ON THREE-SHEETED POLYNOMIAL MAPPINGS OF $\mathbf{C}^2$

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Abstract. It is proved that the Jacobian of a 3-fold polynomial mapping $\mathbf{C}^2 \to \mathbf{C}^2$ cannot be constant

§1. Statement of the result

Let $\widetilde{\mathbf{C}}^2$ and $\mathbf{C}^2$ be two copies of the complex plane, and let $f : \widetilde{\mathbf{C}}^2 \to \mathbf{C}^2$ be a locally invertible polynomial mapping, i.e. $f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$, where f_1 and f_2 are polynomials in x_1 and x_2 with complex coefficients such that

$$\frac{\partial(f_1, f_2)}{\partial(x_1, x_2)} = \text{cont} \neq 0. \tag{1}$$

The well-known Jacobian Conjecture states that such a mapping is polynomially invertible (surveys on this subject may be found in [1] and [2]). Polynomial invertibility of f satisfying (1) is obviously equivalent to the fact that a generic point of $\mathbf{C}^2$ has at most one preimage (see, for example [2], Theorem 2.1). We shall call the number of preimages of a generic point the *multiplicity* of the mapping.

The main result of this paper is

Theorem 1.1. *The Jacobian of a two-sheeted or three-sheeted polynomial mapping $\widetilde{\mathbf{C}}^2 \to \mathbf{C}^2$ vanishes.*

In other words, the multiplicity N of the above mapping cannot equal to two or three. The fact that $N \neq 2$ is well-known (see, for example, [2], Theorem 2.1, (a) $\Leftrightarrow$ (g)), and it follows immediately from the following arguments. If we compactify $\widetilde{\mathbf{C}}^2$ and $\mathbf{C}^2$, and then, using σ-processes (blowups) resolve all the indeterminacy points of the obtained rational mapping, then we get a branched finite covering $\tilde{f}$ of some projective surface $\widetilde{X}$ over $\mathbf{C}P^2$ such that the preimage of $\mathbf{C}^2$ contains $\widetilde{\mathbf{C}}^2$ as a Zariski open subset. The image of the branching curve is an algebraic curve. Let us denote it by K. This already implies that the mapping cannot be two-fold. Indeed, if it were then $\tilde{f}^{-1}(K) \cap \widetilde{\mathbf{C}}^2 = \varnothing$, hence, the variety $\tilde{f}^{-1}(\mathbf{C}^2 - K)$ would be simply connected (since it contains $\widetilde{\mathbf{C}}^2$) and would be a finite unbranched covering of $\mathbf{C}^2 - K$. This is impossible because the fundamental group $\pi_1(\mathbf{C}^2 - K)$ is infinite.

This kind of purely topological arguments does not extend even to the three-fold case. Vitushkin [3] constructed an example of a three-fold simplicial (not analytic) branched covering of $\mathbf{C}^2$ satisfying all the above conditions except the condition that K is algebraic.

In order to prove Theorem 1.1 for three-fold mappings, first we show that if $N = 3$ then K is a nonsingular irreducible curve (we do it by means of Euler

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characteristic computation and a more detailed study of the regularisation process). Then it follows from Abhyankar-Moh theorem that K is unknotted in $\mathbf{C}^2$, and we can argue as in the two-fold case.

We hope that some of our results may appear to be of some use in proving the Jacobian Conjecture. Since a lot of wrong proofs of the Jacobian Conjecture have appeared (see [2], Sect. I.3), we shall try, as far as possible, for all our assertions, either to provide proofs in maximal (sometimes, maybe, excessive) detail or to refer to such proofs. In particular, we prove some statements which were used by others but whose proofs were omitted or sketched because of their evidence.

