COUNTER-EXAMPLES TO THE "JACOBIAN CONJECTURE AT INFINITY"

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To Anatoliy Georgievich Vitushkin for his 70-th birthday

Introduction

The well-known Jacobian Conjecture (see surveys [17], [3]) is as follows:

Jacobian Conjecture (*JC*). *Let $P(x,y)$ and $Q(x,y)$ be polynomials with complex coefficients whose jacobian $P'_xQ'_y - P'_yQ'_x$ is identically equal to one. Then the mapping $\mathbf{C}^2 \to \mathbf{C}^2$, $(x,y) \mapsto (u,v) = \big(P(x,y), Q(x,y)\big)$ is one-to-one (or, in algebraic language: the ring homomorphism $\mathbf{C}[u,v] \to \mathbf{C}[x,y]$, $u \mapsto P(x,y)$, $v \mapsto Q(x,y)$ is an isomorphism).*

Definition 0.1. A pair (U,l) where U is a smooth analytic surface and $l \subset U$ a smooth compact (i.e. isomorphic to $\mathbf{CP}^1$) curve of self-intersection $+1$, will be called a $(+1)$-*pair*.

Let us call a $(+1)$-pair (U,l) *flat* if U is biholomorphically equivalent to a subset of $\mathbf{CP}^2$ (it is clear, that such a biholomorphism maps l onto a line).

A *meromorphic immersion* (respectively, *embedding*) of a $(+1)$-pair (U,l) into $\mathbf{C}^2$ is a pair functions meromorphic on U such that the both of them are holomorphic on $U \setminus l$ and the mapping $U \setminus l \to \mathbf{C}^2$ defined by these functions is an immersion (respectively, embedding).

The *index* of a meromorphic immersion of a $(+1)$-pair $f : U \setminus l \to \mathbf{C}^2$ is by defined as the degree of the Gauss mapping $G_f : M \to S^3$ where $M = -\partial V$ is the boundary of a tubular neighbourhood V of l with the reversed orientation (the mapping G_f takes $p \in M$ into the positive normal vector to the hyperplane $f_*(T_pM)$).

The Jacobian Conjecture can be equivalently reformulated as follows:

Any meromorphic immersion of a flat $(+1)$-pair into $\mathbf{C}^2$ is an embedding.

Indeed, if (U,l) is a flat $(+1)$-pair then one may consider l as the infinite line in $\mathbf{C}^2$ and U as its neighbourhood. Then any function, holomorphic on $U \setminus l$, is extendable to the whole $\mathbf{C}^2$ by the theorem of removing compact singularities. Moreover, if it is meromorphic on U, it is a polynomial.

A natural question arises:[1] can one omit the hypothesis that the $(+1)$-pair is flat? In other words, does the following conjecture hold:

[1]This question (maybe, not so concretely formulated) was posed to me by A.G. Vitushkin when I was his graduate student.

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Weak Jacobian Conjecture at Infinity (WJC_∞). *Any meromorphic immersion of a $(+1)$-pair into $\mathbf{C}^2$ is an embedding.*

In the paper [13], I constructed a counter-example to this conjecture. Later, I constructed many other analogous counter-examples to WJC_∞ (unpublished) but all of them were not extendable to counter-examples to JC because they had too big index.

But it is clear that *The index of a meromorphic immersion of a flat $(+1)$-pair is equal to one.* Indeed, it is equal to $DG(F|_{S^3_r})$ for $r \gg 1$ where $F : \mathbf{C}^2 \to \mathbf{C}^2$ is a polynomial mapping whose jacobian is equal to one, S^3_r is the sphere of radius r (oriented as the boundary of a ball), and $DG(\varphi)$ denotes the degree of Gauss mapping associated to an immersion φ. It remains to note that the function $g(r) = DG(F|_{S^3_r})$ is continuous, hence, constant and that $F|_{S^3_r}$ is an embedding for $r \ll 1$, i.e. $g(r) = 1$.

Because of this, I formulated a new conjecture which I announced at several conferences:

Jacobian Conjecture at Infinity (JC_∞). *Any meromorphic immersion of a $(+1)$-pair into $\mathbf{C}^2$ whose index is equal to one, is an embedding.*

In this paper we construct a counter-example to this conjecture also:

Proposition 0.2. *There exists a $(+1)$-pair (U, l) and its meromorphic immersion $f : U \setminus l \to \mathbf{C}^2$ of index 1, which is not an embedding.*

Such a meromorphic immersion of a (+1)-pair is constructed in §3. The (+1)-pair (U, l) constructed in §3 is not flat, i.e. it can not provide a counter-example to JC. It is proved in §4, Sections 4.1 – 4.2 analysing the coefficients of polynomials $P(x, y)$ and $Q(x, y)$ which could realize the given immersion. (in Sect. 4.3, we give a simple but not rigorous topological explanation of this fact). Since we can always choose a strictly pseudo-convex tubular neighbourhood of l (see. [13; §2]), the non-extendibility of the constructed counter-example to JC_∞ up to a counter-example to JC implies an amazing consequence. To formulate it, we need one more definition.

Definition 0.3. In immersion f of a smooth oriented $(2n-1)$-manifold Z to a complex n-manifold Y is called *strictly pseudo-convex* if any point $z \in Z$ has a neighbourhood $V \subset Z$ such that $f(V)$ is a part of the boundary (taking in account the orientations) of some strictly pseudo-convex domain in Y. Recall, that a *regular homotopy* is such a homotopy $\{f_t\}_{t\in[0,1]}$ that f_t is an immersion for any t. If in addition, each f_t is strictly pseudo-convex then such a homotopy is called *strictly pseudo-convex.*

Proposition 0.4. *There exists a strictly pseudo-convex immersion of the sphere $f : S^3 \to \mathbf{C}^2$ which is regularly homotopic to an embedding but is not strictly pseudo-convexly homotopic to an embedding.*

This proposition is proved in §5. At the same time, we prove Proposition 5.6 on the uniqueness of an extension of a pseudo-convex immersion of the 3-sphere up to an immersion of the 4-ball.

In the paper [13], we gave a complete proof that the example constructed there satisfies the required properties. But the construction was exposed, using the school geometry language, without an "analysis of the problem". Probably, this caused some difficulties to understand how the example was constructed and how to construct other similar examples. In this paper, I tried to fill this gap by adding Sect. 2.4. In this section we also discuss some parallelism between our approach to JC and those from the papers [10] and [9].

It is P. Cassou-Nogues who called my attention to some correspondence between [13] and [9]. I am grateful to her for this and for other useful discussions. I am grateful also to my teacher A.G. Vitushkin due to whom I started to work on the Jacobian Conjecture.

§1. Preliminaries

1.1. Dual graphs of reducible curves and their splice diagrams.
Let D be a curve on a smooth analytic surface such that all its irreducible components $D_1, \dots, D_n$ are isomorphic to $\mathbf{CP}^1$, meet each other transversally and at most pairwise. We call *dual graph* or just *graph* of D the graph Γ_D whose vertices correspond to irreducible components of D and edges correspond to their intersections. To each vertex we associate its *weight* which is equal to the self-intersection of the corresponding irreducible component. If it does not lead to a misunderstanding, we shall use the same notation for a curve and its graph.

If C is a smooth curve (not necessarily compact) meeting transversally D then we define the *graph of C near D* as the graph $\Gamma_{D,C}$ obtained from the graph of D by adding vertices corresponding to local branches $C_1, \dots, C_r$ of C near D (we depict these vertices as arrowheads). The vertex corresponding to a local branch C_i is connected by a single edge to the vertex corresponding to the component D_j which meets C_i. The weight of C_i is not defined.

Example. If D and C are a line and a conic on $\mathbf{CP}^2$ then $\Gamma_{D,C} = \longleftarrow\overset{+1}{\circ}\longrightarrow$.

The *determinant of a curve D* is by definition the determinant of the minus intersection matrix: $\det D = \det \|-D_iD_j\|_{i,j=1}^n$.

From now on, we assume that the graph of D is a tree (i.e. a connected graph without cycles). We call a *branch of D at a vertex D_i* a connected component of the closure of $D \setminus D_i$.

A linear chain is a graph with vertices $v_1, \dots, v_n$ and edges $[v_1, v_2]$, $[v_2, v_3], \dots, [v_{n-1}, v_n]$.

A splice diagram of a curve D (respectively, of a curve C near a curve D) is defined as a graph Δ_D (respectively, $\Delta_{D,C}$), obtained from Γ_D (respectively, from $\Gamma_{D,C}$) by replacing some (for instance, all) linear chains by a single edge. To each beginning of edge coming from a non-end vertex D_i, we associate the number equal to the determinant of the branch of D at the vertex D_i which grows to the direction of this edge (this definition slightly differs from the original definition of splice diagram introduced by Eisenbud and Neumann in [6]).

Proposition 1.1. (Edge determinant formula; see [6], [11]). *Let Δ_D be a splice diagram of a tree D of curves with simple normal crossings. Let u and v be vertices of Δ_D connected to each other by an edge. Let E be the linear chain of irreducible components of D corresponding to the edge uv (the curves corresponding to the vertices u and v themselves are not included into E). Suppose that Δ looks as in Fig. 1 near the edge uv. Then*

$$\det D \cdot \det E = a_0 b_0 - (a_1 \dots a_k) \cdot (b_1 \dots b_n).$$

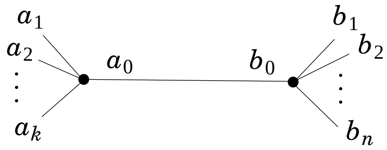


FIG. 1

1.2. Transformation of the determinant of the intersection matrix under a proper analytic mapping. The object of this subsection is to prove the following not difficult statement (it was used in [5]).

Proposition 1.2. *Let* $f : \widetilde{X} \to X$ *be a proper analytic mapping of smooth complex surfaces. Let* $D = D_1 \cup \dots \cup D_r$ *be compact curves on* X *and* $\widetilde{D} = \widetilde{D}_1 \cup \dots \cup \widetilde{D}_{\tilde{r}} = f^{-1}(D)$. *Then:*

(a). $\det \widetilde{D} = 0$ *if and only if* $\det D = 0$.

(b). Suppose that $\widetilde{D}_1$ *is the only irreducible component of the curve* $f^{-1}(D_1)$ *which is not constructed to a single point by* f. *Let us denote the closures of* $D \setminus D_1$ *and* $\widetilde{D} \setminus \widetilde{D}_1$ *by* D' *and* $\widetilde{D}'$ *respectively. Let* m *be the degree of* $f|_{\widetilde{D}_1}$ *and* n *the branching order of* f *along* $\widetilde{D}_1$ *(i.e. the jacobian of* f *has zero of order* $n-1$ *on* $\widetilde{D}_1$*). Then*

$$\frac{\det \widetilde{D}'}{\det \widetilde{D}} = \frac{n}{m} \cdot \frac{\det D'}{\det D}\,. \tag{1}$$

Moreover, if one of the denominators in (1) is zero then the other also is zero.

Proof. (a). First, let us prove that $\det D = 0$ implies $\det \widetilde{D} = 0$. Indeed, if $\det D = 0$ then there exists a non-zero divisor $E = \sum x_i D_i$ such that $ED_1 = \dots = ED_r = 0$. Then $f^*(E)$ is a non-zero divisor whose support is contained in $\widetilde{D}$ and $f^*(E) \cdot \widetilde{D}_j = E \cdot f_*(\widetilde{D}_j) = 0$ for all j. Hence, $\det \widetilde{D} = 0$.

Now, let us prove that $\det \widetilde{D} = 0$ implies $\det D = 0$. Indeed, if $\det \widetilde{D} = 0$ then there exists a non-zero divisor $\widetilde{E} = \sum \tilde{x}_i \widetilde{D}_i$ such that

$$\widetilde{E}\widetilde{D}_1 = \dots = \widetilde{E}\widetilde{D}_{\tilde{r}} = 0. \tag{2}$$

Then $f_*(\widetilde{E})$ is a divisor whose support is contained in D and $f_*(\widetilde{E}) \cdot D_j = \widetilde{E} \cdot f^*(D_j) = 0$ for all j. Hence, the equality $\det D = 0$ would follow from the fact that $f_*(\widetilde{E}) \neq 0$. Suppose that $f_*(\widetilde{E}) = 0$. This means that the support of the divisor $\widetilde{E}$ is concentrated in the preimage of a finite set of points. But the intersection matrix of irreducible components of a compact curve contractible to a point by an analytic mapping is negative definite. Hence, $\widetilde{E}^2 < 0$. This contradicts to (2).

(b). In virtue of (a), we may assume that the both denominators in (1) are non-zero. Let us denote by $E = \sum x_i D_i$ and $\widetilde{E} = \sum \tilde{x}_i \widetilde{D}_i$ the divisors with rational coefficients, dual to D_1 and $\widetilde{D}_1$ respectively. It means that

$$E \cdot D_1 = 1, \quad E \cdot D_i = 0 \text{ for } i > 1; \qquad \widetilde{E} \cdot \widetilde{D}_1 = 1, \quad \widetilde{E} \cdot \widetilde{D}_i = 0 \text{ for } i > 1. \tag{3}$$

The existence of the divisors E and $\widetilde{E}$ easily follows from the fact that the intersection matrices are non-degenerate. Indeed,

$$(x_1, \dots, x_r) = B(1, 0, \dots, 0), \qquad (\tilde{x}_1, \dots, \tilde{x}_{\tilde{r}}) = \widetilde{B}(1, 0, \dots, 0), \tag{4}$$

where $B = \|b_{ij}\| = A^{-1}$, $A = \|D_iD_j\|_{i,j=1}^r$, $\widetilde{B} = \|\tilde{b}_{ij}\| = \widetilde{A}^{-1}$ and $\widetilde{A} = \|\widetilde{D}_i\widetilde{D}_j\|_{i,j=1}^{\tilde{r}}$.

We have $f_*(\widetilde{D}_1) = mD_1$, hence, $f_*(\widetilde{E}) = m\tilde{x}_1D_1 + F$ where $D_1 \notin \operatorname{supp} F$, and hence, by (3),

$$E \cdot f_*(\widetilde{E}) = m\tilde{x}_1. \tag{5}$$

Analogously, $f^*(D_1) = n\widetilde{D}_1 + \widetilde{F}_1$ where $\widetilde{D}_1 \notin \operatorname{supp} \widetilde{F}_1$ and hence, $f^*(E) = nx_1\widetilde{D}_1 + \widetilde{F}_2$ where $\widetilde{D}_1 \notin \operatorname{supp} \widetilde{F}_2$. Hence, by (3) we have

$$f^*(E) \cdot \widetilde{E} = nx_1. \tag{6}$$

Putting (5) and (6) into the equality $E \cdot f_*(\widetilde{E}) = f^*(E) \cdot \widetilde{E}$, we get $\tilde{x}_1 = (n/m) \cdot x_1$. Note that $x_1 = b_{11}$ and $\tilde{x}_1 = \tilde{b}_{11}$ by (4). Finally, by Cramer rule, we have

$$b_{11} = \frac{\det D'}{\det D} \qquad \text{and} \qquad \tilde{b}_{11} = \frac{\det \widetilde{D}'}{\det \widetilde{D}}.$$

1.3. A formula for the canonical class of a blown up (+1)-pair. Let (U, l) be some (+1)-pair (for example, l is the infinite line of the affine plane $\mathbf{C}^2$) and let $\sigma : X \to U$ be a composition of blow-ups "at infinity", i.e. $\sigma|_{X\setminus L} : X \setminus L \to U \setminus l$ is an isomorphism where $L = \sigma^{-1}(l)$. Let L_0 be the proper preimage of the line l.

