varphi_*\big((e_1 - e_3)e_2 + (e_2 - e_4)e_3\big)\\ &= \big(2a + (n-4)b\big)(e_1e_2 - e_3e_4) + (a-b)(e_1 + e_2 - e_3 - e_4)e\end{aligned}$$

Since $\{e_ie_j\}_{i\le j}$ is a base of $\operatorname{Sym}^2 V$, it follows that $2a + (n-4)b = n-2$ and $a - b = 0$ whence $a = b = 1$. $\square$

Lemma 2.3. *Let φ_1 and φ_2 be equivalent homomorphisms $\mathbf{B}'_n \to \mathbf{B}_n$. Then $\mu\varphi_1$ and $\mu\varphi_2$ are conjugate.*

Proof. This fact immediately follows from Dyer – Grossman's [5] classification of automorphisms of $\mathbf{B}_n$ (see the beginning of the introduction) because $\mu\Lambda = \mu$. $\square$

Proposition 2.4. (Kordek and Margalit [11, §3, Proof of Thm. 1.1, Cases 1–3 and Step 1 of Case 4]). *There exists $f \in \operatorname{Aut}(\mathbf{B}_n)$ such that $f\varphi(c_i) = c_i$ for each odd i in the range $3 \le i < n$ (recall that we assume $n \ge 5$).*

This proposition implies, in particular, that $\mu\varphi$ is non-trivial, hence by Lin's result [12, Theorem C] $\mu\varphi$ is conjugate either to μ' or to $\nu\mu'$ (when $n = 6$) where ν is the restriction to $\mathbf{A}_6$ of the automorphism of $\mathbf{S}_6$ given by $(12) \mapsto (12)(34)(56)$, $(123456) \mapsto (123)(45)$ (it represents the only nontrivial element of $\operatorname{Out}(\mathbf{S}_6)$).

Lemma 2.5. *If $n = 6$, then $\mu\varphi$ is not conjugate to $\nu\mu'$.*

Proof. Let H be the subgroup generated by c_3 and c_5. By Lemma 2.3 and Proposition 2.4 we may assume that $\varphi|_H = \mathrm{id}$. Then we have

$$\mu'(H) = \mu\varphi(H) = \{\mathrm{id}, (12)(34), (12)(56), (34)(56)\}.$$

In particular, no element of $\{1,\dots,6\}$ is fixed by all elements of $\mu\varphi(H)$. A straightforward computation shows that

$$\nu\mu'(H) = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}, \tag{2}$$

thus 5 and 6 are fixed by all elements of $\nu\mu'(H)$. Hence these subgroups are not conjugate in $\mathbf{S}_6$. $\square$

Lemma 2.6. *There exists $f \in \operatorname{Aut}(\mathbf{B}_n)$ such that $f\varphi(c) = c$ and $\mu f\varphi = \mu'$.*

Proof. By Proposition 2.4 we may assume that

$$\varphi(c) = c. \tag{3}$$

Then $\mu\varphi$ is non-trivial, hence, by [12, Thm. C] combined with Lemma 2.5, it is conjugate to μ', i.e. there exists $\pi \in \mathbf{S}_n$ such that $\tilde\pi\mu\varphi = \mu'$, i.e. $\pi\mu(\varphi(x)) = \mu(x)\pi$ for each $x \in \mathbf{B}'_n$. For $x = c$ this implies by (3) that π commutes with $(12)(34)$, hence $\pi = \pi_1\pi_2$ where $\pi_1 \in V_4$ (the group in the right hand side of (2)) and $\pi_2(i) = i$ for $i \in \{1,2,3,4\}$. Let $\tilde V_4 = \{1, c, \Delta_4, c\Delta_4\}$. This is not a subgroup but we have $\mu(\tilde V_4) = V_4$. We can choose $y_1 \in \tilde V_4$ and $y_2 \in \langle\sigma_5,\dots,\sigma_{n-1}\rangle$ so that $\mu(y_j) = \pi_j$,

$j = 1, 2$. Let $y = y_1 y_2$. Then we have $\tilde{y}(c) = c^{\pm 1}$ and $\mu\tilde{y}\varphi = \tilde{\pi}\mu\varphi = \mu'$. Thus, for $f = \Lambda^k \tilde{y}$, $k \in \{0, 1\}$, we have $f\varphi(c) = c$ and $\mu f \varphi = \mu'$. □

Due to Lemma 2.6, from now on we assume that $\mu\varphi = \mu'$ and $\varphi(c) = c$. Then, by Lemma 2.2, we have $\varphi_* = \mathrm{id}$, hence (see Lemma 2.1)

$$\mathrm{lk}_{ij}(x) = \mathrm{lk}_{ij}(\varphi(x)) \qquad \text{for any } x \in \mathbf{J}_n \text{ and } 1 \le i < j \le n. \tag{4}$$

Starting at this point, the proof of [17, Thm. 1.1] given in [17, §5], can be repeated almost word-by-word in our setting. The only exception is the proof of [17, Lemma 5.8] (which is Lemma 2.11 below) where the invariance of the isomorphism type of centralizers of certain elements is used as well as Dyer–Grossman result [5]. However, as pointed out in [17, Remark 5.15] (there is a misprint there: $n \ge 6$ should be replaced by $n \ge 5$), there is another, even simpler, proof of Lemma 2.11 based on Lemma 2.7 (see below). This proof was not included in [17] by the following reason. At that time we new only Garside-theoretic proof of Lemma 2.7 while the rest of the proof of the main theorem for $n \ge 6$ used only Nielsen-Thurston theory and results of [12]. So we wanted to make the proofs (at least for $n \ge 6$) better accessible for readers who are not familiar with the Garside theory. Now we learned from [11] that when we wrote that paper, Lemma 2.7 had been already known for a rather long time [2, Lemma 4.9] and the proof in [2] is based on Nielsen-Thurston theory.