§2. Regularization of the mapping f

We shall consider $\mathbf{C}^2$ embedded in the product of two complex projective lines $\mathbf{P}^1\times\mathbf{P}^1$ and $\widetilde{\mathbf{C}}^2$ in the complex projective plane $\widetilde{\mathbf{P}}^2$. Then the mapping f extends to a rational map $F:\widetilde{\mathbf{P}}^2\to\mathbf{P}^1\times\mathbf{P}^1$, possibly having points of indeterminacy. By means of a finite number of σ-processes, F may be made regular. Since $F|_{\widetilde{\mathbf{C}}^2}$ is regular, the σ-processes can be done only "at infinity", i.e. at points lying on the total preimage of the line $\widetilde{\mathbf{P}}^2-\widetilde{\mathbf{C}}^2$ under previous σ-processes. As a result we obtain a nonsingular compact variety $\widetilde{X}$ containing $\widetilde{\mathbf{C}}^2$ and a regular birational morphism $\sigma:\widetilde{X}\to\widetilde{\mathbf{P}}^2$ whose restriction on $\widetilde{\mathbf{C}}^2$ is the identity, and such that the mapping $F\circ\sigma:\widetilde{X}\to\mathbf{P}^1\times\mathbf{P}^1$ is regular. Since $F\circ\sigma$ is an extension of f to $\widetilde{X}$, we denote it also by f. We also introduce the following notation: $L=\widetilde{X}-\widetilde{\mathbf{C}}^2$, $L_\infty=f^{-1}(\mathbf{P}^1\times\mathbf{P}^1-\widetilde{\mathbf{C}}^2)$, and $L_{FC}=\overline{L-L_\infty}$. Since $f(\widetilde{\mathbf{C}}^2)\subset\mathbf{C}^2$, we have the inclusion $L_\infty\subset L$. Finally we denote by L_C the union of those irreducible components of L_{FC} on which f is constant, and by L_F the curve $\overline{L_{FC}-L_C}$.

It is clear that the irreducible components of L are nonsingular rational curves intersecting transversally and at most pairwise at any point. To each (reducible) curve it is convenient to associate a graph whose vertices correspond to the irreducible components of the curve and whose edges correspond to the intersection points of the corresponding components. It is easy to verify that the graph corresponding to L is a tree, i.e. a connected graph which has no cycles.

The following assertion (in slightly different form) was formulated for example in [3].

Lemma 2.1. *The variety $\widetilde{X}$ may be chosen so that each connected component K of L_{FC} has the following properties:*

a) *K intersects L_∞ at a unique point p, and $f(K-p)\subset\widetilde{\mathbf{C}}^2$.*

b) *The graph corresponding to K is linear, with p lying on a curve corresponding to its endpoint. This implies that the irreducible components $l_1,\dots,l_k$ of K can be indexed so that l_i intersects l_j if and only if $|i-j|=1$ and p lies on l_k.*

c) *$l_k\subset L_F$, and $l_i\subset L_C$ for $1\le i\le k-1$.*

Before proving Lemma 2.1, we prove the following auxiliary result.

Lemma 2.2. *Let g be a rational function on $\mathbf{P}^2$ whose restriction to $\mathbf{C}^2$ is polynomial. Let X be a manifold obtained from $\mathbf{P}^2$ by several σ-processes, and g_X be the lift of g to X. Then at each point p on the intersection of two components of the support of polar divisor of g_X, the multiplicity of the zero divisor is less than the multiplicity of the polar divisor.*

By the multiplicity of a divisor at a point we mean the order of zero of the restriction of a function locally defining the divisor to a generic line through the point.

Proof. The polar divisor of g is supported by the line at infinity $\mathbf{P}^2 - \mathbf{C}^2$. Therefore, the support of the polar divisor of g_X is contained in the total preimage of this line, which is the union of nonsingular curves meeting each other transversally and at most pairwise at any point. Let us consider a point $p \in X$ lying in the intersection of two components C_1 and C_2 of the support of the polar divisor of g_X. Choose a holomorphic coordinate system (x_1, x_2) in a neighbourhood U of p such that $C_i = \{x_i = 0\}$. If the neighbourhood is sufficiently small, g_X may be represented in the form $g_X(x_1, x_2) = x_1^{-n_1} x_2^{-n_2} h(x_1, x_2)$ where h is holomorphic at p and nonvanishing on $(C_1 \cup C_2 - p) \cap U$. Let m_1 and m_2 be the orders of the zeros at p of the restrictions of h onto the coordinate axes C_1 and C_2 respectively. It is clear that the numbers n_1, n_2, m_1, and m_2 do not depend on the choice of coordinate system.

Now let us prove by induction on the number of σ-processes the following assertion (A), which implies Lemma 2.2:

Assertion (A). *For any point p lying on the intersection of components of the support of the polar divisor, we have either $n_1 > m_1$ or $n_2 > m_2$.*

In order to better visualize Assertion (A) and its formal proof given below, it is convenient to depict the Newton diagrams of the numerator and the denominator of g_X and observe how they are transformed under σ-processes.