Proclaim 1.3. *(a). L be the line of rational curves,* $\det L = -1$.

(b). The determinant of any branch of L at the vertex L_0 is equal to one.

(c). Let L_1 is an irreducible component of L, different from L_0. Consider the branches of L at the vertex L_1 which does not contain L_0. Among these branches, there is at most one whose determinant is not equal to one.

Proof. Induction by the number of the blow-ups. □

Let $L_0, \dots, L_n$ be the irreducible components of L. Suppose that the canonical class K_X of X is representable by a divisor supported by L, i.e. there is a meromorphic 2-form ω on X which neither has zeros nor poles outside of L. Let

$$K_X = \sum k_jL_j.$$

We are still assuming that L_0 is the proper transform of l. The irreducible components are numbered arbitrarily, hence any irreducible component different from L_0 can be considered as the curve L_1 in the next proposition.

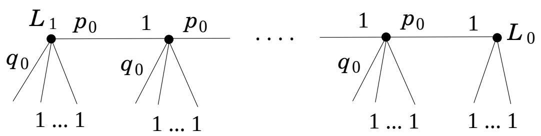


FIG. 2

Proposition 1.4. (see. [14]). *(a).* $k_0 = -3$.

(b). Let us denote the weights of the splice diagram of L, situated along the shortest path from L_0 to L_1, as in Fig. 2 (see Proposition 1.3). Then

$$k_1 = -1 - q_0 - p_0 + \sum_{j=1}^{m} q_0 \dots q_{j-1}(q_j - 1)(p_j - 1).$$

1.4. Coverings branched along linear chains. As in [13], we shall use the language of toric varieties to describe coverings branched along linear chains of rational curves. An equivalent description not involving toric varieties see in [2; III, §5]. Since we need a very small portion of the theory of toric varieties, for the reader's convenience we give all the definitions and statements that we use.

1.4.1. Fans and toric surfaces. Let us identify $\mathbf{Z}^2 \wedge \mathbf{Z}^2 = \mathbf{Z}$, i.e. $(a, b) \wedge (c, d)$ will denote $ad - cb$. For $e_1, e_2 \in \mathbf{Z}^2$, let us denote the cone $\{x_1e_1 + x_2e_2 \mid x_i \in \mathbf{R}, x_i > 0\}$ by $\langle e_1, e_2\rangle$, and let $\langle e\rangle = \langle e, e\rangle$ (the ray in the direction e). A vector $e \in \mathbf{Z}^2$ is called *primitive* if it cannot be presented in the form me', $e' \in \mathbf{Z}^2$, $m \in \mathbf{Z}$. We call a *fan* a collection of distinct primitive integral vectors $\Sigma = (e_0, \dots, e_{r+1}) \subset \mathbf{Z}^2$ such that $e_i \wedge e_{i+1} > 0$ for all $i = 0, \dots, r$ and the cones $\langle e_0, e_1\rangle, \dots, \langle e_r, e_{r+1}\rangle$ are pairwise disjoint. If $e_i \wedge e_{i+1} = 1$ for all $i = 0, \dots, r$ then the fan is called *primitive.*

Let us denote $u_j = e_j$ and $v_j = e_{j+1}$. *The toric surface associated to a primitive fan* Σ is the smooth algebraic surface X_Σ glued out of charts $U_0, \dots, U_r$ isomorphic to $\mathbf{C}^2$. The chart U_j with coordinates (x_j, y_j) corresponds to the cone $\langle u_j, v_j\rangle$ and the transition functions are:

$$\begin{cases} x_i = x_j^a\, y_j^c \\ y_i = x_j^b\, y_j^d \end{cases} \qquad \text{where} \qquad \begin{cases} u_j = au_i + bv_i \\ v_j = cu_i + dv_i \end{cases}$$

It is clear that X contains a Zariski open subset isomorphic to $\mathbf{T}^2 = (\mathbf{C} \setminus 0)^2$ which is defined by the inequality $x_iy_i \neq 0$ in any coordinates x_i, y_i.

Proposition 1.5. *Let Σ be a primitive fan and $E = X_\Sigma \setminus \mathbf{T}^2$. Then $E = E_0 \cup \dots \cup E_{r+1}$. Moreover,*

(a). E_j is defined by $x_j = 0$ in the coordinates (x_j, y_j), by $y_{j-1} = 0$ in the coordinates (x_{j-1}, y_{j-1}), and E_j does not meet the other charts.

(b). $E_0 \cong E_{r+1} \cong \mathbf{C}$ and $E_1 \cong \dots \cong E_r \cong \mathbf{CP}^1$.

(c). The self-intersection E_j^2 of E_j is equal to $-e_{j-1} \wedge e_{j+1}$, $j = 1, \dots, r$.

(d). $\det \|-E_iE_j\|_{i,j=1}^r = e_0 \wedge e_{r+1}$.

Proof. (a) – (c) follow immediately from the definitions; (d) is proved by induction, using (c). □

1.4.2. Mappings of toric surfaces. To a linear mapping $A : \mathbf{Z}^2 \to \mathbf{Z}^2$ and primitive fans $\tilde\Sigma = (\tilde e_0, \dots, \tilde e_{\tilde r+1})$ and $\Sigma = (e_0, \dots, e_{r+1})$, we associate a birational mapping $f = A_* : X_{\tilde\Sigma} \to X_\Sigma$. In coordinates $(\tilde x_j, \tilde y_j)$ on $X_{\tilde\Sigma}$ and (x_i, y_i) on X_Σ, it is defined by

$$f(\tilde x_j, \tilde y_j) = (x_i, y_i), \qquad \begin{cases} x_i = \tilde x_j^a\, \tilde y_j^c \\ y_i = \tilde x_j^b\, \tilde y_j^d \end{cases} \qquad \text{where} \qquad \begin{cases} A(\tilde u_j) = au_i + bv_i \\ A(\tilde v_j) = cu_i + dv_i \end{cases}$$

(As above, here $u_i = e_i$, $v_i = e_{i+1}$, and also $\tilde{u}_j = \tilde{e}_j$, $\tilde{v}_j = \tilde{e}_{j+1}$). A *regular mapping* of a fan $\widetilde{\Sigma}$ to a fan Σ is called a linear mapping $A : \mathbf{Z}^2 \to \mathbf{Z}^2$ such that for any cone $\langle\widetilde{u}_j, \widetilde{v}_j\rangle$ there is a cone $\langle u_i, v_i\rangle$ such that $f\big(\langle\widetilde{u}_j, \widetilde{v}_j\rangle\big) \subset \langle u_i, v_i\rangle$. It is easy to check that in this case A_* is a regular (i.e. without indeterminacy points) mapping $X_{\widetilde{\Sigma}} \to X_\Sigma$.

The following properties follow immediately from the definitions and from Proposition 1.5.

Proposition 1.6. *Let $A : \mathbf{Z}^2 \to \mathbf{Z}^2$ be a regular mapping of primitive fans $\widetilde{\Sigma} \to \Sigma$, and let $f = A_* : X_{\widetilde{\Sigma}} \to X_\Sigma$. Suppose also that $A(\langle\tilde{e}_0\rangle) = \langle e_0\rangle$ and $A(\langle\tilde{e}_{\tilde{r}+1}\rangle) = \langle e_{r+1}\rangle$. Let us denote:*

N the degree of f;

n_0 the order of branching of f along $\widetilde{E}_0$;

n_1 the order of branching of f along $\widetilde{E}_{\tilde{r}+1}$

(i.e. the jacobian of f has zero of order $n_0 - 1$ on $\widetilde{E}_0$ and zero of order $n_1 - 1$ on $\widetilde{E}_{\tilde{r}+1}$);

m_0 the branching order of $f|_{\widetilde{E}_0}$ at the point $\widetilde{E}_0 \cap \widetilde{E}_1$;

m_1 the branching order of $f|_{\widetilde{E}_{\tilde{r}+1}}$ at the point $\widetilde{E}_{\tilde{r}} \cap \widetilde{E}_{\tilde{r}+1}$.

$\Delta = \det \|-E_iE_j\|_{i,j=1}^r, \quad \widetilde{\Delta} = \det \|-\widetilde{E}_i\widetilde{E}_j\|_{i,j=1}^{\tilde{r}}$

Then:

(a). $\det A = N = m_0n_0 = m_1n_1$;

(b). $A(\tilde{e}_0) = n_0e_0$ *and* $A(\tilde{e}_{\tilde{r}+1}) = n_1e_{r+1}$;

(c).

$$\widetilde{\Delta} = \frac{n_0n_1}{N}\Delta.$$

Corollary 1.7. *Let the notation be as in 1.6. If the mapping f is not branched along $\widetilde{E}_{\tilde{r}}$ then $\Delta = m_0\widetilde{\Delta}$.* □

A fan $\Sigma' = (e'_0, \dots, e'_{r'+1})$ is called a *subdivision* of a fan $\Sigma = (e_0, \dots, e_{r+1})$, if $e'_0 = e_0$, $e'_{r'+1} = e_{r+1}$ and the identity mapping $\mathrm{id} : \mathbf{Z}^2 \to \mathbf{Z}^2$ is a regular mapping of fans $\Sigma' \to \Sigma$.

Lemma 1.8. *Any fan Σ has a primitive subdivision Σ'.*

Proof. For each 2-dimensional cone σ, let us add as new generators all the vectors lying on compact sides of the convex hull of the set $(\mathbf{Z}^2 \cap \bar{\sigma}) \setminus \{0\}$. □

Propositions 1.9. *Let $\widetilde{\Sigma} = (e_0, \dots, \tilde{e}_{\tilde{r}+1})$ and $\Sigma = (e_0, \dots, e_r)$ be two fans and let $A : \mathbf{Z}^2 \to \mathbf{Z}^2$ be a linear mapping such that $A(\langle\tilde{e}_0\rangle) = \langle e_0\rangle$ and $A(\langle\tilde{e}_{\tilde{r}+1}\rangle) = \langle e_{r+1}\rangle$. Then there exist subdivisions $\widetilde{\Sigma}'$ and Σ' of the fans $\widetilde{\Sigma}$ and Σ such that A is a regular mapping $\widetilde{\Sigma}' \to \Sigma'$.*

Proof.

1). Let us add to Σ the integral generators of the rays $A(\langle\tilde{e}_0\rangle), \dots, A(\langle\tilde{e}_{\tilde{r}_1}\rangle)$ and subdivide the obtained fan up to a primitive one $\Sigma' = (e'_0, \dots, e'_{r'+1})$.

2). Let us add to $\widetilde{\Sigma}$ the integral generators of the rays $A^{-1}(\langle\tilde{e}'_0\rangle), \dots, A^{-1}(\langle\tilde{e}'_{r'_1}\rangle)$ and subdivide the obtained fan up to a primitive one $\widetilde{\Sigma}'$. □

§2. Regular compactification at infinity of a meromorphic immersion of a $(+1)$-pair

2.1. Compactification at infinity. Dicritical components. Let us consider some algebraic compactification X of the complex plane $\mathbf{C}^2$ with simple normal crossings of the curve at the infinity, i.e. X is a projective surface which contains a curve L (generally, reducible) such that $X \setminus L = \mathbf{C}^2$. All such compactifications are obtained from $\mathbf{CP}^2$ with a chosen infinite line by blow-ups and blow-downs at the infinity. The dual graph of L is a tree.

Let (U, l) be a $(+1)$-pair and $U \setminus l \to \mathbf{C}^2$ a meromorphic immersion of it into $\mathbf{C}^2$. Blowing up points of l, it can be presented in the form $f \circ \sigma^{-1}$ where $\sigma : \widetilde{X} \to U$ is a composition of blow-ups and $f : \widetilde{X} \to X$ a holomorphic mapping. The triple $(\widetilde{X}, X, f)$ is called a *regular compactification of the meromorphic immersion of the $(+1)$-pair (U, l) into* $\mathbf{C}^2$. Let us denote

$$\widetilde{L} = \sigma^{-1}(l), \qquad \widetilde{L}_\infty = f^{-1}(L), \qquad \widetilde{L}_{FC} = \overline{\widetilde{L} \setminus \widetilde{L}_\infty},$$

$$D = f(\widetilde{L}_{FC}), \qquad \widetilde{D} = \widetilde{D}_1 \cup \dots \cup \widetilde{D}_d, \qquad \widetilde{L}_C = \overline{\widetilde{L}_{FC} \setminus \widetilde{D}}$$

where $\widetilde{D}_1, \dots, \widetilde{D}_d$ are the *dicritical components* of the mapping f, i.e. the irreducible components of the curve $\widetilde{L}_{FC}$ such that f is not constant on them. The curve D will be called the branching curve.

Proposition 2.1. (cp. [12]) *(a). Irreducible components of the curve $\tilde{L}$ are rational curves and the dual graph of $\tilde{L}$ is a tree.*

(b). The curve $\widetilde{L}_\infty$ is connected.

(c). $\widetilde{L}_{FC}$ has d connected components $\widetilde{L}^{(1)}_{FC}, \dots, \widetilde{L}^{(d)}_{FC}$.

(d). The dual graph of $\widetilde{L}^{(i)}_{(FC)}$ $(i = 1, \dots, d)$ is a linear chain (possibly, with a single vertex) one of whose end vertices corresponds to the dicritical component $\widetilde{D}_i$.

(e). The curve $\widetilde{L}^{(i)}_{FC}$ $(i = 1, \dots, d)$ cuts $\widetilde{L}_\infty$ at a single point and this point belongs to $\widetilde{D}_i$ $(i = 1, \dots, d)$.

Let n_i, $i = 1, \dots, d$, be the branching order of f along $\widetilde{D}_i$, i.e. the jacobian of f vanishes on $\widetilde{D}_i$ with the multiplicity $n_i - 1$.

Proposition 2.2. *The canonical class $K_{\widetilde{X}}$ of $\widetilde{X}$ can be represented by a divisor supported by $\widetilde{L}$ and the multiplicity of a dicritical component D_i in this divisor is $n_i - 1$.*

Proof. $K_{\widetilde{X}}$ is represented by the divisor of the form $f^*(dx \wedge dy)$ where x, y are the affine coordinates in $\mathbf{C}^2$.