In the rest of this section, for the reader's convenience we re-expose Section 5.1 of [17] (Sections 5.2–5.3 can be left without any change). In this re-exposition we give another proof of [17, Lemma 5.8] and omit the lemmas which are no longer needed due to Proposition 2.4.

We shall consider $\mathbf{B}_n$ as a mapping class group of n-punctured disk $\mathbb{D}$. We assume that $\mathbb{D}$ is a round disk in $\mathbb{C}$ and the set of the punctures is $\{1, 2, \dots, n\}$. Given an embedded segment I in $\mathbb{D}$ with endpoints at two punctures, we denote with σ_I the positive half-twist along the boundary of a small neighborhood of I. The set of all such braids is the conjugacy class of σ_1 in $\mathbf{B}_n$. The arguments in the rest of this section are based on Nielsen-Thurston theory. The main tool are the canonical reduction systems. One can use [3], [6], or [10] as a general introduction to the subject. In [17] we gave all precise definitions and statements needed there (using the language and notation inspired mostly by [8]).

Lemma 2.7. ([2, Lemma 4.9], [17, Lemma A.2]). *Let $x, y \in \mathbf{B}_n$ be such that $xyx = yxy$ and each of x and y is conjugate to σ_1. Then there exists $u \in \mathbf{B}_n$ such that $\tilde{u}(x) = \sigma_1$ and $\tilde{u}(y) = \sigma_2$.*

Let $\mathrm{sh}_2 : \mathbf{B}_{n-2} \to \mathbf{B}_n$ be the homomorphism $\mathrm{sh}_2(\sigma_i) = \sigma_{i+2}$. We set

$$\tau = \sigma_1^{(n-2)(n-3)} \, \mathrm{sh}_2(\Delta_{n-2}^{-2}).$$

We have $\tau \in \mathbf{J}_n$ (in the notation of [17], $\tau = \psi_{2,n-2}(1; \sigma_1^{(n-2)(n-3)}, \Delta^{-2})$). Recall that we assume $\varphi(c) = c$, $\mu\varphi = \mu'$, and hence (4) holds.

Lemma 2.8. *Let I and J be two disjoint embedded segments with endpoints at punctures. Then $\varphi(\sigma_I^{-1}\sigma_J) = \sigma_{I_1}^{-1}\sigma_{J_1}$ where I_1 and J_1 are disjoint embedded segments such that $\partial I_1 = \partial I$ and $\partial J_1 = \partial J$.*

Proof. The braid $\sigma_I^{-1}\sigma_J$ is conjugate to c, hence so is its image (because $\varphi(c) = c$). Therefore $\varphi(\sigma_I^{-1}\sigma_J) = \sigma_{I_1}^{-1}\sigma_{J_1}$ for some disjoint I_1 and J_1. The matching of the boundaries follows from (4) applied to $\sigma_I^{-2}\sigma_J^2$. □

Lemma 2.9. (cf. [17, Lemmas 5.1 and 5.3]). *Let C_1 be a component of the canonical reduction system of $\varphi(\tau)$. Then C_1 cannot separate the punctures 1 and 2, and it cannot separate the punctures i and j for $3 \le i < j \le n$.*

Proof. Let $u = \sigma_1^{-1}\sigma_{ij}$, $3 \le i < j \le n$. By Lemma 2.8, $\varphi(u) = \sigma_I^{-1}\sigma_J$ with $\partial I = \{1,2\}$ and $\partial J = \{i,j\}$. Since $\varphi(u)$ commutes with $\varphi(\tau)$, the result follows. □

Lemma 2.10. (cf. [17, Lemma 5.7]). *$\varphi(\tau)$ is conjugate in $\mathbf{P}_n$ to τ.*

Proof. $\varphi(\tau)$ cannot be pseudo-Anosov because it commutes with $\varphi(c)$ which is c by our assumption, hence it is reducible.

If $\varphi(\tau)$ were periodic, then it would be a power of Δ^2 because it is a pure braid. This contradicts (4), hence $\varphi(\tau)$ is reducible non-periodic.

Let C be the canonical reduction system for $\varphi(\tau)$. By Lemma 2.9, one of the following three cases occurs.

Case 1. C is connected, the punctures 1 and 2 are inside C, all the other punctures are outside C. Then the restriction of $\varphi(\tau)$ (viewed as a diffeomorphism of $\mathbb{D}$) to the exterior of C cannot be pseudo-Anosov because $\varphi(\tau)$ commutes with $\varphi(c) = c$, hence it preserves a circle which separates 3 and 4 from $5,\dots,n$. Hence $\varphi(\tau)$ is periodic which contradicts (4). Thus this case is impossible.

Case 2. C is connected, the punctures 1 and 2 are outside C, all the other punctures are inside C. This case is also impossible and the proof is almost the same as in Case 1. To show that $\varphi(\tau)$ cannot be pseudo-Anosov, we note that it preserves a curve which encircles only 1 and 2.