Proof of Assertion (A). If no σ-process has been performed then Assertion (A) is true because there are no intersection points of components of the polar divisor. Assume that (A) holds for some surface X and let $\sigma : X' \to X$ be a σ-process centered at p_0 with exceptional curve $C' = \sigma^{-1}(p_0)$. We consider separately two cases: 1) p_0 lies on just one component C_1 of the support of the polar divisor, and 2) p_0 lies on the intersection of two components, C_1 and C_2. It is clear that in any other cases, new intersection points of polar divisor components cannot arise, so these are irrelevant to us. We choose the coordinate system (x_1, x_2) and define the function h and the numbers n_1, n_2, m_1, and m_2 as above with the only difference that, in our case 1), we take for C_2 an arbitrary curve meeting C_1 transversally and not contained in the zero divisor of g_X.

Case 1. ($n_1 > 1$, $n_2 = 0$). We represent h in the form

$$h(x_1, x_2) = \sum_{i=k}^{\infty} h_i(x_1, x_2), \qquad h_k \neq 0,$$

where h_i is a homogeneous polynomial of degree i, and k is the multiplicity of the zero divisor at p_0. If $k \geq n_1$, no new components of the polar divisor appear after the σ-process, so the induction step is complete. So, let $k < n_1$. Then after the σ-process there appear one new component $(n_1 - k)C'$ and the only new intersection point of the components of the support of the polar divisor is $p_1 = C' \cap C_1'$ where C_1' is the strict transform of C_1 (i.e. $\overline{\sigma^{-1}(C_1 - p_0)}$). By the definition of σ-process, in a neighbourhood of p_1 we have a coordinate system (x_1', x_2') such that the mapping

σ takes the form $x_1 = x_1' x_2'$, $x_2 = x_2'$ In these coordinates we have $C_1' = \{x_1' = 0\}$, $C' = \{x_2' = 0\}$, and, further,

$$\begin{aligned} g_{X'} &= {x_1'}^{-n_1} {x_2'}^{-n_1} \sum_{i=k}^{\infty} h_i(x_1' x_2', x_2') \\ &= {x_1'}^{-n_1} {x_2'}^{-(n_1-k)} \left(h_k(x_1', 1) + \sum_{i=k+1}^{\infty} {x_2'}^{i-k} h_i(x_1', 1) \right) . \end{aligned}$$

In other words, if n_1' and m_1' are the numbers which mean for $g_{X'}$ the same as n_1 and m_1 mean for g_X, then $n_1' = n_1$ and m_1' is the multiplicity with which the polynomial $h_k(x_1', 1)$ vanishes at the point $x_1' = 0$. Therefore, $m_1' \le \deg_{x_1'} h_k(x_1', 1) \le k < n_1 = n_1'$, as it was required to prove.

Case 2. ($n_1 > 0$, $n_2 > 0$). In this case by the induction hypothesis the zero multiplicity at p_0 is less than the polar multiplicity; therefore after the σ-process the exceptional curve C' becomes part of the polar divisor and two new intersection points are added: $p_1 = c_1' \cap C'$ and $p_2 = c_2' \cap C'$. We shall consider only p_1 since the case of p_2 is completely analogous. As in Case 1, we introduce in a neighbourhood of p_2 coordinates (x_1', x_2') such that $x_1 = x_1' x_2'$, $x_2 = x_2'$, $C_1' = \{x_1' = 0\}$, and $C' = \{x_2' = 0\}$. By the induction hypothesis, we have either $m_1 < n_1$ or $m_2 < n_2$. If $m_1 < n_1$ then $k < m_1 < n_1$ and we get $m_1' < n_1'$ by the same considerations as in Case 1. If $m_2 < n_2$ then $m_2' = m_2 - k$, $n_2' = n_1 + n_2 - k$, hence, $m_2' < n_2'$. The lemma is proved.