2.2. Formula for the index of a meromorphic immersion of a $(+1)$-pair.

Definition 2.3. *The local multiplicity at a point $x \in X$* of a continuous mapping of topological spaces $\phi : X \to Y$ is called $\mu_x \phi = \min_U \deg\left(\phi_{\widetilde{U}(x)}\right)$ where the minimum is taken over all neighbourhoods U of $\phi(x)$ and $\widetilde{U}(x)$ denotes the connected component of $f^{-1}(U)$ which contains x.

Let the notation be as in the previous subsection. Let X^* be the one-point compactification of $\mathbf{C}^2$. Denote by $\widetilde{X}^*$ the singular surface obtained from $\widetilde{X}$ if each connected component of each set $f^{-1}(x)$, $x \in X$ is contracted to a single point and also the curve $\widetilde{L}_\infty$ is contracted to a single point (which we denote by ∞). Then there exists a unique mapping $f^* : X^* \to X$ such that the following diagram commutes

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ \widetilde{X}^* & \xrightarrow{f^*} & X^* \end{array}$$

(the vertical arrows are the natural projections). Let us denote the image of $\widetilde{D}_i$ on $\widetilde{X}^*$ by $\widetilde{D}_i^*$.

Let, as in Sect. 2.1, n_i, be the branching order of f along $\widetilde{D}_i$, i.e. $n_i = \mu_x f$ for a generic point $x \in \widetilde{D}_i$.

Proposition 2.4. *The index* ind *of the meromorphic immersion* $(U, l) \to \mathbf{C}^2$ *is equal to*

$$\text{ind} = \mu_\infty f^* - \sum_{i=1}^{d} \Big(n_i + \sum_{x \in D_i^* \setminus \{\infty\}} (\mu_x f^* - n_i) \Big). \tag{7}$$

Proof. Let us choose coordinates (z, w) in $\mathbf{C}^2$ in such a way that the line $z = 0$ does not meet the curve D at the infinity under the standard inclusion of $\mathbf{C}^2$ into $\mathbf{CP}^2$. Denote the projection $(z, w) \mapsto z$ by $\pi : \mathbf{C}^2 \to \mathbf{C}$.

Denote the branch points of the mapping $\pi|_D : D \to \mathbf{C}$ by $p_1, \ldots, p_n$, the order of the branching at p_i by m_i $(i = 1, \ldots, n)$, and the degree of the curve D by m_∞. Let B_i, $(i = 1, \ldots, n)$, be a ball of a sufficiently small radius centred at p_i, and let B_∞ be a ball of a sufficiently large radius centred at the origin.

Let V be a tubular neighbourhood of D whose radius is small with respect to the radii of the spheres S_i. Let $T = B_\infty \cap \big((-\partial V) \setminus (B_1 \cup \cdots \cup B_n)\big)$ and $S = (-\partial B_1) \cup \cdots \cup (-\partial B_n) \cup \partial B_\infty$ (the minus means the orientation reversing). Let R_i, $i \in \{1, \ldots, n, \infty\}$ be a hypersurface with a boundary (homeomorphic to several copies of $S^1 \times S^1 \times [0, 1]$) which smoothes the corner between ∂B_i and T as it is shown in Fig. 3.

Each sphere ∂B_i has exactly one point where the positive normal vector is equal to $(1, 0)$, moreover, its index (i.e. the contribution into the degree of the Gauss mapping) is equal to -1 for $i = 1, \ldots, n$ and $+1$ for $i = \infty$. If the coordinates (z, w) are generic then this points is outside V. The surface T has no such points, and each surface R_i has m_i such points of index $+1$ for $i = 1, \ldots, n$ and -1 for $i = \infty$.

By the definition, ind is equal to the degree of the Gauss map associated to $f|_{-M}$ where M is the boundary of the tubular neighbourhood of l in U or, which is the same, the boundary of a neighbourhood of $\widetilde{D}^*$ in $\widetilde{X}^*$. The minus before M means the reversing of the orientation.

The immersion $f|_{-M}$ can be deformed into an immersion whose image is in $S \cup R \cup T$. Extend the mapping $\pi \circ f^*|_{\widetilde{D}_j^* \setminus \infty} : \widetilde{D}_j^* \setminus \infty \to \mathbf{C}$ up to a mapping $f_j^* : \widetilde{D}_j^* \to \mathbf{C} \cup \{\infty\}$. The contributions of the surfaces into the degree of the Gauss

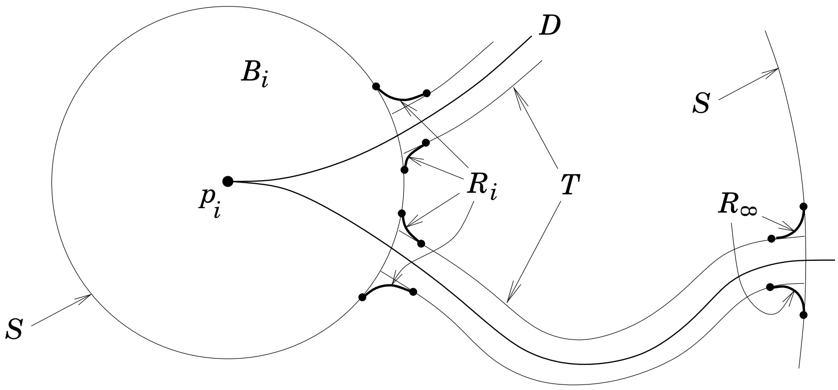


FIG. 3

mapping are:

$$S_\infty \to \mu_\infty f^*, \qquad R_\infty \to -\sum_{j=1}^{d} n_j \cdot \mu_\infty f_j^*, \qquad T \to 0,$$

$$S_i \to -\sum_{\substack{x\in\widetilde{D}^*\\ f^*(x)=p_i}} \mu_x f^*, \qquad R_i \to \sum_{j=1}^{d} \sum_{\substack{x\in\widetilde{D}_j^*\\ f^*(x)=p_i}} n_j \cdot \mu_x f_j^*.$$

Thus, denoting $P=\{p_1,\dots,p_n\}$, we have

$$\begin{aligned}
\mathrm{ind} &= \mu_\infty f^* - \sum_{i=1}^{n} \sum_{\substack{x\in\widetilde{D}^*\\ f^*(x)=p_i}} \mu_x f^* - \sum_{j=1}^{d} n_j\Big(\mu_\infty f_j^* - \sum_{i=1}^{n} \sum_{\substack{x\in\widetilde{D}_j^*\\ f^*(x)=p_i}} \mu_x f_j^*\Big)\\
&= \mu_\infty f^* - \sum_{j=1}^{d} \sum_{\substack{x\in\widetilde{D}_j^*\\ f^*(x)\in P}} \mu_x f^* - \sum_{j=1}^{d} n_j\Big(\mu_\infty f_j^* - \sum_{\substack{x\in\widetilde{D}_j^*\\ f^*(x)\in P}} \mu_x f_j^*\Big)\\
&= \mu_\infty f^* - \sum_{j=1}^{d}\Big(\sum_{\substack{x\in\widetilde{D}_j^*\\ f^*(x)\in P}} (\mu_x f^* - n_j) + n_j\Big\{\mu_\infty f_j^* - \sum_{\substack{x\in\widetilde{D}_j^*\\ f^*(x)\in P}} (\mu_x f_j^* - 1)\Big\}\Big).
\end{aligned}$$

It remains to note that by Riemann-Hurwitz formula applied to the branched covering f_j^*, the expression in the braces is equal to one. □

Remark. In the case of a meromorphic immersion of a flat (+1)-pair defined by a polynomial mapping $\mathbf{C}^2 \to \mathbf{C}^2$ with a constant jacobian, the fact that the right hand side of (7) is equal to one was proved in [12] by computing the Euler characteristic. Proposition 2.4 is a generalization of this fact to the case of meromorphic immersions into $\mathbf{C}^2$ of arbitrary (+1)-pairs.

2.3. Properties of splice diagrams of L and $\widetilde{L}$.

We may assume that L meets D transversally (otherwise we blow up $D \cap L$ several times). Then the formulas given in §1 together with Proposition 2.2 impose rather strong restrictions for the splice diagrams of $\widetilde{L}$ and $L \cup D$. We apply the formulas from §1 as follows:

(1) we apply Proposition 1.3 to the splice diagrams of L and $\widetilde{L}$;
(2) we apply Proposition 1.6(c) to each edge of the splice diagrams;
(3) we apply Proposition 1.1 to each edge of the splice diagrams between vertices of the valence ≥ 3;
(4) we apply Propositions 1.4 and 2.2 to the dicritical components;
(5) we apply Proposition 1.2 (if it is applicable) to each non-linear connected component of the graph of L from which some vertices of the valence ≥ 3 are removed.

In the papers [5], [4], it is shown that these restrictions are sufficient to prove that there are no counter-examples to the Jacobian Conjecture provided by a mapping of the topological degree $N \leq 4$ (for $N = 2$ this is evident, and for $N = 3$ this follows from Abhyankar-Moh-Suzuki theorem, see [12]).

2.4. The case of an irreducible branching curve with two characteristic pairs. Suppose that $\widetilde{L}$ has a single dicritical component $\widetilde{D}$, and that the branching curve $D = f(\widetilde{D})$ has two characteristic pairs at the infinity. This means that after the resolution of the singularity of D at the infinity, its splice diagram near L has the form $-\circ-\circ\to$ (with a vertical edge at each vertex). Moreover, we shall suppose that the following additional condition holds:

($*$) There exists an irreducible component of L whose preimage has only one irreducible component which is not contractible by f into a single point (compare with Proposition 1.2(b)).

Under these assumptions, the splice diagrams of $\widetilde{D}$ near $\widetilde{L}_\infty$, of D near L, and of $\widetilde{L}$ have the form depicted in Fig. 4, Fig. 5 and Fig. 6. The black vertex denotes the proper transform of l under the mapping $\sigma : (\widetilde{X}, \widetilde{L}) \to (U, l)$.

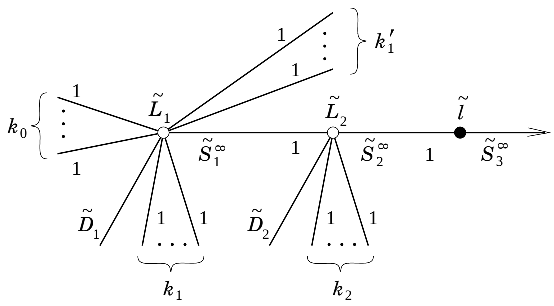


FIG. 4. SPLICE DIAGRAM $\Delta_{\widetilde{L}_\infty, \widetilde{D}}$

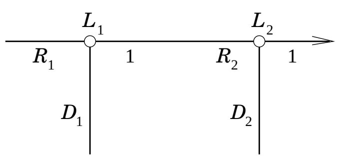


FIG. 5. SPLICE D. $\Delta_{L,D}$

Let us introduce the following notation. Let Q_2 and $\widetilde{Q}_2$ be the determinants of the edges L_1L_2 and $\widetilde{L}_1\widetilde{L}_2$ of the splice diagrams Δ_L and $\widetilde{\Delta}_L$, i.e. Q_2 (respectively, $\widetilde{Q}_2$) is the determinant of that component of the closure of the curve $L \setminus (L_1 \cup L_2)$, (the curve $\widetilde{L} \setminus (\widetilde{L}_1 \cup \widetilde{L}_2)$) which is between L_1 and L_2 (between $\widetilde{L}_1$ and $\widetilde{L}_2$).

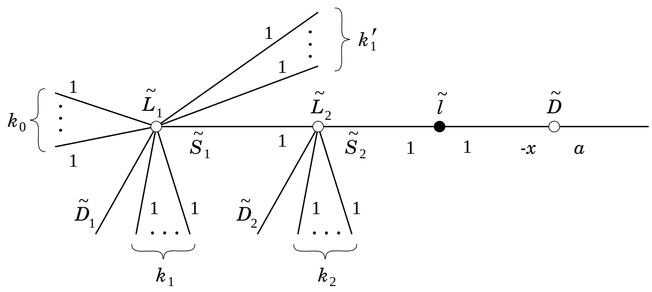


FIG. 6. SPLICE DIAGRAM $\Delta_{\widetilde L}$

For $j = 1, 2$ let us denote the degree of $f|_{\widetilde L_j}$ by m_j, the branching order of f along $\widetilde L_1$ by n_j, and let m_j' (respectively, d_j) be the branching order of $f|_{\widetilde L_j}$ at the point of intersection of $\widetilde L_j$ with that branch of $\widetilde L$ at the vertex L_j which contains $\tilde l$, (respectively, which has the determinant $\widetilde D_j$). Let us also denote the degree of $f|_{\widetilde D}$ and the branching order of f along $\widetilde D$ by m and n.

All the introduced integers are positive (the positivity of R_j see [11]; from the theory of approximation roots[1], it follows also that $R_j > 1$). They satisfy the following relations.

$$\gcd(D_1, R_1) = \gcd(D_2, R_2) = 1, \tag{8}$$

the edge determinant formula (Proposition 1.1) yields

$$-Q_2 = R_2 - R_1 D_1 D_2 \qquad \text{(the edge } L_1L_2 \text{ in } L), \tag{9}$$

$$-x\,\widetilde Q_2 = \widetilde S_1^\infty - \widetilde S_2^\infty \widetilde D_1 \widetilde D_2 \qquad \text{(the edge } \widetilde L_1 \widetilde L_2 \text{ in } \widetilde L_\infty). \tag{10}$$

By Proposition 1.6 and Corollary 1.7, we have

$$D_1 = d_1 \widetilde D_1, \quad D_2 = d_2 \widetilde D_2, \quad \frac{\widetilde Q_2}{Q_2} = \frac{n_1}{m_2} = \frac{n_2}{m_1'}, \quad \widetilde S_2^\infty = \frac{n_2}{m} = \frac{n}{m_2'}, \tag{11}$$

and also see that the branching orders at the points of $\widetilde L_1$ and $\widetilde L_2$, corresponding to the edges of the splice diagram are:

$$\underbrace{R_1, \dots, R_1}_{k_0}, \qquad d_1, \underbrace{D_1, \dots, D_1}_{k_1}, \qquad m_1', \underbrace{1, \dots, 1}_{k_1'} \qquad \text{for the curve } \widetilde L_1,$$

$$m_2, \qquad d_2, \underbrace{D_2, \dots, D_2}_{k_2}, \qquad m_2' \qquad \text{for the curve } \widetilde L_2,$$

this implies

$$k_0 R_1 = d_1 + k_1 D_1 = m_1' + k_1' = m_1, \qquad d_2 + k_2 D_2 = m_2, \tag{12}$$

$$m_1' = k_1 + k_0; \qquad m_2' = k_2 + 1 \tag{13}$$

(the relation (13) is obtained from (12) and Riemann-Hurwitz formula).