Case 3. C has two components: C_1 and C_2 which encircle $\{1,2\}$ and $\{3,\dots,n\}$ respectively. Let α be the interior braid of C_2 (that is $\varphi(\tau)$ with the strings 1 and 2 removed). It cannot be pseudo-Anosov by the same reasons as in Case 1: because $\varphi(\tau)$ preserves a circle separating 3 and 4 from $5,\dots,n$. Hence α is periodic. Using (4), we conclude that $\varphi(\tau)$ is a conjugate of τ. Since the elements of $Z(\tau;\mathbf{B}_n)$ realize any permutation of $\{1,2\}$ and of $\{3,\dots,n\}$, the conjugating element can be chosen in $\mathbf{P}_n$. □

Lemma 2.11. (cf. [17, Lemma 5.8]). *There exists $u \in \mathbf{P}_n$ such that $\varphi(c_i) = \tilde{u}(c_i)$ for each $i = 3,\dots,n-1$.*

Proof. Due to Lemma 2.10, without loss of generality we may assume that $\varphi(\tau) = \tau$ and $\tau(C) = C$ where C is the canonical reduction system for τ consisting of two round circles C_1 and C_2 which encircle $\{1,2\}$ and $\{3,\dots,n\}$ respectively. Since the conjugating element in Lemma 2.10 is chosen in $\mathbf{P}_n$, we may assume that (4) still holds.

By Lemma 2.8, for each $i = 3,\dots,n-1$, we have $\varphi(c_i) = \sigma_{I_i}^{-1}\sigma_{J_i}$ with $\partial I_i = \{1,2\}$ and $\partial J_i = \{i,i+1\}$. Since τ commutes with each c_i, the segments I_i and J_i can be chosen disjoint from the circles C_1 and C_2. Hence $\sigma_{I_i} = \sigma_1$ for each i, and all the segments J_i are inside C_2.

Therefore the braids $\sigma_{J_3},\dots,\sigma_{J_{n-1}}$ satisfy the same braid relations as $\sigma_3,\dots,\sigma_{n-1}$. Hence, by Lemma 2.7 combined with [17, Lemma 5.13], $J_3 \cup \dots \cup J_{n-1}$ is an embedded segment. Hence it can be transformed to the straight line segment $[3,n]$ by a diffeomorphism identical on the exterior of C_2. Hence for the braid u represented by this diffeomorphism we have $\tilde{u}(c_i) = c_i$, $i \ge 3$. The condition $\partial J_i = \{i,i+1\}$ implies that $u \in \mathbf{P}_n$. □

The rest of the proof of Theorem 1.1 repeats word-by-word [17, §§5.2–5.3].

Remark 2.12. Besides Nielsen-Thurston theory, in the case $n = 5$, the arguments in [17, §5.3] use an auxiliary result [17, Lemma A.1] for which the only proof we know is based on a slight modification of the main theorem of [16] which is proven there using the Garside theory.

3. The case $n = 4$

We shall use the same notation as in [17, §6]. The groups $\mathbf{B}_3'$ and $\mathbf{B}_4'$ were computed in [9], namely $\mathbf{B}_3'$ is freely generated by $u = \sigma_2\sigma_1^{-1}$ and $t = \sigma_1^{-1}\sigma_2$, and $\mathbf{B}_4' = \mathbf{K}_4 \rtimes \mathbf{B}_3'$ where $\mathbf{K}_4 = \ker R$ (see (1)). The group $\mathbf{K}_4$ is freely generated by $c = \sigma_3\sigma_1^{-1}$ and $w = \sigma_2 c \sigma_2^{-1}$. The action of $\mathbf{B}_3'$ on $\mathbf{K}_4$ by conjugation is given by

$$ucu^{-1} = w, \qquad uwu^{-1} = w^2c^{-1}w, \qquad tct^{-1} = cw, \qquad twt^{-1} = cw^2. \tag{5}$$

The action of σ_1 and σ_2 on $\mathbf{K}_4$ is given by

$$\sigma_1 c\sigma_1^{-1} = c, \quad \sigma_1 w\sigma_1^{-1} = c^{-1}w, \quad \sigma_2 c\sigma_2^{-1} = w, \quad \sigma_2 w\sigma_2^{-1} = wc^{-1}w. \tag{6}$$

So, we also have $\mathbf{B}_4 = \mathbf{K}_4 \rtimes \mathbf{B}_3$.

Besides the elements c, w, u, t of $\mathbf{B}_4'$, we consider also

$$d = \Delta\sigma_1^{-3}\sigma_3^{-3} \qquad \text{and} \qquad g = R(d) = \Delta_3^2\sigma_1^{-6}$$

(here and below $\Delta = \Delta_4$). One has (see Figure 1)

$$d = [c^{-1}t, u^{-1}], \qquad g = [t, u^{-1}]. \tag{7}$$

We denote the subgroup generated by c and d by H and the subgroup generated by c and g by G.

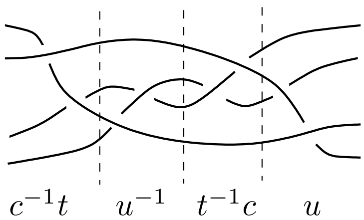

Figure 1. The identity $d = [c^{-1}t, u^{-1}]$.

Let $\varphi : \mathbf{B}_4' \to \mathbf{B}_4$ be a homomorphism such that $\mathbf{K}_4 \not\subset \ker\varphi$.

Lemma 3.1. *The restriction of φ to H is injective, $\varphi(H) \subset \mathbf{B}_4'$, and $\varphi(G) \subset \mathbf{B}_4'$.*

Proof. We have $H = \langle c\rangle \rtimes \langle d\rangle$ and d acts on c by $dcd^{-1} = c^{-1}$. Hence any non-trivial normal subgroup of H contains a power of c. Thus, if $\varphi|_H$ were not injective, $\ker\varphi$ would contain a power of c and hence c itself because the target group $\mathbf{B}_4$ does not have elements of finite order. Then we also have $w \in \ker\varphi$ because $w = ucu^{-1}$. This contradicts the assumption $\mathbf{K}_4 = \langle c, w\rangle \not\subset \ker\varphi$, thus $\varphi|_H$ is injective.