Proof of Lemma 2.1. The mapping $f : \widetilde{\mathbf{P}}^2 \to \mathbf{P}^1 \times \mathbf{P}^1$ may be considered as two rational functions f_1 and f_2 on $\widetilde{\mathbf{P}}^2$. Denote by L_i ($i = 1, 2$) the union of those irreducible components of L which are not in the support of the polar divisor of f_i. Then $L_{FC} = L_1 \cap L_2$. It follows from Lemma 2.2 that each connected component of L_i corresponds to a branch of the graph of L; i.e. a subgraph which is connected with its complement by a single edge. If two branches of a tree intersect each other then either one contains the other or their union is the whole graph. But the latter case is impossible since the strict transform of $\widetilde{\mathbf{P}}^2 - \widetilde{\mathbf{C}}^2$ lies neither in L_1 nor in L_2. Therefore, each connected component K of L_{FC} is a connected component of one of the curves L_i, i.e. of a branch of the graph of L. Part (a) of the lemma is proved. Without loss of generality, we may assume that K is a connected component of L_1. Then by induction on the number of σ-processes it is easy to show that the part (b) holds for all intermediate σ-processes. For this we use Lemma 2.2 applied for f_1 and the fact that it suffices to perform σ-processes only at points lying on the intersection of the zero divisor and the polar divisor of one of the functions f_i. Part (c) follows from (a), since a function on l_i without poles is constant.

§3. Structure of a mapping at a branch point in general position

The following is well known.

Lemma 3.1. *Let $u(x, y)$ and $v(x, y)$ be holomorphic functions defined in a neighbourhood U of the origin in $\mathbf{C}^2$, giving a mapping $g = (u, v) : U \to \mathbf{C}^2$ such that each point has a finite number of preimages. Assume also that $g(0, 0) = (0, 0)$, $\partial(u, v)/\partial(x, y) \ne 0$ for $y \ne 0$, and $g(\{y = 0\}) \subset \{v = 0\}$. Then, in suitable holomorphic coordinates (x', y'), the mapping g takes the form $u = x'$, $v = {y'}^k$.*

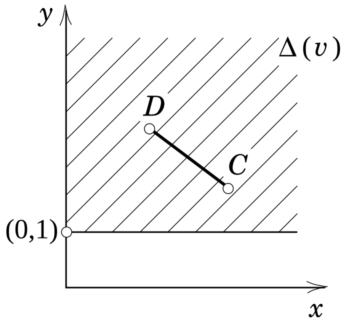

FIGURE 1

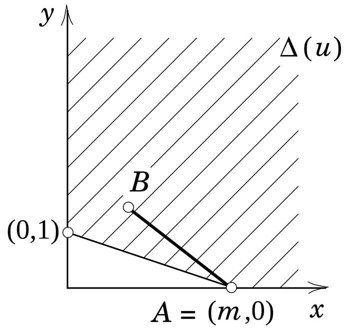

FIGURE 2

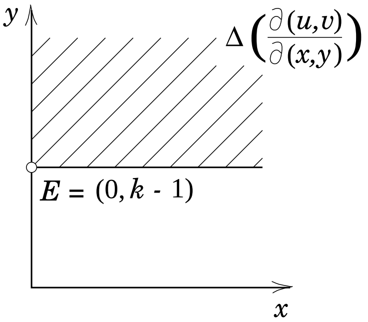

FIGURE 3

As usual, by the Newton diagram $\Delta(h)$ of a holomorphic function $h(x,y) = \sum a_{mn}x^m y^n$ with respect to coordinates (x,y) we mean the convex hull of the union of the sets $(m,n) + \mathbf{R}^2_+ \subset \mathbf{R}^2$ where $\mathbf{R}^2_+$ is the positive quadrant and $a_{mn} \neq 0$.