Applying Proposition 1.2 to the curve L itself and to its branch at the vertex L_1 containing L_2, we get

$$\frac{\widetilde{D}_1\widetilde{S}_1^\infty}{x} = \frac{n_1}{m_1}\cdot\frac{R_1D_1}{1}, \qquad \frac{\widetilde{D}_2\widetilde{Q}_2\widetilde{S}_2^\infty}{\widetilde{S}_1^\infty} = \frac{n_2}{m_2}\cdot\frac{Q_2D_2}{1}. \tag{14}$$

Finally, by Proposition 2.2, the order of the jacobian of f on $\widetilde{D}$ is equal to $n-1$ and by Proposition 1.4 it is equal to $-1-a+x$, i.e.

$$x = a + n. \tag{15}$$

If the considered meromorphic immersion is realizable by polynomials $P(x,y)$, $Q(x,y)$ then (see Sect. 4.1)

$$\deg P(x,y) = k_0\widetilde{D}_1\widetilde{D}_2 + aD_1D_2; \quad \deg Q(x,y) = k_1\widetilde{D}_1\widetilde{D}_2 + \widetilde{D}_2 + aR_1D_2. \tag{16}$$

A first arbitrarily found solution of simultaneous equations allowed me to construct a counter-example to WJC_∞ in [13].

Proposition 2.5. *The simultaneous equations (8) – (15) have a finite number of positive integral solutions under the condition that* $m_1n_1 = N = \text{const}$.

Proof. From the equations (11), (14) and $k_0R_0 = m_1$ (see (12)) we get $m_1 = mxd_1d_2R_1$, $n_1 = md_2\widetilde{S}_1^\infty$, hence, $m^2xd_1d_2^2R_1\widetilde{S}_1^\infty = N$. The other parameters also can be easily estimated via N. □

I wrote a simple computer program which finds all positive integral solutions of (8) – (15) for any given $N = m_1n_1$. For $N < 9$, there is no solution. For $N = 9$, there is a unique solution

$$\begin{array}{llllll} R_1 = 3, & D_1 = \widetilde{D}_1 = 4, & m_1 = 9, & n_1 = 1, & \widetilde{Q}_2 = 5, & n = 2, \\ R_2 = 23, & D_2 = \widetilde{D}_2 = 4, & m_2 = 5, & n_2 = 1, & Q_2 = 25, & a = 1. \end{array} \tag{17}$$

This solution allows one to construct the simplest counter-example to WJC_∞. If this solution were a counter-example to JC then, by (16), the degrees of P and Q would be 48 and 64. The solution (17) is discussed in [9; §3]. There is exactly four solutions with $\max(\deg P, \deg Q) < 100$. They correspond exactly to the four difficult cases in Moh's paper [10].

The example which we construct in §3 corresponds to the solution

$$\begin{array}{llllll} R_1 = 3, & D_1 = \widetilde{D}_1 = 20, & m_1 = 21, & n_1 = 1, & \widetilde{Q}_2 = 17, & n = 4, \\ R_2 = 112, & D_2 = \widetilde{D}_2 = 3, & m_2 = 4, & n_2 = 2, & Q_2 = 68, & a = 3. \end{array}$$

§3. Construction of the example

3.1. The mapping of graphs. We shall construct a meromorphic immersion of a (+1)-pair into $\mathbf{C}^2$ with a single dicritical component $\widetilde{D}$ whose image $D = f(\widetilde{D})$ has two characteristic pairs at the infinity and whose regular compactification at the infinity induces the mapping of the graphs of $\widetilde{L}$ and $D \cup L$ depicted in Fig. 7. The black vertices "•" in Fig. 7 correspond to the curves which are contracted by f into a single point. The mapping of linear chains $\widetilde{\Gamma} \to \Gamma$ is given in Sect. 3.3.2.

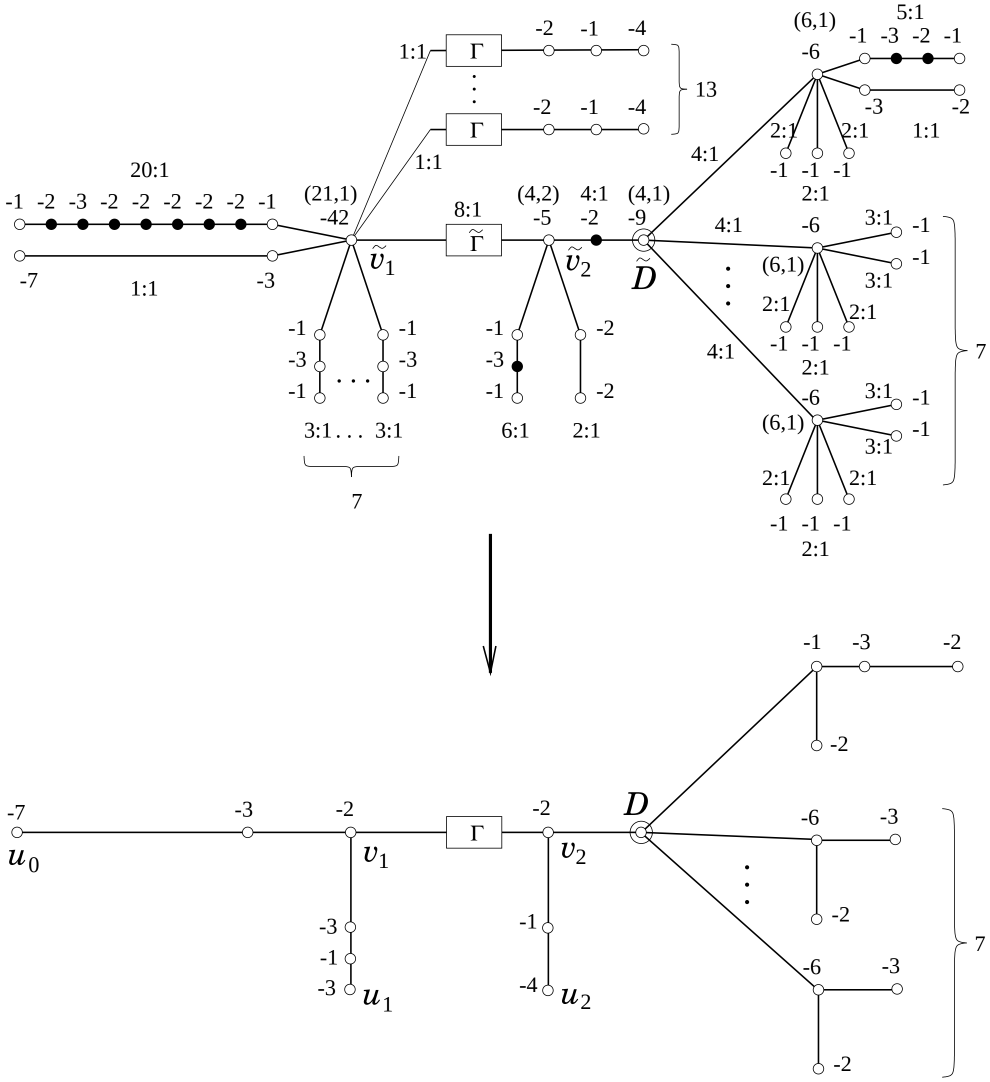


FIG. 7

In Fig. 7, we also show the resolutions of local branches of the curve D and the regularization of f over them.

In the picture of the graph of $\widetilde{L}$ (Fig. 7, upper part) is also equipped with the following information concerning the mapping f. A mark of the form $N:1$ near a linear chain (an edge of the splice diagram) means that the degree of f at its neighbourhood is N. A mark of the form (m,n) near a vertex of the valence ≥ 3 (denote the corresponding curve by A) means that the degree of $f|_A$ is m and the branching order of f along A is n.

3.2. Construction of the branching curve.

Lemma 3.1. *There exists a curve D, parametrized by a polynomial mapping g : $\mathbf{C} \to \mathbf{C}^2$, $t \mapsto \big(p(t), q(t)\big)$ where $p(t)$ and $q(t)$ are polynomials of degrees* 60 *and* 9 *respectively, and pairwise distinct points* $t_1, \ldots, t_7 \in \mathbf{C} \setminus \{0\}$ *such that*

(a). $g(0) = (0,0)$, *and* $g|_{\mathbf{C}\setminus\{0,t_1,\ldots,t_7\}}$ *is an immersion, i.e.* $p'(t) \neq 0$ *and* $q'(t) \neq 0$ *for* $t \notin \{0, t_1, \ldots, t_7\}$*;*

(b). *the splice diagram of* D *at the infinity is as in Fig. 8;*

(c). *the local branch of* D *parametrized by a neighbourhood of* 0 *has the splice diagram at the origin depicted in Fig. 9, i.e. it has the singularity defined by* $u^2 = v^5$ *in some local coordinates (the singularity of the type* A_4*);*

(d). *for all* $k = 1, \ldots, 7$*, the local branch of* D*, parametrized by a neighbourhood of* t_k *has the splice diagram depicted in Fig. 10, i.e. it has the singularity defined by* $u^2 = v^3$ *in some local coordinates (the singularity of the type* A_2*);*

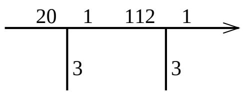


FIG. 8 $(t = \infty)$

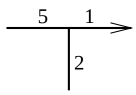


FIG. 9 $(t = 0)$

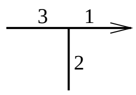


FIG. 10 $(t = t_k)$

The conditions (b) and (c) of Lemma 3.1 are equivalent to the fact that the graph of resolution of the curve at the infinity (of the local branch at $t = 0$, at $t = t_k$) is as in Fig. 11 (in Fig. 12, in Fig. 13).

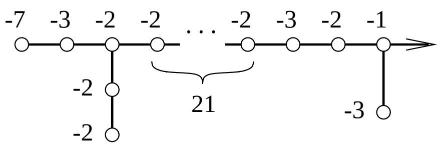


FIG. 11 $(t = \infty)$

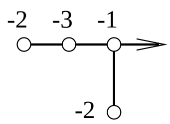


FIG. 12 $(t = 0)$

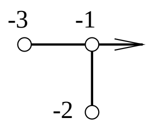


FIG. 13 $(t = t_k)$

Proof. By linear changes of coordinates, the polynomials p and q can be put into the form $p(t) = t^{60} + \ldots$ and $q(t) = t^9 + \ldots$. The condition (b) of Lemma 3.1 means that there exists a polynomial $G(u, v)$ of the form

$$G(u,v) = u^3 - v^{20} + \sum_{\substack{20i+3j<60 \\ i,j\geq 0}} C_{ij} u^i v^j$$

such that

$$\deg_t G\big(p(t), q(t)\big) = 112. \tag{18}$$

The condition (c) of Lemma 3.1 means that there exist constants c_1 and c_2 such that

$$\mathrm{ord}_{t=0} \left(p(t) - c_1 q(t) - c_1 q(t)^2 \right) = 5. \tag{19}$$

The condition (d) of Lemma 3.1 means that for $k = 1, \ldots, 7$,

$$p'(t_k) = q'(t_k) = 0, \tag{20}$$

$$p''(t_k) q'''(t_k) \neq q''(t_k) p'''(t_k). \tag{21}$$

Since $\deg q' = 8$ and $q'(0) = 0$, the condition (20) is equivalent to the fact that $\{0, t_1, \dots, t_7\}$ are the roots of q', and there exists a polynomial $r(t)$ of degree 51 such that

$$p'(t) = r(t) \cdot q'(t). \tag{22}$$

The condition (18), (19), (21) and (22) provide a system of simultaneous equations and inequalities for the unknowns c_1, c_2 and for the coefficients of q, r and G (the coefficients of p can be found from (22) and $p(0) = 0$).

This system has a solution:

$$c_1 = -\frac{32368762111892400}{90684846733}, \qquad c_2 = \frac{30833889663060}{410338673},$$

$$q(t) = t^9 + 3\,t^6 + \tfrac{54}{17}\,t^3 + 3\beta t^2, \qquad \text{where} \quad \beta = -\tfrac{36}{17}\,\sqrt[3]{\tfrac{147}{34}}$$

$$\begin{aligned}
r(t) = \tfrac{20}{3}\big(&t^{51} + 17\,t^{48} + 137\,t^{45} + 17\beta\,t^{44} + 681\,t^{42} + 238\beta\,t^{41} + \tfrac{501054}{221}\,t^{39} + \\
&+ 1561\beta\,t^{38} + 119\beta^2\,t^{37} + \tfrac{86262}{17}\,t^{36} + 6160\beta\,t^{35} + 1309\beta^2\,t^{34} + \tfrac{1932222}{289}\,t^{33} + \\
&+ \tfrac{263516}{17}\beta\,t^{32} + 6622\beta^2\,t^{31} - \tfrac{90093894}{4913}\,t^{30} + \tfrac{397232}{17}\beta\,t^{29} + 19250\beta^2\,t^{28} - \\
&- \tfrac{846133035}{4913}\,t^{27} + \tfrac{3005266}{289}\beta\,t^{26} + \tfrac{543884}{17}\beta^2\,t^{25} - \tfrac{49957930891}{83521}\,t^{24} - \\
&- \tfrac{420538624}{4913}\beta\,t^{23} + \tfrac{353276}{17}\beta^2\,t^{22} - \tfrac{91298290787}{83521}\,t^{21} - \tfrac{101367959}{289}\beta\,t^{20} - \\
&- \tfrac{9780232}{289}\beta^2\,t^{19} - \tfrac{1162052993707}{1419857}\,t^{18} - \tfrac{56603024170}{83521}\beta\,t^{17} - \tfrac{807377032}{4913}\beta^2\,t^{16} \\
&+ \tfrac{766218083922}{1419857}\,t^{15} - \tfrac{45450918051}{83521}\beta\,t^{14} - \tfrac{1442603701}{4913}\beta^2\,t^{13} + \tfrac{46921116263706}{24137569}\,t^{12} + \\
&+ \tfrac{230339555352}{1419857}\beta\,t^{11} - \tfrac{19395246255}{83521}\beta^2\,t^{10} + \tfrac{1413944975064438}{410338673}\,t^{9} + \\
&+ \tfrac{621393338592}{1419857}\beta\,t^{8} + \tfrac{6245611218}{83521}\beta^2\,t^{7} + \tfrac{554559678398538}{410338673}\,t^{6} + \\
&+ \tfrac{8204586193656}{24137569}\beta\,t^{5} + \tfrac{4273175790}{83521}\beta^2\,t^{4} - \tfrac{3524860866785148}{6975757441}\,t^{3} + \\
&+ \tfrac{27750500696754}{410338673}\beta\,t^{2} - \tfrac{4855314316783860}{90684846733}\big)\,;
\end{aligned}$$

$$\begin{aligned}
G(u,v) = u^3 &+ \tfrac{125694347226670235997}{52415841411674}\,u^2 + \\
&+ u \cdot \big(\tfrac{36795}{442}\,v^{12} + \tfrac{143479380}{63869}\,v^{11} - \tfrac{67444266186}{1085773}\,v^{10} - \tfrac{137300874259560}{313788397}\,v^{9} + \\
&\quad + \tfrac{25807260467075948851}{181369693466}\,v^{8} + \tfrac{1284725195106192180}{1541642394461}\,v^{7} - \tfrac{1112181512392833344850}{34271896307633}\,v^{6}\big) - \\
&- v^{20} + \tfrac{440}{17}\,v^{19} + \tfrac{137380}{289}\,v^{18} - \tfrac{1557031600}{83521}\,v^{17} + \tfrac{78582704594145}{24955406632}\,v^{16} + \\
&+ \tfrac{5956885860588398964}{901514064581}\,v^{15} + \tfrac{15055946520458773867110}{260537564663909}\,v^{14} - \\
&- \tfrac{3286758414687687627327372}{75295356187869701}\,v^{13}\,.
\end{aligned}$$

One can check that the polynomial $t^{-1}q'(t)$ is irreducible over $\mathbf{Q}(\beta)$, hence, its roots $t_1, \dots, t_7$ are distinct. Lemma 3.1 is proved $\square$

Remark 3.2. After the change of coordinates $t = \beta s$, the coefficients of polynomials $p(t)$ and $q(t)$ becomes rational.