We have $dcd^{-1} = c^{-1}$, hence the image of $\varphi(c)$ under the abelianization $e : \mathbf{B}_4 \to \mathbb{Z}$ is zero, i.e., $\varphi(c) \in \mathbf{B}_4'$. By (7) we also have $\varphi(d) \in \mathbf{B}_4'$ and $\varphi(g) \in \mathbf{B}_4'$, thus $\varphi(H) \subset \mathbf{B}_4'$ and $\varphi(G) \subset \mathbf{B}_4'$. □

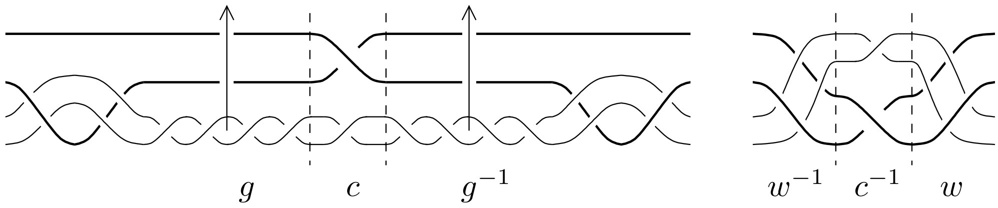

FIGURE 2. The identity $gcg^{-1} = w^{-1}c^{-1}w$.

Lemma 3.2. *$\varphi(c)$ and $\varphi(g)$ do not commute.*

Proof. Suppose that $\varphi(c)$ and $\varphi(g)$ commute. Then $\varphi(c) = \varphi(gcg^{-1})$. Hence (see Figure 2) $\varphi(c) = \varphi(w^{-1}c^{-1}w)$, i.e., φ factors through the quotient of $\mathbf{B}_4'$ by the relation $wc = c^{-1}w$. Let us denote this quotient group by $\hat{\mathbf{B}}_4'$.

The relation $wc = c^{-1}w$ allows us to put any word $\prod_j c^{k_j}w^{l_j}$ with $l_j = \pm 1$ into the normal form $c^{k_1-k_2+k_3-\cdots}w^{l_1+l_2+l_3+\cdots}$ in $\hat{\mathbf{B}}_4'$. Due to (5), the conjugation by t of the word $w^{-1}cwc$ (which is equal to 1 in $\hat{\mathbf{B}}_4'$) yields

$$1 = t(w^{-1}cwc)t^{-1} = (w^{-2}c^{-1})(cw)(cw^2)(cw) = w^{-1}cw^2cw = c^{-2}w^2$$

(here in the last step we put the word into the above normal form). Conjugating once more by t and putting the result into the normal form, we get

$$1 = t(c^{-2}w^2)t^{-1} = (w^{-1}c^{-1})(w^{-1}c^{-1})(cw^2)(cw^2) = w^{-1}c^{-1}wcw^2 = c^2w^2.$$

Thus $c^{-2}w^2 = c^2w^2 = 1$, i.e., $c^4 = 1$ in $\hat{\mathbf{B}}_4'$, hence $\varphi(c^4) = 1$ which contradicts Lemma 3.1. $\square$

As in [17], we denote the stabilizer of 1 under the natural action of $\mathbf{B}_3$ on $\{1,2,3\}$ by $\mathbf{B}_{1,2}$. It is well-known (and easy to prove by Reidemeister-Schreier method) that $\mathbf{B}_{1,2}$ is isomorphic to the Artin group of type B_2, that is $\langle x, y \mid xyxy = yxyx\rangle$. The Artin generators x and y of the latter group correspond to σ_1^2 and σ_2.

Lemma 3.3. (cf. [17, Lemma 6.2]) *We have $G = Z(d^2c^6; \mathbf{B}_4')$ and this group is generated by g and c subject to the defining relation $gcgc = cgcg$.*

Proof. The centralizer of d^2c^6 in $\mathbf{B}_4$ is the stabilizer of its canonical reduction system which is shown in Figure 4, and (see [8, Thm. 5.10]) it is the image of the injective homomorphism $\mathbf{B}_{1,2} \times \mathbb{Z} \to \mathbf{B}_4$, $(X, n) \mapsto Y\sigma_1^n$, where the 4-braid Y is obtained from the 3-braid X by doubling the first strand. It follows that $Z(d^2c^6; \mathbf{B}_4')$ is the isomorphic image of $\mathbf{B}_{1,2}$ under the homomorphism $\psi : \mathbf{B}_{1,2} \to \mathbf{B}_4'$ defined on the generators by $\psi(\sigma_1^2) = g$, $\psi(\sigma_2) = c$ (see Figure 3), thus $Z(d^2c^6; \mathbf{B}_4') = G$. As we have pointed out above, $\mathbf{B}_{1,2}$ is the Artin group of type B_2, hence so is G and $gcgc = cgcg$ is its defining relation. $\square$

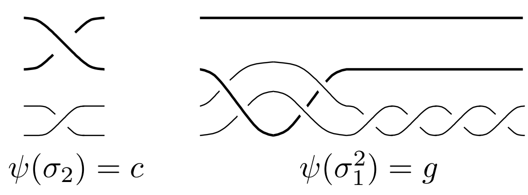

FIGURE 3. The images of the generators under $\psi : \mathbf{B}_{1,2} \to \mathbf{B}_4'$.