Proof. The assumption $g(\{y=0\}) \subset \{v=0\}$ implies that $\Delta(v(x,y))$ is contained in the shaded area of Figure 1. The point $(0,0)$ has a finite number of preimages, therefore $\Delta(u(x,y))$ is contained in the shaded area of Figure 2 where $A = (m,0) \in \Delta(u)$ for some $m > 0$. By assumption, $\Delta(\partial(u,v)/\partial(x,y))$ coincides with the shaded area in Figure 3 ($k > 0$). Let AB be the edge of $\Delta(u)$ adjacent to A (we have $B \neq A$, but possibly $B = (m,\infty)$), and CD the edge of $\Delta(v)$ parallel to AB (possibly, $C = D$). Then, since the point $E \overset{\text{def}}{=} A + C - (1,1)$ lies in $\mathbf{R}^2_+$, it must be the lower endpoint of some (nonhorizontal!) edge of $\Delta(\partial(u,v)/\partial(x,y))$ parallel to AB (this edge may be degenerate also). Hence $E = (0,k-1)$. This is possible only for $A = (0,1)$ and $C = (0,k)$. This means that $u = xu_1(x,y) + yu_2(x,y)$ where u_1 and u_2 are holomorphic functions and $u_1(0,0) \neq 0$. Moreover, if C were the rightmost endpoint of a nonhorizontal edge CF, then $F + A - (1,1)$ would lie in $\Delta(\partial(u,v)/\partial(x,y))$ below E, which is impossible (see Figure 3). Therefore $\Delta(u) = C + \mathbf{R}^2_+$, i.e. $v = y^k v_1(x,y)$ and $v_1(0,0) \neq 0$. Let us choose a holomorphic branch v_2 of the function $\sqrt[k]{v_1}$ in a neighbourhood of $(0,0)$. Then $x' = u(x,y)$ and $y' = yv_2(x,y)$ is the desired coordinate change. Indeed,

$$\begin{aligned}\frac{\partial(x',y')}{\partial(x,y)}(0,0) &= \left[\left(u_1 + x\frac{\partial u_1}{\partial x} + y\frac{\partial u_2}{\partial x}\right)\left(v_2 + y\frac{\partial v_2}{\partial y}\right) - \frac{\partial u}{\partial y}\cdot y\frac{\partial v_2}{\partial x}\right]_{(0,0)} \\ &= u_1(0,0)v_2(0,0) \neq 0.\end{aligned}$$

The lemma is proved.

Lemma 3.2. *Let $g : \mathbf{C}^2 \to \mathbf{C}^2$ be a mapping given by the formula $g(x,y) = (u,v)$ where $u = x$ and $v = y^k$, and let S be a smooth real hypersurface in $\mathbf{C}^2$ intersecting the complex line $v = 0$ transversally. Then $\tilde{S} \overset{\text{def}}{=} g^{-1}(S)$ is a smooth real hypersurface transversally intersecting $y = 0$ and is a branched covering of S branched along $\tilde{S} \cap \{y = 0\}$.*

Proof. In a neighbourhood of a point not lying on $y = 0$, the lemma obviously holds. Let us prove it in a neighbourhood of an arbitrary point p lying on $S \cap \{y = 0\}$. Without loss of generality we may assume that $p = (0,0)$. Since S is transversal to the line $v = 0$, the implicit function theorem implies that there exist a smooth parametrization $\varphi : U \to \mathbf{C}^2$ of the surface S where U is a neighbourhood of $(0,0)$

in $\mathbf{R}\times\mathbf{C}$ and φ is a mapping given by $\varphi(t,w)=(u,v)$, $u=t+iF(t,w)$, and $v=w$ (here $i=\sqrt{-1}$, and F is a smooth real-valued function). Then the mapping $\psi(s,z)=(x,y)$, $x=s+iF(s,z^k)$, and $y=z$ is a parametrization of the surface $\tilde{S}$; moreover, $g\circ\psi=\phi\circ\lambda$ where $\lambda:U\to U$ is the mapping given by $\lambda(s,z)=(t,w)$, $t=s$, $w=z^k$.

4. Estimate of the number of branch curves whose image lie in $\mathbf{C}^2$

Let $\varphi:A\to B$ be a continuous mapping of topological spaces. By the multiplicity of φ at $x\in A$ we mean the largest number $k=\mu_x\varphi$ such that in every neighbourhood of x, there are points $x_1,\dots,x_k$ such that $\varphi(x_1)=\dots=\varphi(x_k)$. If $\varphi(A)=B$ and for all $b\in B$

$$\sum_{a\in\varphi^{-1}(b)}\mu_a\varphi=\mu_\varphi<\infty \tag{2}$$

where μ_φ is independent of b, then we say that φ is a *constant-multiplicity mapping*, and we call μ_φ the *multiplicity* of φ.

Given a finite simplicial complex, for each subset A being a union of open simplices (possibly of different dimensions) we denote by $\chi_0(A)$ the number $\sum_\sigma(-1)^{\dim\sigma}$ where the summation is taken over all the open simplices contained in A. If A is a closed subcomplex then $\chi_0(A)$ is equal to the Euler characteristic $\chi(A)$ (this may not be true in other cases). For convenience, let us formulate the standard method of computing the Euler characteristic of a branched covering in the form of the following lemma, which becomes the Riemann-Hurwitz formula in the case of Riemann surfaces.