Remark 3.3. The number of the unknowns in the system (18), (19), (22) exceeds the number of the equations. Therefore, to simplify the computations, we set the coefficients of t^8, t^7, t^5 and t^4 in $q(t)$ to be zero from the very beginning.

3.3. Construction of the covering of the edges of the splice diagram. For each linear chain in the graphs in Figures 11–13, we shall construct a covering over a neighbourhood of the union of the corresponding curves.

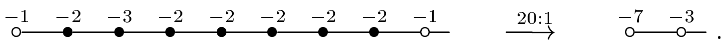
3.3.1. Covering over the edge $\overset{-7}{\circ}\!\!-\!\!-\!\!\overset{-3}{\circ}\!-$ *(Fig. 11).*
According to Proposition 1.5, a neighbourhood of the corresponding curve can be embedded into a toric surface associated to the fan

$$\Sigma = \begin{bmatrix} 1 & 1 & 6 & 17 \\ 0 & 1 & 7 & 20 \end{bmatrix}, \qquad \text{and let} \quad A = \begin{bmatrix} 1 & 17 \\ 0 & 20 \end{bmatrix}, \quad \widetilde{\Sigma} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Here and further, the fans are represented by matrices whose columns correspond to the vectors. Applying to A, $\widetilde{\Sigma}$ and Σ the procedure described in Proposition 1.9, we obtain primitive fans

$$\Sigma' = \Sigma \quad \text{and} \quad \widetilde{\Sigma}' = \begin{bmatrix} \mathbf{1} & \mathbf{3} & 2 & 1 & 1 & 1 & 1 & 1 & 1 & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & 1 & 1 & 2 & 3 & 4 & 5 & 6 & \mathbf{7} & \mathbf{1} \end{bmatrix}.$$

Here and further, we use the bold font for the vectors $\tilde{e}'_j$ of the fan $\widetilde{\Sigma}'$ such that $A(\mathbf{R}\tilde{e}'_j) = \mathbf{R}e'_i$ for some $e'_i \in \Sigma'$. The mapping of graphs has the form:



Here and later in Sections 3.3.1 – 3.3.8 the black vertices correspond to those irreducible components which are contracted to a point by the considered mapping. By 1.6, we have $N = \det A = 20$, $m_0 = m_1 = 20$, $n_0 = n_1 = 1$.

3.3.2. Covering over the edge $\underbrace{-\overset{-2}{\circ}-\cdots-\overset{-2}{\circ}}_{21}\!-\!\overset{-3}{\circ}\!-\!\overset{-2}{\circ}-$ *(Fig. 11).*

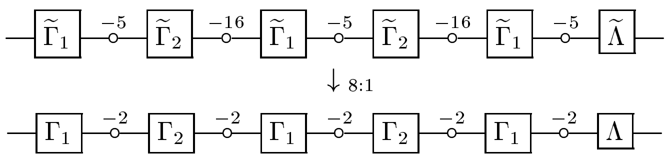
$$\Sigma = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 & 2 & 3 \\ 0 & 1 & \dots & 21 & 22 & 45 & 68 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix}, \quad \widetilde{\Sigma} = \begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 5 & 8 & 11 & 14 & 17 \end{bmatrix}$$

$$\Sigma' = \begin{bmatrix} 1 & 1 & \dots & 1 & 3 & 2 & 1 & 3 & 5 & 2 & 3 \\ 0 & 1 & \dots & 21 & 64 & 43 & 22 & 67 & 112 & 45 & 68 \end{bmatrix}$$

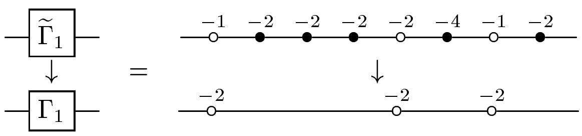
$$\widetilde{\Sigma}' = \left[\begin{matrix} \mathbf{1} & \mathbf{8} & 7 & 6 & 5 & \mathbf{4} & 3 & \mathbf{8} & 5 & \mathbf{2} & 5 & \mathbf{8} & 3 & \mathbf{4} & 5 & 6 & 7 & \mathbf{8} & \mathbf{1} & \mathbf{8} & 7 & 6 & 5 & \mathbf{4} & 3 & \mathbf{8} & 5 & \mathbf{2} & 5 & \mathbf{8} & 3 & \mathbf{4} & 5 & 6 & 7 & \mathbf{8} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & 1 & 1 & 1 & \mathbf{1} & 1 & \mathbf{3} & 2 & \mathbf{1} & 3 & \mathbf{5} & 2 & \mathbf{3} & 4 & 5 & 6 & \mathbf{7} & \mathbf{1} & \mathbf{9} & 8 & 7 & 6 & \mathbf{5} & 4 & \mathbf{11} & 7 & \mathbf{3} & 8 & \mathbf{13} & 5 & \mathbf{7} & 9 & 11 & 13 & \mathbf{15} & \mathbf{2} \end{matrix}\right.$$

$$\left.\begin{matrix} \mathbf{8} & 7 & 6 & 5 & \mathbf{4} & 3 & \mathbf{8} & 5 & \mathbf{2} & 5 & \mathbf{8} & \mathbf{3} & \mathbf{16} & 13 & 10 & 7 & \mathbf{4} & 9 & 14 & 19 & \mathbf{24} & \mathbf{5} & \mathbf{16} & 11 & \mathbf{6} \\ \mathbf{17} & 15 & 13 & 11 & \mathbf{9} & 7 & \mathbf{19} & 12 & \mathbf{5} & 13 & \mathbf{21} & \mathbf{8} & \mathbf{43} & 35 & 27 & 19 & \mathbf{11} & 25 & 39 & 53 & \mathbf{67} & \mathbf{14} & \mathbf{45} & 31 & \mathbf{17} \end{matrix}\right]$$

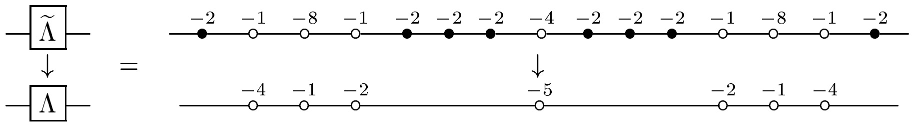
The mapping of graphs has the form:



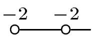
where



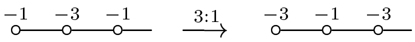
Γ_2, $\widetilde{\Gamma}_2$ are the mirror images of Γ_1, $\widetilde{\Gamma}_1$, and

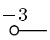


By Proposition 1.6, we have $N = \det A = 8$, $m_0 = 8$, $m_1 = 4$, $n_0 = 1$, $n_1 = 2$.

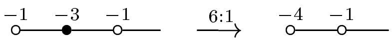
3.3.3. Covering over the vertical edge  *(Fig. 11).*

$$\Sigma = \begin{bmatrix} 1&1&1&1 \\ 0&1&2&3 \end{bmatrix}, \quad A = \begin{bmatrix} 1&1 \\ 0&3 \end{bmatrix}, \quad \widetilde{\Sigma} = \begin{bmatrix} 1&0 \\ 0&1 \end{bmatrix}, \quad \Sigma' = \begin{bmatrix} 1&1&2&1&1 \\ 0&1&3&2&3 \end{bmatrix}, \quad \widetilde{\Sigma}' = \begin{bmatrix} \mathbf{1}&\mathbf{2}&\mathbf{1}&\mathbf{1}&\mathbf{0} \\ \mathbf{0}&\mathbf{1}&\mathbf{1}&\mathbf{2}&\mathbf{1} \end{bmatrix}$$

The mapping of graphs has the form:  .
By Proposition 1.6, we have $N = \det A = 3$, $m_0 = m_1 = 3$, $n_0 = n_1 = 1$.

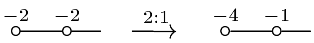
3.3.4. Two coverings over the vertical edge  *(Fig. 11).*
The first covering:

$$\Sigma = \begin{bmatrix} 1&1&2 \\ 0&1&3 \end{bmatrix}, \quad A = \begin{bmatrix} 1&4 \\ 0&6 \end{bmatrix}, \quad \widetilde{\Sigma} = \begin{bmatrix} 1&0 \\ 0&1 \end{bmatrix}, \quad \Sigma' = \begin{bmatrix} 1&1&3&2 \\ 0&1&4&3 \end{bmatrix}, \quad \widetilde{\Sigma}' = \begin{bmatrix} \mathbf{1}&\mathbf{2}&1&\mathbf{1}&\mathbf{0} \\ \mathbf{0}&\mathbf{1}&1&\mathbf{2}&\mathbf{1} \end{bmatrix}$$

The mapping of graphs has the form:  .
By Proposition 1.6, we have $N = \det A = 6$, $m_0 = 6$, $m_1 = 3$, $n_0 = 1$, $n_1 = 2$.

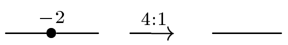
The second covering:

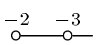
$$\Sigma = \begin{bmatrix} 1&1&2 \\ 0&1&3 \end{bmatrix}, \quad A = \begin{bmatrix} 1&1 \\ 0&2 \end{bmatrix}, \quad \widetilde{\Sigma} = \widetilde{\Sigma}' = \begin{bmatrix} \mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1} \\ \mathbf{0}&\mathbf{1}&\mathbf{2}&\mathbf{3} \end{bmatrix}, \quad \Sigma' = \begin{bmatrix} 1&1&3&2 \\ 0&1&4&3 \end{bmatrix}$$

The mapping of graphs has the form:  .
By Proposition 1.6, we have $N = \det A = 2$, $m_0 = 2$, $m_1 = 1$, $n_0 = 1$, $n_1 = 2$.

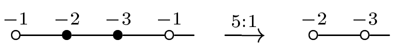
3.3.5. Covering over the edge "$\longrightarrow$" (Fig. 11).

$$\Sigma = \Sigma' = \begin{bmatrix} 1&0 \\ 0&1 \end{bmatrix}, \quad A = \begin{bmatrix} 2&-1 \\ 0&2 \end{bmatrix}, \quad \widetilde{\Sigma} = \widetilde{\Sigma}' = \begin{bmatrix} \mathbf{1}&1&\mathbf{1} \\ \mathbf{0}&1&\mathbf{2} \end{bmatrix}$$

The mapping of graphs has the form:  .
By Proposition 1.6, we have $N = \det A = 4$, $m_0 = 2$, $m_1 = 1$, $n_0 = 2$, $n_1 = 4$.

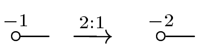
3.3.6. Covering over the edge  *(Fig. 12).*

$$\Sigma = \Sigma' = \begin{bmatrix} 1&1&1&2 \\ 0&1&2&5 \end{bmatrix}, \quad A = \begin{bmatrix} 1&2 \\ 0&5 \end{bmatrix}, \quad \widetilde{\Sigma} = \begin{bmatrix} 1&0 \\ 0&1 \end{bmatrix}, \quad \widetilde{\Sigma}' = \begin{bmatrix} \mathbf{1}&\mathbf{3}&2&1&\mathbf{1}&\mathbf{0} \\ \mathbf{0}&\mathbf{1}&1&1&\mathbf{2}&\mathbf{1} \end{bmatrix}$$

The mapping of graphs has the form: 
By Proposition 1.6, we have $N = \det A = 5$, $m_0 = m_1 = 5$, $n_0 = n_1 = 1$.

3.3.7. Covering over vertical the edge  *(Fig. 12 and Fig. 13).*

$$\Sigma = \Sigma' = \begin{bmatrix} 1&1&1 \\ 0&1&2 \end{bmatrix}, \quad A = \begin{bmatrix} 1&1 \\ 0&2 \end{bmatrix}, \quad \widetilde{\Sigma} = \begin{bmatrix} 1&0 \\ 0&1 \end{bmatrix}, \quad \widetilde{\Sigma}' = \begin{bmatrix} \mathbf{1}&\mathbf{1}&\mathbf{0} \\ \mathbf{0}&\mathbf{1}&\mathbf{1} \end{bmatrix}$$

The mapping of graphs has the form:  .
By Proposition 1.6, we have $N = \det A = 2$, $m_0 = m_1 = 2$, $n_0 = n_1 = 1$.

3.3.8. Covering over the edge $\overset{-3}{\circ\!\!-}$ *(Fig. 13).*

$$\Sigma = \Sigma' = \begin{bmatrix} 1&1&2 \\ 0&1&3 \end{bmatrix}, \quad A = \begin{bmatrix} 1&2 \\ 0&3 \end{bmatrix}, \quad \widetilde{\Sigma} = \begin{bmatrix} 1&0 \\ 0&1 \end{bmatrix}, \quad \widetilde{\Sigma}' = \begin{bmatrix} \mathbf{1}&\mathbf{1}&\mathbf{0} \\ \mathbf{0}&\mathbf{1}&\mathbf{1} \end{bmatrix}$$

The mapping of graphs has the form: $\overset{-1}{\circ\!\!-} \;\overset{3:1}{\longrightarrow}\; \overset{-3}{\circ\!\!-}$.
By Proposition 1.6, we have $N = \det A = 3$, $m_0 = m_1 = 3$, $n_0 = n_1 = 1$.