Lemma 3.4. *$\varphi(d^2c^6)$ is conjugate in $\mathbf{B}_4$ to d^{2k}, $d^{2k}c^{6k}$, or h^k for some integer $k \neq 0$, where $h = \Delta^2\Delta_3^{-4} = \Delta_3^{-2}\sigma_3\sigma_2\sigma_1^2\sigma_2\sigma_3$.*

Proof. Let $x = d^2c^6$. By Lemma 3.3, $G = Z(x; \mathbf{B}_4')$, hence $\varphi(G) \subset Z(\varphi(x); \mathbf{B}_4)$. By Lemma 3.1 we also have $\varphi(G) \subset \mathbf{B}_4'$, hence $\varphi(G) \subset Z(\varphi(x); \mathbf{B}_4')$. Then it follows from Lemma 3.2 that $Z(\varphi(x); \mathbf{B}_4')$ is non-commutative. The isomorphism classes of the centralizers (in $\mathbf{B}_4'$) of all elements of $\mathbf{B}_4'$ are computed in [17, Table 6.1]. We see in this table that $Z(\varphi(x); \mathbf{B}_4')$ is non-commutative only in the required cases (see the corresponding canonical reduction systems in Figure 4) unless $\varphi(x) = 1$. However the latter case is impossible by Lemma 3.1. □

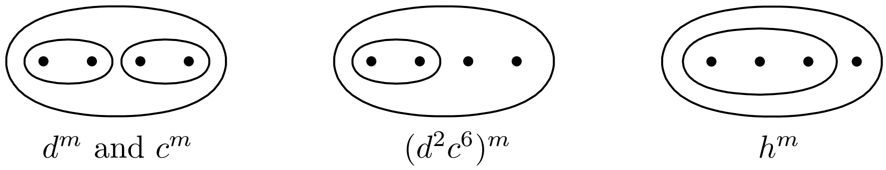

FIGURE 4. Canonical reduc. systems for d^m, c^m, $(d^2c^6)^m$, h^m, $m \neq 0$.

Lemma 3.5. *There exists an automorphism of $\mathbf{B}_4$ which takes $\varphi(c)$ and $\varphi(d)$ to c^k and d^k respectively for an odd positive integer k.*

Proof. Let $x = d^2c^6$ and $y = d^2c^{-6}$. Since $y = dxd^{-1}$, the images of x and y are conjugate and both of them belong to one of the conjugacy classes indicated in Lemma 3.4. The canonical reduction systems for d^{2k}, $d^{2k}c^{6k}$, and h^k for $k \neq 0$ are shown in Figure 4. Since x and y commute, the canonical reduction systems of their images can be chosen disjoint from each other. Hence, up to composing φ with an inner automorphism of $\mathbf{B}_4$, $\big(\varphi(x), \varphi(y)\big)$ is either (h^{k_1}, h^{k_2}) or $\big(d^{2k_1}c^{l_1}, d^{2k_2}c^{l_2}\big)$ where $l_j \in \{0, \pm 6k_j\}$, $j = 1, 2$. Since x and y are conjugate, by comparing the linking numbers between different pairs of strings, we deduce that $k_1 = k_2$ and (in the second case) $l_1 = \pm l_2$. Moreover, $\varphi(x) \neq \varphi(y)$ by Lemma 3.1. Hence, up to exchange of x and y (which is realizable by composing φ with $\tilde{d}$), we have $\varphi(x) = d^{2k}c^{6k}$ and $\varphi(y) = d^{2k}c^{-6k}$ whence, using that $xy^{-1} = c^{12}$, we obtain $\varphi(c^{12}) = \varphi(xy^{-1}) = c^{12k}$. Since the canonical reduction systems of any braid and its non-zero power coincide (see, e.g., [7, Lemmas 2.1–2.3]), we obtain $\varphi(c) = c^k$ and $\varphi(d) = d^k$. By composing φ with Λ if necessary, we can arrive to $k > 0$. The relation $d^kc^kd^{-k} = c^{-k}$ combined with Lemma 3.1 implies that k is odd. □

Lemma 3.6. $\varphi(\mathbf{K}_4) \subset \mathbf{K}_4$.

Proof. Lemma 3.5 implies that c^k is mapped to $\varphi(c)$ by an automorphism of $\mathbf{B}_4$. Since $\mathbf{K}_4$ is a characteristic subgroup of $\mathbf{B}_4'$ (see [17, Lemma 6.5][1]) and $\mathbf{B}_4'$ is a characteristic subgroup of $\mathbf{B}_4$, we deduce that $\varphi(c) \in \mathbf{K}_4$. The same arguments can be applied to any other homomorphism of $\mathbf{B}_4'$ to $\mathbf{B}_4$ whose kernel does not contain $\mathbf{K}_4$, in particular, they can be applied to $\varphi\tilde{u}$ whence $\varphi\tilde{u}(c) \in \mathbf{K}_4$. Since $\varphi(w) = \varphi\tilde{u}(c)$, we conclude that $\varphi(\mathbf{K}_4) = \langle\varphi(c), \varphi(w)\rangle \subset \mathbf{K}_4$. □

Let
$$F = G \cap \mathbf{K}_4.$$

[1]It is based on [17, Lemma 6.3] whose proof should be considered as a hint rather than a proof.

Lemma 3.7. *(a). The group F is freely generated by c and $c_1 = w^{-1}c^{-1}w$.*

(b). Let $a_1, \dots, a_{m-1}$ and $b_1, \dots, b_m$ be non-zero integers, and let a_0 and a_m be any integers. Then $c^{a_0}w^{b_1}c^{a_1}\dots w^{b_m}c^{a_m}$ is in F if and only if m is even and $b_j = (-1)^j$ for each $j = 1, \dots, m$.