Lemma 4.1. *Let $\varphi:A\to B$ be a constant-multiplicity simplicial mapping of finite simplicial complexes, and let each set $A_k=\{a\in A\mid\mu_a\varphi=k\}$ be the union of open simplices. Then*

$$\chi(A)=\chi(B)\times\mu_\varphi-\sum_k(k-1)\chi_0(A_k). \tag{3}$$

Proof. Let τ be an open simplex in B. Denote the number of simplices $\sigma\subset\varphi^{-1}(\tau)$ by $N(\tau)$. Since φ is of constant multiplicity, we have, first, $\dim\sigma=\dim\varphi(\sigma)$, and, second, by virtue of (2),

$$N(\tau)=\mu_\varphi-\sum_{\sigma\subset\varphi^{-1}(\tau)}(\mu_\sigma\varphi-1).$$

Therefore,

$$\begin{aligned}\chi(A)&=\sum_{\tau\subset B}(-1)^{\dim\tau}N(\tau)\\&=\sum_{\tau\subset B}(-1)^{\dim\tau}\left[\mu_\varphi-\sum_{\sigma\subset\varphi^{-1}(\tau)}(\mu_\sigma\varphi-1)\right]\\&=\sum_{\tau\subset B}(-1)^{\dim\tau}\mu_\varphi-\sum_{\sigma\subset A}(-1)^{\dim\sigma}(\mu_\sigma\varphi-1)\\&=\chi(B)\mu_\varphi-\sum_k(k-1)\chi_0(A).\end{aligned}$$

The lemma is proved.

Let $\tilde{f} : \widetilde{X} \to X$ be the mapping considered in §2. Recall that L_C is the union of those irreducible components of the curve L_{FC} on which f is constant. Let $\widetilde{X}^*$ be the topological space obtained from $\widetilde{X}$ by contracting the curve L_∞ into one point which we denote by ∞, and contracting each connected component of the curve L_C to a single point. Let $\pi : \widetilde{X} \to \widetilde{X}^*$ be the natural projection. Also, let us denote the one-point compactification of $\mathbf{C}^2$ by X^*. It is clear that f induces a continuous constant-multiplicity mapping $f^* : tX^* \to X^*$. Let us denote its multiplicity by N.

Using results on triangularizability of analytic sets [4], and Lemma 3.1, we can easily show that the spaces $\widetilde{X}^*$ and X^* admit triangulations relative to which the mapping f^* is simplicial. It follows from Lemma 3.1 that for each irreducible curve $l \subset \widetilde{X}$, for almost all points $x \in \pi(l)$, except for a finite set, the multiplicity $\mu_x f^*$ is equal to the same number which we denote by $\mu_l f^*$.

Lemma 4.2.

$$\sum_{l \subset L_F} \left[\mu_l f^* + \sum_{x \in \pi(l) - \{\infty\}} (\mu_x f^* - \mu_l f^*) \right] = N - 1 \tag{4}$$

where the outer sum is takes over all irreducible components of L_F.

Although the inner sum in (4) is taken over all points of $\pi(l) - \{\infty\}$, it is clear that it can contain only finitely many nonzero summands. Moreover, from an obvious property of the semicontinuity of the multiplicity, it follows that each summand in the inner sum is nonnegative, and we get

Corollary 4.3.

$$\sum_{l \subset L_F} \mu_l f^* \leq N - 1 \tag{5}$$

where the equality is attained if and only if $\mu_x f^* = \mu_l f^*$ for each irreducible component $l \subset L_F$ and for all points $x \in \pi(l)$, $x \neq \infty$.