3.4. Covering over trivalent vertices of the splice diagram.

Lemma 3.4. *Let* $(m_1^{(1)}, \dots, m_{k_1}^{(1)};\ m_1^{(2)}, \dots, m_{k_2}^{(2)};\ m_1^{(3)}, \dots, m_{k_3}^{(3)})$ *be one of the following collections of integers (here* n^m *denotes* m *times* $n, \dots, n$*):*

$$(20, 1;\ 3^7;\ 8, 1^{13}), \quad (4;\ 3, 1;\ 2, 1^2), \quad (5, 1;\ 2^3;\ 4, 1^2), \quad (3^2;\ 2^3;\ 4, 1^2).$$

Then there exists a branch covering $\varphi : S^2 \to S^2$ *which has three critical values* p_1*,* p_2*,* p_3*, i.e.* φ *is unbranched over* $\varphi^{-1}(\{p_1, p_2, p_3\})$*, such that* $m_1^{(i)}, \dots, m_{k_i}^{(i)}$ *are the multiplicities of* φ *at the preimages of* p_i*.*

Proof. Let us connect the points p_2 and p_3 by an embedded segment I. Let Γ be the graph embedded into S^2 in one of the ways depicted in Fig. 14. Let us define a mapping $\varphi : \Gamma \to I$ which takes the black vertices into p_2, the white ones into p_3, and maps the edges homeomorphically onto I. Let us extend φ to the whole sphere so that each component of the complement of Γ covers$S^2 \setminus I$ with a single branch point which is over p_1. $\square$

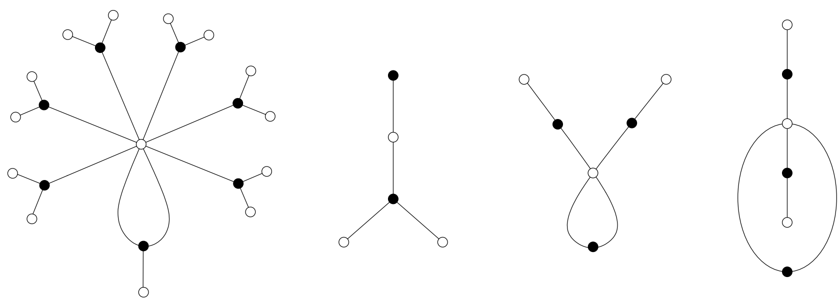


FIG. 14

For all the trivalent vertices of the graph in Fig. 7(lower part) except v_2, we extend the covering of the sphere constructed in Lemma 3.4 up to a covering which is branched only along the curves corresponding to the neighbouring vertices and unbranched along the curve corresponding to the trivalent vertex itself.

For the vertex v_2, we should extend the covering more carefully. It must have double branching along the curve corresponding to v_2 (denote it by A) because $n_1 = 2$ in the coverings 3.3.2, 3.3.4 and $n_0 = 2$ in the coverings 3.3.5. Let B_1, B_2, B_3 be the curves corresponding to the neighbouring vertices and let V be a sufficiently small tubular neighbourhood of A. Let α, β_1, β_2, β_3 be the elements of the fundamental group $\Pi = \pi_1\big(V \setminus (A \cup B_1 \cup B_2 \cup B_3)\big)$ defined by the meridians of A, B_1, B_2 and B_3 respectively (we call a meridian the positive loop along the boundary of a small transversal disk). Since $A^2 = -2$, under a certain choice of the paths connecting the meridians to a common base point, we have the following relations:

$$\alpha^2 = \beta_1\beta_2\beta_3, \qquad \alpha\beta_j = \beta_j\alpha, \quad j = 1, 2, 3 \tag{23}$$

The covering is defined by a homomorphism to the symmetric group $\Pi \to S(8)$, i.e. by an action of Π on the preimages of the base point.

Analysing the coverings from 3.3.2, 3.3.4 and 3.3.5, we see that the monodromy of the covering over some neighbourhoods of the points $A \cap B_i$ should be as in Fig. 15. It remains to establish a correspondence between the points so that the relations (23) are satisfied. This can be achieved by the numbering of the vertices depicted in Fig. 15.

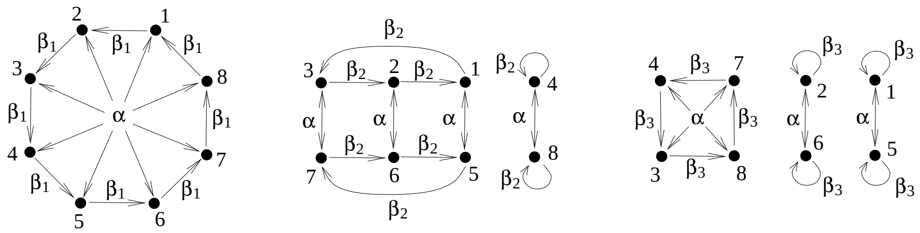
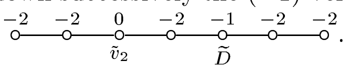


FIG. 15

3.5. Proof of Proposition 0.2. The same way as in [13] (or somehow else), one can check that the polynomial immersion $\mathbf{C} \to \mathbf{C}^2$ constructed in Section 3.2 is extendable up to an immersion of some neighbourhood of $\tilde{D}$ (see Fig. 7). To see this, one should compute the degree of the normal bundle of the curve D on the surface corresponding to the graph in the lower

Blowing down successively the (-1)-vertices in Fig. 7(upper part), we obtain the linear chain .

Blowing up four times the point corresponding to the edge to the right of $\tilde{v}_2$, we obtain the graph depicted in Fig. 16. This graph can be blown down to a single $(+1)$-vertex $\tilde{l}$. Thus, we obtain a meromorphic immersion of a $(+1)$-pair.

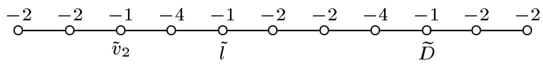


FIG. 16

It remains to note that by Proposition 2.4, the index of the constructed immersion is equal to one. Indeed, introducing the notation as in Sect. 2.2, we have

$$\mu_\infty f^* = 21, \quad d = 1, \quad n_1 = 4, \quad \text{ and } \sum_{x \in D_1^* \setminus \{\infty\}} (\mu_x f^* - n_1) = 8 \times (6 - 4).$$

§4. NON-EXISTENCE OF A COUNTER-EXAMPLE TO THE JACOBIAN CONJECTURE WITH THE GIVEN BEHAVIOUR AT THE INFINITY

In this section we prove

Proposition 4.1. *There does not exist a polynomial mapping* $\mathbf{C}^2 \to \mathbf{C}^2$ *realizing the meromorphic immersion constructed in §3. In particular, the* $(+1)$*-pair constructed in §3 is not flat.*

4.1. Reduction to a system of simultaneous equations and inequalities. Suppose, there exist polynomials $P(x,y)$ and $Q(x,y)$ such that $P'_xQ'_y - P'_yQ'_x = 1$ and the mapping $(x,y) \mapsto (u,v) = \big(P(x,y), Q(x,y)\big)$ at the infinity is as it is described in §3. By linear changes of coordinates one can achieve that the lines $u = \text{const}$ and $v = \text{const}$ meets transversally the curves corresponding to the vertices u_0 and u_1 in Fig. 7(lower part).

According to 3.3.1, 3.3.3, the restriction of f onto each of the curves corresponding to the preimages of u_0 and u_1 is one-to-one. Therefore, each preimage of v_0 (respectively, of v_1) meets the curve $P(x,y) = \text{const}$ (respectively, $Q(x,y) = \text{const}$) once and transversally. All the other intersections of $P = \text{const}$ and $Q = \text{const}$ with the infinite curve are concentrated in the dicritical component $\tilde{D}$. Moreover, since the polynomials parametrizing the branch curve D are of degrees 9 and 60, the curves $P = \text{const}$ and $Q = \text{const}$ have 9 and 60 intersections with $\tilde{D}$.

Blowing down successively extra (-1)-curves in Fig. 7(upper part), we obtain a common resolution graph for the curves $P = \text{const}$ and $Q = \text{const}$ at the infinity which is depicted in Fig. 17. Hence, the splice diagrams of these curves at the infinity are as in Figures 18 and 19. This implies, in particular (see [11]), that

$$\deg P(x,y) = 600, \qquad \deg Q(x,y) = 90. \tag{24}$$

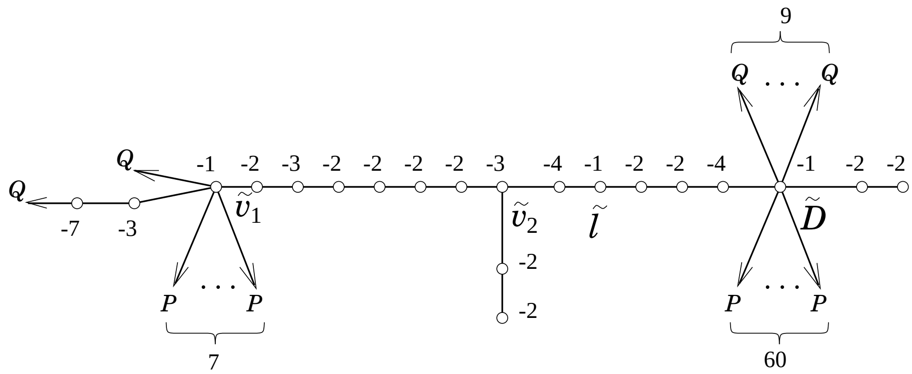


FIG. 17

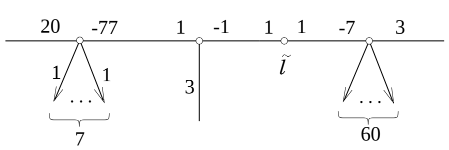


FIG. 18

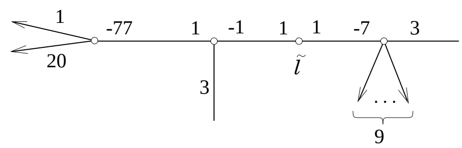


FIG. 19

Let

$$X_{19} \xrightarrow{\sigma_{19}} X_{18} \xrightarrow{\sigma_{18}} \dots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_0 = \mathbf{CP}^2$$

Be the sequence of the σ-processes which blows down successively all the vertices in Fig. 17 except $\tilde{l}$. Let us denote the infinite line in $\mathbf{CP}^2$ by E_0, and let E_j be the exceptional curve of the σ-process (blow-up) σ_j, $j = 1, \dots, 19$. We shall use the same notation for a curve and all its proper transforms on the other surfaces. The mutual position of the curves E_j is depicted in Fig. 20. The numbers in the parentheses near the vertices have the following meaning. $P \circ \sigma_{19}$ and $Q \circ \sigma_{19}$ are rational functions on X_{19}. Let (P) and (Q) be their divisors. They are of the form

$$(P) = (P)_0^{\text{aff}} - \sum_{j=0}^{19} p_j E_j, \qquad (Q) = (Q)_0^{\text{aff}} - \sum_{j=0}^{19} q_j E_j,$$

where $(P)_0^{\text{aff}}$ and $(Q)_0^{\text{aff}}$ are the closures of the affine curves $\{P = 0\}$ and $\{Q = 0\}$. The numbers in the parentheses near a vertex E_j in Fig. 20 are (p_j, q_j).

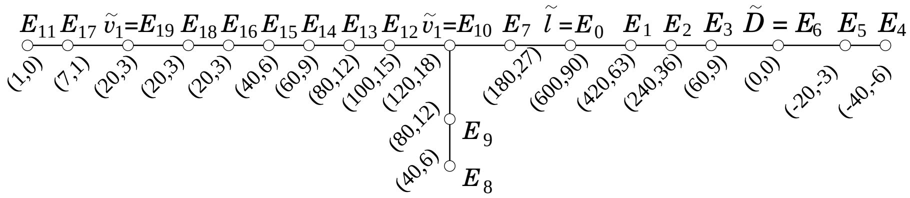


FIG. 20

Let us choose coordinates x, y in $\mathbf{C}^2$ so that the centre of the blow-ups σ_1 and σ_2 are at the infinite point of the axis $y = 0$ and the centres of the blow-ups σ_7 and σ_8 are are at the infinite point of the $axis x = 0$. Let us choose coordinates charts on X_j called *standard* as follows. As the standard charts on $X_0 = \mathbf{CP}^2$, we choose (x, y), $(1/x, y/x)$ and $(x/y, 1/y)$. If the centre of the blowup σ_j is at the origin of one of the standard coordinate charts (x', y') on X_{j-1} then we replace this chart on X_j by the two charts $(x'/y', y')$ and $(x', y'/x')$. The only three blow-ups where the choice of the standard charts is ambiguous, are σ_3, σ_4 and σ_{11}.

Let (x_2, y_2) be the standard coordinates on X_2 in which $E_2 = \{x_2 = 0\}$ and $\{y = 0\} = \{y_2 = 0\}$, i.e. $x_2 = x^{-1}$, $y_2 = xy$. In these coordinates, the curve E_2 is the coordinate axis $x_2 = 0$. Since the centre of σ_3 lies on E_2, its coordinates are $(x_2, y_2) = (0, \alpha_2)$. As the standard coordinates at this point, we chose the coordinates $x_{2'} = x_2 = x^{-1}$ and $y_{2'} = y_2 - \alpha_2 = xy - \alpha_2$. Let (x_3, y_3) be the standard coordinates on X_3 such that $x_3 = x_{2'} = x^{-1}$ and $y_3 = y_{2'}/x_{2'} = (xy - \alpha_2)x$. In these coordinates, the curve E_3 is the coordinate axis $x_3 = 0$. Since the centre of σ_4 lies on E_3, its coordinates are $(x_3, y_3) = (0, \alpha_3)$. As the standard coordinates at this point, we chose the coordinates $x_{3'} = x_3 = x^{-1}$ and $y_{3'} = y_3 - \alpha_3 = x^2 y - \alpha_2 x - \alpha_3$.

Analogously, let (x_{10}, y_{10}) be the standard coordinates on X_{10} in which $E_{10} = \{x_{10} = 0\}$ and $E_7 = \{y_{10} = 0\}$, i.e. $x_{10} = x$, $y_{10} = x^{-3} y^{-1}$. Rescaling if necessary, the axis x, we may assume that the centre of σ_{11} is at the point $(x_{10}, y_{10}) = (0, 1) \in E_{10}$. As the standard coordinates at this point, we chose the coordinates $x_{10'} = x_{10}$ and $y_{10'} = y_{10} - 1$.

When depicting the Newton polygons, we shall use the following convention. If the depicted polygon Δ is not completely known then we show a polygon which contains Δ. In this case, we depict the vertices which are known to belong to Δ, as a small black circle "$\bullet$".

Lemma 4.1. *Let R stands for one of P or Q and let us set $a = 20$ when $R = P$ and $a = 3$ when $R = Q$.*

(a). The Newton polygon of $R(x, y)$ is the quadrangle depicted in Fig. 21 (left upper part).

(b). Passing to the coordinates (x_j, y_j) for $j = 2, 2', 3, 3'$ or 10, the polynomial R becomes a Laurent polynomial which we denote by $R_j(x_j, y_j)$. The Newton polygons of these Laurent polynomials are depicted in Fig. 21.

(c). Passing to the coordinates $(x_{10'}, y_{10'})$, the polynomial R becomes a rational function of the form $(1+y_{10'})^{-6a} R_{10'}(x_{10'}, y_{10'})$, where $R_{10'}$ is a Laurent polynomial whose Newton polygon is depicted in Fig. 22.