Proof. The relation on g and c in Lemma 3.3 is equivalent to

$$g^{-1}cgc = cgcg^{-1}. \tag{8}$$

Recall that $G = \langle c, g\rangle$. We have $R(c) = 1$ and, by (7), $g = R(d) \in \mathbf{B}_3'$ whence $R(g) = g$. Hence $R(G)$ is generated by g. By definition, $F = \ker(R|_G)$, hence F is the normal closure of c in G, i.e., F is generated by the elements $\tilde g^k(c)$, $k \in \mathbb{Z}$. We have $\tilde g(c) = c_1$ (see Figure 2) and

$$\tilde g(c_1) = \tilde g^2(c) = g\,c^{-1}(cgcg^{-1})g^{-1} \overset{\text{by (8)}}{=} g\,c^{-1}(g^{-1}cgc)g^{-1} = c_1^{-1}c\,c_1$$

whence by induction we obtain $\tilde g^k(c) \in \langle c, c_1\rangle$ for all positive k. Similarly,

$$\tilde g^{-1}(c) = (g^{-1}cgc)c^{-1} \overset{\text{by (8)}}{=} (cgcg^{-1})c^{-1} = c(gcg^{-1})c^{-1} = c\,c_1c^{-1}$$

and $\tilde g^{-1}(c_1) = c$ whence $\tilde g^k(c) \in \langle c, c_1\rangle$ for all negative k. Thus $F = \langle c, c_1\rangle$.

To check that c and c_1 is a free base of F (which completes the proof of (a)), it is enough to observe that if, in a reduced word in x, y, we replace each x^k with c^k and each y^k with $w^{-1}c^{-k}w$, then we obtain a reduced word in c and w. The statement (b) also easily follows from this observation. □

Lemma 3.8. *If $x \in F$ and $x = [w^{-1}, A]$ with $A \in \mathbf{K}_4$, then $x = [w^{-1}, c^k]$, $k \in \mathbb{Z}$.*

Proof. Let $A = w^{b_1}c^{a_1}\dots w^{b_m}c^{a_m}w^{b_{m+1}}$, $m \ge 0$, where $a_1, \dots, a_m$ and $b_2, \dots, b_m$ are non-zero while b_1 and b_{m+1} may or may not be zero. If $m = 0$, then $[w^{-1}, A] = 1 = [w^{-1}, c^0]$ and we are done. If $m = 1$, then $[w^{-1}, A] = w^{b_1-1}c^{a_1}w\,c^{-a_1}w^{-b_1}$ where, by Lemma 3.7(b), we must have $b_1 = 0$, hence $[w^{-1}, A] = [w^{-1}, c^{a_1}]$ as required. Suppose that $m \ge 2$. Then

$$[w^{-1}, A] = w^{b_1-1}c^{a_1}\dots w^{b_m}c^{a_m}w\,c^{-a_m}w^{-b_m}\dots c^{-a_1}w^{-b_1}$$

and this is a reduced word in c, w. Hence, by Lemma 3.7(b), the sequence of the exponents of w in this word (starting form $b_1 - 1$ when $b_1 \ne 1$ or from b_2 when $b_1 = 1$) should be $(-1, 1, -1, 1, \dots, -1, 1)$. Such a sequence cannot contain $(\dots, b_m, 1, -b_m, \dots)$. A contradiction. □

Lemma 3.9. *If $\varphi(d^2) = d^2$ and $\varphi(c) = c$, then $w^{-1}\varphi(w) \in F$.*

Proof. For any $k \in \mathbb{Z}$ we have

$$\sigma_3^k w = \sigma_3^k(\sigma_2\sigma_3)(\sigma_1^{-1}\sigma_2^{-1}) = (\sigma_2\sigma_3)\sigma_2^k(\sigma_1^{-1}\sigma_2^{-1}) = (\sigma_2\sigma_3)(\sigma_1^{-1}\sigma_2^{-1})\sigma_1^k = w\sigma_1^k,$$

hence $\sigma_3^k w\sigma_1^{-k} = w = \sigma_3^{-k}w\sigma_1^k$ and we obtain

$$d^2wd^{-2} = \Delta^2\sigma_1^{-6}(\sigma_3^{-6}w\sigma_1^6)\sigma_3^6\Delta^{-2} = \sigma_1^{-6}(\sigma_3^6w\sigma_1^{-6})\sigma_3^6 = c^6wc^6. \tag{9}$$

Set $x = w^{-1}\varphi(w)$, i.e., $\varphi(w) = wx$. The relation (9) combined with our hypothesis on c and d^2 implies

$$c^6wxc^6 = \varphi(c^6wc^6) = \varphi(\tilde d^2(w)) = \tilde d^2(wx) = \tilde d^2(w)\tilde d^2(x) = c^6wc^6d^2xd^{-2}$$

whence $x(c^6d^2) = (c^6d^2)x$, i.e., $x \in Z(d^2c^6)$. On the other hand, $\varphi(w) \in \mathbf{K}_4$ by Lemma 3.6, hence $x = w^{-1}\varphi(w) \in \mathbf{K}_4$. By Lemma 3.3 we have $Z(d^2c^6; \mathbf{B}_4') = G$, thus $x \in Z(d^2c^6) \cap \mathbf{K}_4 = G \cap \mathbf{K}_4 = F$. □

Lemma 3.10. *There exists $f \in \mathrm{Aut}(\mathbf{B}_4)$ and a homomorphism $\tau : \mathbf{B}_4' \to Z(\mathbf{B}_4)$ such that $f\varphi(c) = c$, $f\varphi(d^2) = d^2$, and $Rf\varphi = R\,\mathrm{id}_{[\tau]}$.*