Proof of Lemma 4.2. Denote the number of irreducible components of the curve L_F by K. For each irreducible component $l \subset L_F$, let us denote the set $\{x \in \pi(l) - \{\infty\} \,|\, \mu_x f^* \neq \mu_l f^*\}$ by $S(l)$, the number of elements of $S(l)$ by $s(l)$, and the set $\pi(l) - \{\infty\} - S(l)$ by l'. Then we have $\chi(\widetilde{X}^*) = 2 + K$, $\chi(X^*) = 2$, and $\mu_\infty f^* = N$. By part (a) of Lemma 2.1, we have $\chi_0(l') = 1 - s(l)$. Substituting these expressions into (3), we get

$$\begin{aligned} \chi(\widetilde{X}^*) =& \chi(X^*)N - (\mu_\infty f^* - 1) \\ & - \sum_{l \subset L_F} (\mu_l f^* - 1)\chi_0(l') - \sum_{l \subset L_F} \sum_{x \in S(l)} (\mu_x f^* - 1), \end{aligned}$$

i.e.

$$2+K = 2N-(N-1)-\sum_{l\subset L_F}\left[(\mu_l f^*-1)(1-s(l))+\sum_{x\in S(l)}(\mu_x f^*-1)\right]$$
$$= N+1-\sum_{l\subset L_F}\left[\mu_l f^*-1-s(l)(\mu_l f^*-1)\right.$$
$$\left.+\sum_{x\in S(l)}(\mu_x f^*-\mu_l f^*)+\sum_{x\in S(l)}(\mu_l f^*-1)\right]$$
$$= N+1+K-\sum_{l\subset L_F}\left[\mu_l f^*-\sum_{x\in S(l)}(\mu_x f^*-\mu_l f^*)\right].$$

The lemma is proved.

§5. Proof of Theorem 1.1

Lemma 5.1. *If a group G contains a subgroup G_1 isomorphic to a free group of rank greater than one, then G cannot contain any subgroup of finite index isomorphic to $\mathbf{Z}$.*

Proof. Let us denote two of free generators of G_1 by a and b. If H is a subgroup of G of finite index, then some powers of the elements a and b must lie in H. Therefore, if H were isomorphic to $\mathbf{Z}$ then there would exist integers m and n such that $a^m = b^n$. But this contradicts the fact that a and b are free generators of G_1.

Lemma 5.2. *Let $x \in \widetilde{X}^*$, $x \neq \infty$ (see §4). Suppose that x lies on the image $l^* = \pi(l)$ of an irreducible component l of L_F. If $\mu_x f^* = \mu_l f^*$, then a germ of the mapping $f^*|_{l^*}$ at x determines a nonsingular branch of the curve $f^*(l^*)$ at $f^*(x)$; that is, there exists a neighbourhood U of x such that $f^*\circ\pi|_{U\cap l}$ is a nonsingular embedding.*

Proof. Let B be a sufficiently small ball centered at the point $y = f^*(x)$ such that its boundary S intersects transversally with $f^*(l^*)$, $f^*(l^*)\cap B$ is homeomorphic to a cone over $f^*(l^*)\cap S$, and each connected component of the set ${f^*}^{-1}(B)$ contains a unique point of ${f^*}^{-1}(y)$. Let $\tilde{B}$ be the connected component of ${f^*}^{-1}(B)$ containing x. Then $\tilde{S} \stackrel{\text{def}}{=} \partial\tilde{B}$ is a smooth real 3-manifold by Lemma 3.2 which is a branched covering of the sphere S with the branching curve $\tilde{K} \stackrel{\text{def}}{=} \tilde{S}\cap l^*$. Let $K = f^*(\tilde{K})$. It follows from the assumption $\mu_x f^* = \mu_l f^*$ that ${f^*}^{-1}(K) = \tilde{K}$ and that $f^*|_{\tilde{S}-\tilde{K}}$ is an unbranched covering of $S-K$ and hence, $H \stackrel{\text{def}}{=} \pi_1(\tilde{S}-\tilde{K})$ is a subgroup of finite index of the group $G \stackrel{\text{def}}{=} \pi_1(S-K)$.

The set $\widetilde{X}^*$ is a smooth complex surface everywhere except the finite set $\{\infty\}\cup\pi(L_C)$. Moreover, by Stein factorisation theorem, $\widetilde{X}^*$ is analytic everywhere except at the point ∞; i.e. for some m there exists an embedding $\iota : \tilde{B}\to\mathbf{C}^m$ such that $\iota(\tilde{B})$ is an analytic subset of $\mathbf{C}^m$, and $\iota\circ\pi : \pi^{-1}(\tilde{B})\to\mathbf{C}^m$ is analytic. Moreover, we may assume all points of $\pi(L_C)$ are singular points of $\widetilde{X}^*$, since otherwise in the construction of $\widetilde{X}$ we would not need the corresponding σ-processes. Using the techniques of [6], it is not difficult to show that $\tilde{B}$ is diffeomerphic to the intersection of $\iota(\tilde{B})$ with a sufficiently small ball in $\mathbf{C}^m$. It is clear that if x is nonsingular in tX^* (i.e. $x \notin \pi(L_C)$) then $H \cong \mathbf{Z}$. It turns out (and this will be proved now) that if x is singular then also $H \cong \mathbf{Z}$.