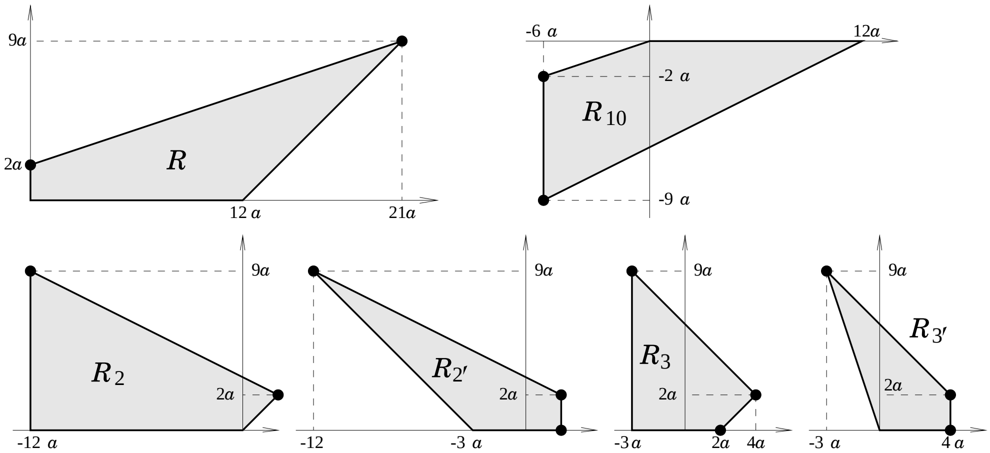


Fig. 21

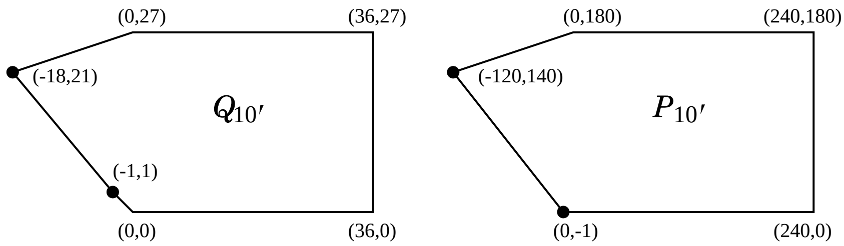


Fig. 22

Proof. It is sufficient to write explicitely all the blow-ups in the standard coordinates and to trace the multiplicities of P and Q on the exceptional curves. For example, $q_0 = 90$ and $q_1 = 27$, hence the divisor $(Q)_0^{\text{aff}}$ has the multiplicity 27 at the centre of σ_1. Therefore, the Newton polygon of Q is contained in the area $x \le 63$, $x + y \le 90$. The further arguments are similar. □

The condition that the coefficients of $Q_{3'}$ and $Q_{10'}$ are zero outside the polygons in Figures 21 and 22 yeilds a system of simultaneous equations for the coefficients of $Q(x, y)$. A straight-forward computation shows that this system has no solution providing non-zero coefficients at the vertices marked as black points •" in Figures 21 and 22. This proves Proposition 4.1. In the next subsection we show how to prove the absence of the solutions without tedious calculations. The idea of the proof is taken from Heitmann's paper [9].

4.2. Proof that there is no solution. We shall proceed analogously to [9; §3]. Let us change the notation denoting the coordinates $(x_{3'}, y_{3'})$ by (t, u), and the coordinates $(x_{10'}, y_{10'})$ by (x, z):

$$x = t^{-1}, \qquad y = \alpha_2 t + \alpha_3 t^2 + u t^2, \qquad y = x^{-3}(1+z)^{-1};$$
$$t = x^{-1}, \qquad u = x^2 y - \alpha_2 x - \alpha_3, \qquad z = x^{-3} y^{-1} - 1.$$

Let us set

$$\tau = xu - 1, \qquad \sigma = u^3 - 2xyu + y + 2\alpha_2 u + \alpha_3 xy - \alpha_2 \alpha_3, \qquad \rho = \sigma^2 \tau^3.$$

These functions are polynomials in (x, y) and Laurent polynomials in (t, u), their Newton polygons are depicted in Fig. 23. In the coordinates (x, z), the functions $(1+z)\tau(x, z)$, $(1+z)^3\sigma(x, z)$ and $(1+z)^9\rho(x, z)$ are Laurent polynomials. Their Newton polygons are also depicted in Fig. 23. In particular, we see that the Newton polygon of ρ^3 is contained in that of Q in all the three coordinate systems (x, y), (x, z), (t, u).

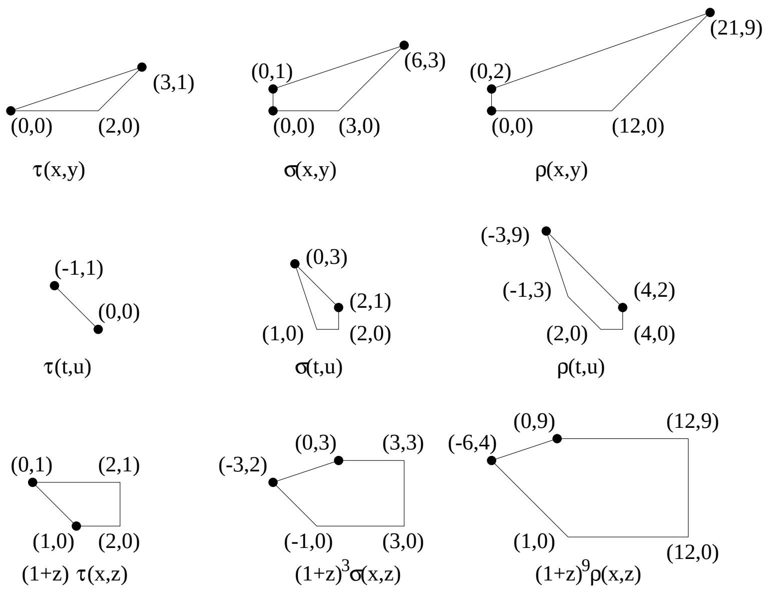


FIG. 23

For a Laurent polynomial $c(t) = \sum_{k=m}^{n} c_k t^k$ such that $c_m \neq 0$ and $c_n \neq 0$, let us denote $\operatorname{ord}_t c = m$ and $\deg_t c = n$.

Lemma 4.2. *Let a be a positive integer and $R(x,y)$ a polynomial whose Newton polygon is contained in the quadrangle $[(0,0),\ (12a,0),\ (21a,9a),\ (0,2a)]$. Let $R(t,u)$ and $R(x,z)$ be the result of the substitution into $R(x,y)$ of the expressions of (x,y) via (x,z) and (t,u). Suppose that the Newton polygon of the Laurent polynomial $R(t,u)$ is contained in the quadrangle $[(0,0),\ (4a,0),\ (4a,2a),\ (-3a,9a)]$ (compare with Fig. 21). Let*

$$(1+z)^{9a} R(x,z) = \sum_{k=-6a}^{12a} b_k(z)\, x^k.$$

Suppose that $b_k = 0$ for $k < m$. Then

$$\deg_z b_m - \operatorname{ord}_z b_m \geq 3/2\,(m+6a),$$

and in the case of the equality sign we have $b_m(z) = z^{-7/6\,m}(1+z)^{3/2\,(m+6a)}$.

Proof. Since $b_k = 0$ for $k < m$, the Newton polygon of $(1+z)^{9a}R(x,z)$ is to the right of the vertical line $x = m$. In the coordinates (x, y_{10}), this condition means that the Newton polygon of $y_{10}^{9a} R(x, y_{10})$ lies in the area shadowed in the left hand side of Fig. 24. Passing from the coordinates (x, y_{10}) to the coordinates (t,u) (see Fig. 24) and back (see Fig. 25), one can trace that the Newton polygon of R must always remain in the shadowed area. Therefore, all non-zero monomials of $y_{10}^{9a} R(x, y_{10})$ lying on the vertical line $x = m$ must be above the segment $[(-6a,0),(0,9a)]$ (it is shown by the dashed line in Figures 24 and 25), i.e. $\operatorname{ord}_{y_{10}} c_m \geq 3/2\,(m+6a)$, where $c_m(y_{10})$ is the coefficient of x^m in the Laurent polynomial $y_{10}^{9a} R(x, y_{10})$. It remains to note that $b_m(z) = c_m(1+z)$, and hence, $\deg_z b_m - \operatorname{ord}_z b_m \geq \operatorname{ord}_{y_{10}} c_m$. □

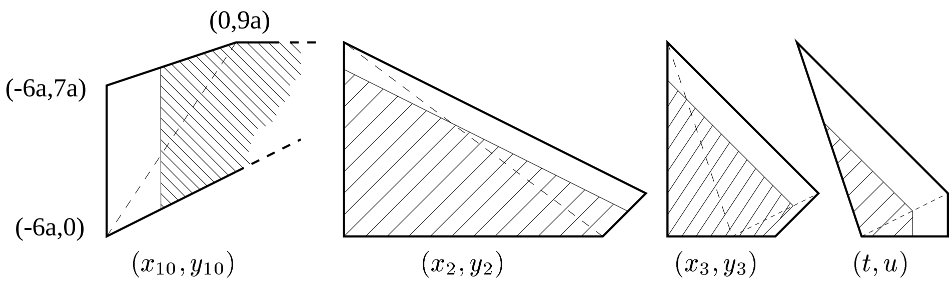


FIG. 24

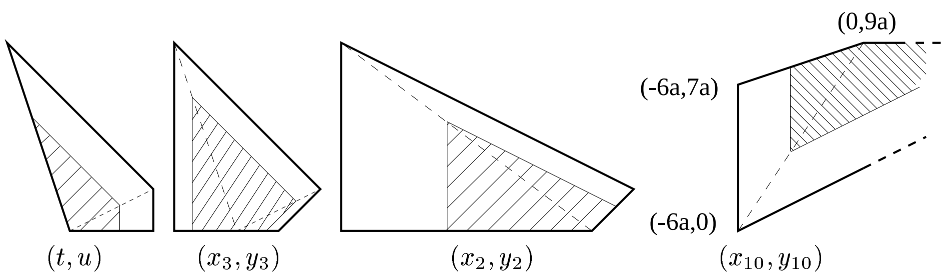


FIG. 25

Without loss of generality, we may assume that the coefficient of $x^{63}y^{27}$ in the polynomial $Q(x,y)$ is equal to one. Applying Lemma 4.2 successively to the polynomial $Q-\rho^3$, we obtain that in the coordinates (x,z), its coefficients of $x^{-18},\dots,x^{-13}$ are zero and the coefficient of x^{-12} is $c_2\,z^{14}\,(z+1)^9$ for some constant c_2.

Still applying Lemma 4.2, this time to the polynomial $Q-\rho^3-c_2\rho^2$, we see that its coefficient of $x^{-12},\dots,x^{-7}$ are zero and the coefficient of x^{-6} is equal to $c_1\,z^7\,(z+1)^{18}$ for some constant c_1.

Finally, applying successively Lemma 4.2 to the polynomial $Q-\rho^3-c_2\rho^2-c_1\rho$, we obtain that it is identically zero, i.e.

$$Q=\rho^3+c_2\rho^2+c_1\rho.$$

It remains to note that the coefficient of $x^{-1}z$ in this polynomial is zero while by Lemma 4.1 it should not be so (see Fig. 22, left hand side). The obtained contradiction proves Proposition 4.1.

4.3. Second proof of Proposition 4.1 (simple but not rigorous). When constructing the branch curve D in Sect. 3.2, we solved an underdeterminate system of simultaneous equations (the number of unknowns was greater than the number of equations: see Remark 3.3). Therefore it is naturally to assume that D admits deformations in the class of curves with the given types of singularities, hence, it can be further degenerated.

Suppose that there exists a degeneration such that a singularity of the type A_2 meets a simple double point and transforms into a singularity of the type A_4 as in the family of curves

$$C_t=\{y^2=x^3(x-t)^2\}\qquad\text{with } t\to 0. \tag{26}$$

Then a counter-example to the Jacobian Conjecture is impossible by the following simple reason of topological nature. Indeed, in this case, there exists a disc Δ continuously embedded into $\mathbf{C}^2$ such that

(i) Δ meets D along its boundary: $\Delta\cap D=\partial\Delta$;

(ii) the path $\partial\Delta$ passes through the simple double point of D and at this point, it passes from one local branch to the other.

For instance, in the situation (26) for $t\in\mathbf{R}$, $t>0$, such a disc can be chosen as $\{(x,y)\in\mathbf{R}^2\,|\,0\le x\le t,\ y^2\le x^3(x-t)^2\}$. Since this disc can be lifted to the covering, the preimages of the both branches meet on the covering which is impossible.

It must be very difficult (if possible at all) to prove the existence of degenerations of the form (26). Nevertheless, it seems that a curve satisfying the conditions of Lemma 3.1 might be obtained by triple application of degenerations of the form (26) to a curve D' which satisfy the conditions (a) and (b) of Lemma 3.1 and has four singular points of the type A_2 and four singular points of the type A_4. Such curves exist, among them, there is a curve which has the symmetry of the fourth order.

§5. Pseudo-convex immersions

Now we shall prove Proposition 0.4.

5.1. Immersions $S^3 \to \mathbf{C}^2$ up to regular homotopy. Let us denote the space of immersions of a manifold X into a manifold Y by $\mathrm{Imm}(X, Y)$. By Smale's theorem [16], the connected components of $\mathrm{Imm}(S^k, \mathbf{R}^n)$ are in a one-to-one correspondence with the elements of the homotopy group $\pi_k(SO(n))$. Hence,

$$\pi_0\big(\mathrm{Imm}(S^3, \mathbf{R}^4)\big) \cong \pi_3(SO(4)) \cong \mathbf{Z}^2. \tag{27}$$

In this subsection, we give an explicite geometric description of this isomorphism when $\mathbf{R}^4$ is equipped by a complex structure (identified with $\mathbf{C}^2$).

For $f \in \mathrm{Imm}(S^3, \mathbf{C}^2)$, denote the degree of the Gauss mapping (see Definition 0.1) by $DG(f)$ and let $CS(f)$ be the homotopy class of the pull-back of the complex tangent field, i.e. the field of tangent 2-planes

$$q \mapsto f_*^{-1}(T \cap iT), \qquad \text{where } q \in S^3 \text{ and } T = f_*(T_q S^3).$$

Proposition 5.1. *Two immersions $f_1, f_2 \in \mathrm{Imm}(S^3, \mathbf{C}^2)$ are regularly homotopic if and only if $DG(f_1) = DG(f_2)$ and $CS(f_1) = CS(f_2)$.*

Proof. Since the sphere S^3 parallelizable, the Smale's isomorphism (27) and the invariants $DG(f)$ and $CS(f)$ admit the following interpretation. Let us identify $\mathbf{C}^2$ with the quaternion body $\mathbf{H}$ by the mapping $(z, w) \mapsto z + wj$. Then $S^3 = \{q \in \mathbf{H} \mid q\bar{q} = 1\}$. Let $\vec{i}$, $\vec{j}$, $\vec{k}$ be the tangent vector fields on S^3 linearly independent at every point and $\vec{n}$ the field of exterior unit normal vectors defined by

$$\vec{i}(q) = qi, \quad \vec{j}(q) = qj, \quad \vec{k}(q) = qk, \quad \vec{n}(q) = q, \qquad \text{where} \quad q \in S^3.$$

To an immersion $f : S^3 \to \mathbf{H}$, we associate a mapping $\alpha(f) : S^3 \to SO(4)$ in the following way. Let us extend f up to an orientation preserving immersion of some neighbourhood of the sphere S^3 in such a way that each vector $f_*(\vec{n}(q))$ is orthogonal to $f_*(T_q S^3)$. Then $\alpha(f) : q \mapsto Q \in SO(4)$, where $q \in S^3$ and Q is the matrix which takes the frame $(1, i, j, k)$ to the orthogonalization of the frame $f_*\big(\vec{n}(q), \vec{i}(q), \vec{j}(q), \vec{k}(q)\big)$. By Smale's theorem, the mapping α induces a bijection $\alpha_* : \pi_0\big(\mathrm{Imm}(S^3, \mathbf{R}^4)\big) \to \pi_3(SO(4))$.