Proof. By Lemma 3.5 we may assume that $\varphi(c) = c^k$ and $\varphi(d) = d^k$ for an odd positive k. For $x \in \mathbf{K}_4$, we denote its image in $\mathbf{K}_4^{\mathfrak{ab}}$ by $\bar{x}$ and we use the additive notation for $\mathbf{K}_4^{\mathfrak{ab}}$. Consider the homomorphism $\pi : \mathbf{B}_4 \to \mathrm{Aut}(\mathbf{K}_4^{\mathfrak{ab}}) = \mathrm{GL}(2,\mathbb{Z})$, where $\pi(x)$ is defined as the automorphism of $\mathbf{K}_4^{\mathfrak{ab}}$ induced by $\tilde{x}$; here we identify $\mathrm{Aut}(\mathbf{K}_4^{\mathfrak{ab}})$ with $\mathrm{GL}(2,\mathbb{Z})$ by choosing $\bar{c}$ and $\bar{w}$ as a base of $\mathbf{K}_4^{\mathfrak{ab}}$. By Lemma 3.6, $\varphi(w) \in \mathbf{K}_4$, hence we may write $\overline{\varphi(w)} = p\bar{c} + q\bar{w}$ with $p, q \in \mathbb{Z}$. Then, for any $x \in \mathbf{B}_4$, we have

$$\pi\varphi(x).P = P.\pi(x) \qquad \text{where} \quad P = \begin{pmatrix} k & p \\ 0 & q \end{pmatrix}. \tag{10}$$

(P is the matrix of the endomorphism of $\mathbf{K}_4^{\mathfrak{ab}}$ induced by $\varphi|_{\mathbf{K}_4}$). By (9) we have

$$\pi(d^2) = \begin{pmatrix} 1 & 12 \\ 0 & 1 \end{pmatrix} \qquad \text{hence} \qquad \pi(d^{2k}).P - P.\pi(d^2) = \begin{pmatrix} 0 & 12k(q-1) \\ 0 & 0 \end{pmatrix}. \tag{11}$$

Since $\varphi(d^2) = d^{2k}$, we obtain from (10) combined with (11) that $q = 1$, i.e., $\overline{\varphi(w)} = p\bar{c} + \bar{w}$. By (5) we have $\varphi(u)c^k\varphi(u)^{-1} = \varphi(ucu^{-1}) = \varphi(w)$, hence

$$k\,\overline{\varphi(u)c\varphi(u)^{-1}} = \overline{\varphi(w)} = p\bar{c} + \bar{w}.$$

Therefore $k = 1$ because $p\bar{c} + \bar{w}$ cannot be a multiple of another element of $\mathbf{K}_4^{\mathfrak{ab}}$. Notice that $\tilde{\sigma}_1(c) = c$, $\tilde{\sigma}_1(d^2) = d^2$, and $\tilde{\sigma}_1(w) = c^{-1}w$ (see (6)). Hence, for $f = \tilde{\sigma}_1^p$, we have

$$f\varphi(c) = c, \qquad f\varphi(d^2) = d^2, \qquad \overline{f\varphi(w)} = \bar{w}. \tag{12}$$

It remains to show that $Rf\varphi = R\,\mathrm{id}_{[\tau]}$ for some $\tau : \mathbf{B}_4' \to Z(\mathbf{B}_4)$. Let $x \in \mathbf{B}_4'$. Since $\mathbf{B}_4' = \mathbf{K}_4 \rtimes \mathbf{B}_3'$ and $\mathbf{B}_4 = \mathbf{K}_4 \rtimes \mathbf{B}_3$, we may write $x = x_1a_1$ and $f\varphi(x) = x_2a_2$ with $x_1 = R(x) \in \mathbf{B}_3'$, $x_2 = Rf\varphi(x) \in \mathbf{B}_3$, and $a_1, a_2 \in \mathbf{K}_4$. The equation (10) for $f\varphi$ (and hence with the identity matrix for P because (12) means that $f\varphi|_{\mathbf{K}_4}$ induces the identity mapping of $\mathbf{K}_4^{\mathfrak{ab}}$) reads $\pi f\varphi(x) = \pi(x)$, that is $\pi(x_2a_2) = \pi(x_1a_1)$. Since $a_1, a_2 \in \mathbf{K}_4 \subset \ker\pi$, this implies that

$$\pi(x_1) = \pi(x_2). \tag{13}$$

Let $S_1 = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S_2 = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. It is well-known that the mapping $\sigma_1 \mapsto S_1$, $\sigma_2 \mapsto S_2$ defines an isomorphism between $\mathbf{B}_3/\langle\Delta_3^4\rangle$ and $\mathrm{SL}(2,\mathbb{Z})$. From (6) we see that $\pi(\sigma_1) = S_1$ and $\pi(\sigma_1^{-1}\sigma_2\sigma_1) = S_2$. Hence $\ker(\pi|_{\mathbf{B}_3}) = \langle\Delta_3^4\rangle = R(Z(\mathbf{B}_4))$. Therefore (13) implies that $x_2 = x_1R(\tau(x))$ for some element $\tau(x)$ of $Z(\mathbf{B}_4)$. It is easy to check that τ is a group homomorphism, thus, recalling that $x_1 = R(x)$ and $x_2 = Rf\varphi(x)$, we get $Rf\varphi(x) = x_2 = x_1R(\tau(x)) = R(x\tau(x)) = R\,\mathrm{id}_{[\tau]}(x)$. □

Lemma 3.11. *If $\varphi|_{\mathbf{K}_4} = \mathrm{id}$ and $R\varphi = R\,\mathrm{id}_{[\tau]}$ for some homomorphism $\tau : \mathbf{B}_4' \to Z(\mathbf{B}_4)$, then $\varphi = \mathrm{id}_{[\tau]}$.*

Proof. Since $\mathbf{B}_4' = \mathbf{K}_4 \rtimes \mathbf{B}_3'$ and $\mathbf{K}_4 \subset \ker\tau$, it is enough to show that $\varphi|_{\mathbf{B}_3'} = \mathrm{id}_{[\tau]}$. So, let $x \in \mathbf{B}_3'$. The condition $R\varphi = R\,\mathrm{id}_{[\tau]}$ means that $\varphi(x) = xa\tau(x)$ with $a \in \mathbf{K}_4$.

Let b be any element of $\mathbf{K}_4$. Then $xbx^{-1} \in \mathbf{K}_4$, hence $\varphi(xbx^{-1}) = xbx^{-1}$ (because $\varphi|_{\mathbf{K}_4} = \mathrm{id}$). Since $\varphi(x) = xa\tau(x)$, $\varphi(b) = b$, and $\tau(x)$ is central, it follows that

$$xbx^{-1} = \varphi(xbx^{-1}) = \varphi(x)b\varphi(x)^{-1} = xa\tau(x)b\tau(x)^{-1}a^{-1}x^{-1} = xaba^{-1}x^{-1}$$

whence $aba^{-1} = b$. This is true for any $b \in \mathbf{K}_4$, thus $a \in Z(\mathbf{K}_4)$. Since $\mathbf{K}_4$ is free, we deduce that $a = 1$, hence $\varphi(x) = x\tau(x) = \mathrm{id}_{[\tau]}(x)$. $\square$

Proof of Theorem 1.4. Recall that we assume in this section that φ is a homomorphism $\mathbf{B}_4' \to \mathbf{B}_4$ such that $\mathbf{K}_4 \not\subset \ker\varphi$.

By Lemma 3.10 we may assume that $\varphi(c) = c$, $\varphi(d^2) = d^2$, and $R\varphi = R\,\mathrm{id}_{[\tau]}$ for some $\tau : \mathbf{B}_4' \to Z(\mathbf{B}_4)$, in particular, $R\varphi(u) = R(u\tau(u))$. The latter condition means that $\varphi(u) = ua\tau(u)$ with $a \in \mathbf{K}_4$. Then, by (5), we have

$$\varphi(w) = \varphi(ucu^{-1}) = uaca^{-1}u^{-1} = \tilde{u}(c\,[c^{-1}, a]) = w[w^{-1}, \tilde{u}(a)],$$

thus $w^{-1}\varphi(w) = [w^{-1}, A]$ for $A = \tilde{u}(a) \in \mathbf{K}_4$. By Lemma 3.9 we have also $w^{-1}\varphi(w) \in F$. Then Lemma 3.8 implies that $w^{-1}\varphi(w) = [w^{-1}, c^k]$ for some integer k, that is $\varphi(w) = c^k w c^{-k}$. Hence, $(\tilde{c}^{-k}\varphi)|_{\mathbf{K}_4} = \mathrm{id}$. Since $c \in \ker R$, we have $R\tilde{c}^{-k} = R$ whence $R\tilde{c}^{-k}\varphi = R\varphi = R\,\mathrm{id}_{[\tau]}$. This fact combined with $(\tilde{c}^{-k}\varphi)|_{\mathbf{K}_4} = \mathrm{id}$ and Lemma 3.11 implies that $\tilde{c}^{-k}\varphi = \mathrm{id}_{[\tau]}$, i.e., φ is equivalent to $\mathrm{id}_{[\tau]}$. $\square$

References

1. E. Artin, *Theory of braids*, Ann. of Math. **48** (1947), 101–126.
2. R. W. Bell, D. Margalit, *Braid groups and the co-Hopfian property*, J. Algebra **303** (2006), 275–294.
3. J. S. Birman, A. Lubotzky, J. McCarthy, *Abelian and solvable subgroups of the mapping class group*, Duke Math. J. **50** (1983), 1107–1120.
4. F. Castel, *Geometric representations of the braid groups*, Astérisque **378** (2016), vi+175.
5. J. L. Dyer, E. K. Grossman, *The automorphism group of the braid groups*, Amer. J. of Math. **103** (1981), 1151–1169.
6. B. Farb, D. Margalit, *A primer on mapping class groups*, volume 49 of Princeton Mathematical Series, Princeton University Press, Princeton, NJ, 2012.
7. J. González-Meneses, *The nth root of a braid is unique up conjugacy*, Algebraic and Geometric Topology **3** (2003), 1103–1118.
8. J. González-Meneses, B. Wiest, *On the structure of the centralizer of a braid*, Ann. Sci. Éc. Norm. Supér. (4) **37** (2004), 729–757.
9. E. A. Gorin, V. Ya. Lin, *Algebraic equations with continuous coefficients and some problems of the algebraic theory of braids*, Math. USSR-Sbornik **7** (1969), 569–596.
10. N. V. Ivanov, *Subgroups of Teichmüller modular groups*, Translations of mathematical monographs, vol. 115, AMS, 1992.
11. K. Kordek, D. Margalit, *Homomorphisms of commutator subgroups of braid groups*, Bull. London Math. Soc. **54** (2022), 95–111.
12. V. Lin, *Braids and permutations*, arXiv:math/0404528.
13. V. Ya. Lin, *Algebraic functions, configuration spaces, Teichmüller spaces, and new holomorphically combinatorial invariants*, Funk. Anal. Prilozh. **45** (2011), no. 3, 55–78 (Russian); English transl., Funct. Anal. Appl. **45** (2011), no. 3, 204–224.
14. V. Lin, *Some problems that I would like to see solved*, Abstract of a talk. Technion, 2015, http://www2.math.technion.ac.il/˜pincho/Lin/Abstracts.pdf.
15. W. Magnus, A. Karrass, D. Solitar, *Combinatorial group theory: presentations of groups in terms of generators and relations*, Interscience Publ., 1966.
16. S. Yu. Orevkov, *Algorithmic recognition of quasipositive braids of algebraic length two*, J. of Algebra **423** (2015), 1080–1108.

17. S. Yu. Orevkov, *Automorphism group of the commutator subgroup of the braid group*, Ann. Faculté des Scie. de Toulouse. Math. (6) **26** (2017), 1137–1161.

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