We shall consider S^3 as a subgroup of the multiplicative group $\mathbf{H} \setminus 0$, and let $S^2 = S^3 \cap (i\mathbf{R} + j\mathbf{R} + k\mathbf{R})$. Let us introduce the following notation:

$$\begin{aligned}
&\iota_1, \iota_2 : S^3 \to S^3 \times S^3, && \iota_1(s) = (s, 1), \quad \iota_2(r) = (1, r),\\
&\tau : S^3 \times S^3 \to SO(4), && \tau(s, r) : q \mapsto s\, q\, \bar{r},\\
&\sigma : SO(4) \to S^3, && \sigma(Q) = Q(1),\\
&\rho : SO(4) \to S^2, && \rho(Q) = Q(i) \cdot \overline{Q(1)},
\end{aligned}$$

($Q \in SO(4)$ is considered as an orthogonal operator $\mathbf{H} \to \mathbf{H}$). Let us fix the natural identifications $\pi_3(S^3) = \pi_3(S^2) = \mathbf{Z}$. The mapping τ is a double covering because it is a group homomorphism and $\mathrm{Ker}\,\tau = \{\pm 1\}$. It induces an isomorphism

$$\tau_* : \mathbf{Z} \oplus \mathbf{Z} = \pi_3(S^3 \times S^3) \to \pi_3(SO(4)).$$

Since $\sigma\tau(s, r) = s\,\bar{r}$, we have

$$\sigma_*\tau_* : \mathbf{Z} \oplus \mathbf{Z} = \pi_3(S^3 \times S^3) \to \pi_3(S^3) = \mathbf{Z}, \qquad (m, n) \mapsto m - n. \tag{28}$$

It is easy to see that $\rho\tau\iota_1 : S^3 \to S^2$ takes s into $s\,i\,\bar{s}$. It is the Hopf fibration. Indeed, $s_1\,i\,\overline{s}_1 = s_2\,i\,\overline{s}_2$ if and only if $s_1 = s_2 \cdot (x+iy)$ for some $x, y \in \mathbf{R}$. Hence there is an isomorphism

$$(\rho\tau\iota_1)_* : \pi_3(S^3) \to \pi_3(S^2).$$

It is clear also that $\rho\tau\iota_2 : S^3 \to S^2$ is a constant map $r \mapsto i$, i.e.

$$\operatorname{im}\Big((\rho\tau\iota_2)_* : \pi_3(S^3) \to \pi_3(S^2)\Big) = 0.$$

Thus, we have

$$\rho_*\tau_* : \mathbf{Z} \oplus \mathbf{Z} = \pi_3(S^3 \times S^3) \to \pi_3(S^2) = \mathbf{Z}, \qquad (m,n) \mapsto m. \tag{29}$$

By definition, $DG(f) = \sigma_*([\alpha(f)]) \in \pi_3(S^3) = \mathbf{Z}$ and $CS(f)$ is defined be the homotopy class of the vector field $q \mapsto f_*^{-1}\big(i\,f_*(\vec{n}(q))\big)$. Since S^3 is parallelizable, non-zero vector fields can be identified with mappings $S^3 \to S^2 \subset T_1(S^3)$. Under this identification, $CS(f)$ corresponds to the homotopy class $\rho_*([\alpha(f)]) \in \pi_3(S^2)$. It follows from (28) and (29) that $[\alpha(f)]$ is determined by its images under the homomorphisms σ_* and ρ_*. $\square$

5.2. On extendibility of an immersion of a sphere to an immersion of a ball.

Let X and Y be connected n-manifolds such that X is compact and has a boundary (not necessary connected), for example, $X = B^n$, $Y = \mathbf{R}^n$. In this subsection, we give a sufficient condition for an immersion of the boundary $\partial X \to Y$ to be extendable to an immersion $X \to Y$. For $n = 2$ it was proved by Francis [8].

Definition. Let Z and Y be manifolds of dimensions $n-1$ and n respectively.

1. An immersion $f : Z \to Y$ is called *Morse*, if for any points $z, z' \in Z$ such that $f(z) = f(z')$ and $f_*(T_z Z) = f_*(T_{z'} Z)$ (we shall call such a pair of points a *self-tangency of* f), there exist neighbourhoods U and U' on which f is an embedding, a neighbourhood V of $y = f(z)$ and a smooth function $\varphi : V \to \mathbf{R}$, such that $\{\varphi = 0\} = V \cup f(U')$ and $\varphi \circ f|_U$ is a Morse function.

2. *A normal bundle* of an immersion $f : Z \to Y$ is called the line bundle $N_f \to Z$ whose fibre over z is $T_{f(z)}Y / f_*(T_z Z)$. *A coorientation* of an immersion f is called an orientation of its normal bundle.

3. Suppose that $f : Z \to Y$ is a cooriented Morse immersion and $z, z' \in Z$ its self-tangency points. The self-tangency at points $z, z' \in Z$ is called *coherent*, if their coorientations are induced by the same orientation of the space $T_{f(z)}Y / f_*(T_z Z) = T_{f(z')}Y / f_*(T_{z'} Z)$, and *opposite* otherwise.

4. The *index* of an opposite self-tangency is called the index of the singular point of the function $\varphi \circ f|_U$ (see above) under the condition that the gradient of φ defines a positive normal vector field $V \cap f(U')$ (in the case of a coherent self-tangency or in the case of a non cooriented immersion, the index is defined only up to the identification of k and $n-k-1$).

5. A regular homotopy $\{h_t\}_{t\in[0,1]}$ is called *Morse*, if for all $t \in [0,1]$, the immersion h_t is Morse. A triple z, z', t where $z, z' \in Z$ are self-tangency points of h_t is called a *passing through a self-tangency.*

6. A passing through a self-tangency z, z', t of a regular homotopy $\{h_t\}$ is called *transversal* if there are neighbourhoods of the points (z, t), (z', t) whose images under the mapping $(z, t) \mapsto (h_t(z), t)$ are transversal to each other in $Y \times [0, 1]$. A regular homotopy is called *transversal* if all its passings through self-tangencies are transversal.

7. A passing through an opposite self-tangency of a cooriented regular homotopy is called *positive* (*negative*) if the velocity of each branch with respect to the other is positive (negative) in the sense of the coorientation.

Fig. 23 illustrates the definitions of the self-tangency types. In Fig. 24, positive passings of self-tangencies of different indices are depicted (the negative ones can be obtained from them by the reversing of time, i.e. for $\varepsilon < 0$). The arrows in the both figures indicate the coorientations.

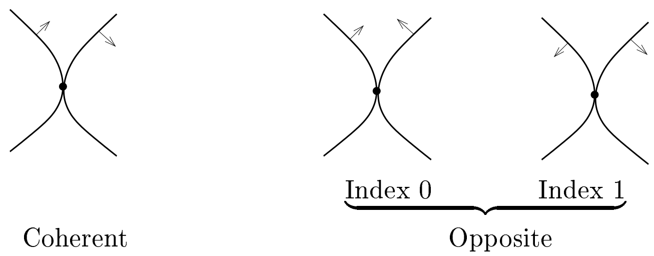


Fig. 23. Self-tangencies

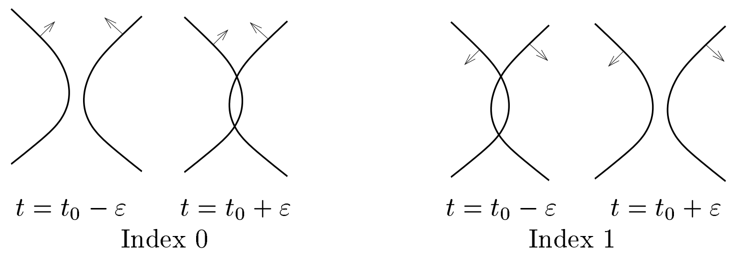


Fig. 24. Positive passings through self-tangencies ($\varepsilon > 0$)

Proposition 5.2. *Let X and Y be connected n-manifolds such that X is compact and has a boundary $Z = \partial X$. Let $H : X \to Y$ be an immersion and $h_t : Z \to Y$, $t \in [0, 1]$ a transversal Morse homotopy such that $h_0 = H|_Z$. Let us fix the coorientation of h_0 defined by the image under the mapping H_* of an exterior normal vector field to Z (it it extends by continuity for all h_t). If all the passings of opposite self-tangencies of index $n - 1$ are positive then there exists a homotopy $H_t : X \to Y$ such that $H_t|_Z = h_t$ and $H_0 = H$.*

For $n = 2$, Proposition 5.2 is proved in [8]. In the general case, the proof is more or less the analogous and we omit it.

Example. In Fig. 25, we depicted a regular homotopy of a circle $\{h_t\}$ such that h_0 is extendable to an immersion of a disc but h_1 is not. One sees that the extendibility fails at the moment of a negative passing through an opposite self-tangency of index 1. By Proposition 5.2, this is the only reason which can break the extendibility.

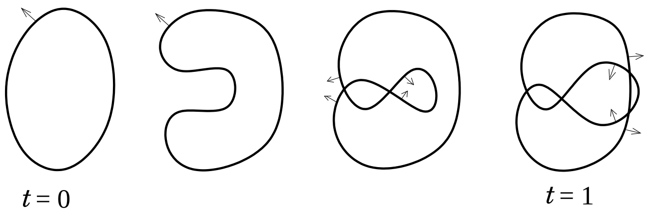


FIG. 25

Definition 5.3. Let X, Y be connected n-manifolds such that X is compact and $f : Z = \partial X \to Y$ is an immersion. Two extensions $F, G : X \to Y$ of f are called *equivalent* if there exists an isotopy $H_t : X \to Z$, $t \in [0,1]$ such that $H_0 = F$, $H_1 = G$, and $H_t|_Z = f$ for all t.

Corollary 5.4. *Let X, Y be connected n-manifolds such that X is compact and $Z = \partial X$. Suppose that $h_t : Z \to Y$ is a transversal Morse regular homotopy which has no opposite self-tangencies of index $n-1$. Then the immersions h_0 and h_1 have the same number of extensions to X up to the equivalence from Definition 5.3.* □

Now let us apply Proposition 5.2 to the problem of extension of a strictly pseudo-convex homotopy of the boundary (see Definition 0.3) to a regular homotopy of the whole manifold.

Corollary 5.5. *Let Y be a smooth complex (or almost complex) manifold of complex dimension $k \geq 2$, and let X a smooth compact oriented manifold of real dimension $n = 2k$ with a boundary $Z = \partial X$. Let $H : X \to Y$ be an immersion and $h_t : Z \to Y$, $t \in [0,1]$ a strictly pseudo-convex homotopy such that $h_0 = H|_Z$. Then there exists a regular homotopy $H_t : X \to Y$ such that $H_t|_Z = h_t$ and $H_0 = H$.*

Proof. By a small perturbation, the homotopy $\{h_t\}$ can be done Morse and transversal. If the perturbation is sufficiently small then the homotopy remains to be strictly pseudo-convex. Comparing the Levi forms of the touching branches, it is easy to see that opposite self-tangencies of index $n-1$ are impossible for Morse pseudo-convex immersions. Hence, Proposition 5.5 follows from Proposition 5.2. □

Corollary 5.6. *Let Y be a smooth complex and let $f : S^3 \to Y$ be a strictly pseudo-convex immersion. Then:*

(a). *The immersion f is extendable to an immersion of a ball if and only if strictly pseudo-convexly homotopic to the standard embedding.*
(b). *Up to equivalency (see Definition 5.3), there exists at most one extension of f to an immersion of the ball $B^4 \to Y$.*

Proof. Suppose that f is extendable to an immersion $F : B^4 \to Y$. By a theorem of Eliashberg [7], there exists a pluri-subharmonic (with respect to the complex structure pulled back from Y) function on B^4 with a single minimum. The level hypersurfaces of this function define a strictly pseudo-convex homotopy between f and an embedding. Since an embedding of a the sphere is uniquely extendable to an embedding of the ball, the required statement follows from Corollary 5.4. □

Remark. For $k = 1$, the conclusion of the Corollary 5.6(b) is wrong: in Fig. 26, we give an example of an immersion $S^1 \to \mathbf{R}^2$ which is non-uniquely extendable to an immersion of the disc. In the paper [15], this example is called Milnor example.

In the 50-th, this example was constructed independently by N.N. Konstantinov. Rotating the curve in Fig. 27 around the axis, we obtain examples of non-uniquely extendable immersions $S^{n-1} \to \mathbf{R}^n$ for all $n \geq 2$, i.e. Corollary 5.6(b) is wrong without the condition that f is pseudo-convex.

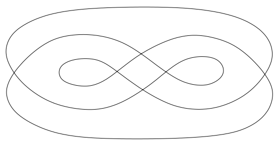


FIG. 26

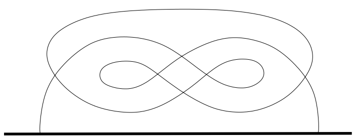


FIG. 27

5.3. Proof of Proposition 0.4.

Proposition 5.1 implies that the immersion f of the sphere $-M$ (see Definition 0.1) corresponding to the meromorphic immersion of the (+1)-pair constructed in §3 is regularly homotopic to the standard embedding. Indeed, $DG(f) = 1$ by the construction $CS(f) = 1$ because being holomorphic, the mapping f preserves the complex tangent field.

Let us prove that there is no strictly pseudo-convex homotopy between f and the standard embedding. Suppose, such a homotopy exists. Then, by Proposition 5.6(a), f would be extendable to a homotopy of a ball. Attaching the ball to M, we obtain a 4-manifold diffeomorphic to $\mathbf{CP}^2$, a 2-sphere L embedded into it, and an immersion $X \setminus L \to \mathbf{C}^2$. Moreover, X has a complex structure in a neighbourhood of L, such that L is a complex line and the functions defining the immersion into $\mathbf{C}^2$ are meromorphic. Let us pull back the complex structure from $\mathbf{C}^2$ to the whole X. Then the obtained complex surface is isomorphic to $\mathbf{CP}^2$, the described construction would give a counter-example to the Jacobian Conjecture which is impossible by Proposition 4.1.

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