In [5], the fundamental group $\pi_1(\tilde{S})$ is computed in terms of a resolution of the singularity at x. If p is an isolated singularity on a surface Y and $\pi : \tilde{Y} \to Y$ its resolution such that $\pi^{-1}(p) = \bigcup l_i$ where l_i are rational curves and the corresponding graph is a tree, then by [5] the fundamental group $\pi_1(M)$ of the boundary M of a sufficiently small spherical neighbourhood of p can be described as follows. Generators of this group are small positively oriented loops α_i around the curves l_i, and the defining relations are

(I_{ij}) α_i commutes with α_j if $l_i \cap l_j = \varnothing$.

(II_i) If a_i is the self-intersection number of l_i and if $l_{j_1}, \dots, l_{j_n}$ are the curves which meet l_i then $\alpha_{j_1} \dots \alpha_{j_n} \alpha_i^{a_i} = 1$.

From the proof of this fact ([5], pp. 12–13), it is easy to see that if T is a circle which is cut out on M by a smooth complex curve transversally intersecting l_{i_0} then the group $\pi_1(M - T)$ also is generated by α_i but to obtain the defining relations for $\pi_1(M - T)$, we must exclude (II_{i_0}) from the above relations for $\pi_1(M)$. In our case, the resolution of the singularity at x is the natural projection $\pi : \widetilde{X} \to \widetilde{X}^*$. By Lemma 2.1 we know that the graph corresponding to the curve $\pi^{-1}(x) = \bigcup_{i=1}^{k-1} l_i$ is linear and $\tilde{K}$ is a circle cut out on $\tilde{S}$ by the curve $l = l_k$, which meets l_{k-1} transversally. Hence, the group $H = \pi_1(\tilde{S} - \tilde{K})$ is defined by the generators $\alpha_1, \dots, \alpha_{k-1}$ and the relations $\alpha_2 = \alpha_1^{-a_1}$, $\alpha_3 = \alpha_2^{-aa_2} \alpha_1^{-1}$, ..., $\alpha_{k-1} = \alpha_{k-2}^{-a_{k-2}} \alpha_{k-3}^{-1}$, where a_j is the self-intersection number of l_j. (In this case the commutativity relations are consequences of the others.) Thus, $\alpha_2, \dots, \alpha_{k-1}$ may be expressed in terms of α_1, and after this, there remains no more relations, i.e. H is isomorphic to $\mathbf{Z}$.

To complete the proof of the lemma, it remains to note that by [6], Corollary 10.2, the commutator subgroup of G is a free group with an even number of generators. Thus, by Lemma 5.1, the knot K is trivial, and hence, by a theorem of Zariski [7], the corresponding branch of the curve $f^*(l^*)$ is nonsingular. The lemma is proved.

Lemma 5.3. *Suppose that $N > 2$, $l = L_F$ (see §2) is irreducible, and $\mu_l f = N - 1$. Then the restriction of f to $l' \overset{\mathrm{def}}{=} l - L_\infty$ is a biregular isomorphism, and hence, the curve $f(l')$ is nonsingular in $\mathbf{C}^2$.*

Proof. The mapping $f|_{l'}$ separates points. Indeed, if $f(x) = f(y)$ for distinct points $x, y \in l'$, then $\mu_x f^* + \mu_y f^* = 2N - 2 > N$ which contradicts the condition (2) of constant multiplicity of f. Hence, first, $f(l')$ is nonsingular since it may not have irreducible singularities by Lemma 5.2, and second, $f|_{l'}$ is an isomorphism. The lemma is proved.

In order to prove Theorem 1.1, we shall show that the multiplicity N of f^* (see §4) cannot equal two or three. The fact that $N \neq 2$ was proved in §1, but we may also derive it as an immediate corollary of Lemma 4.3. If $N = 2$ then, by Corollary 4.3 the multiplicity of f^* at every point of $\pi(L_F)$ except the point ∞ is equal to one, hence, f^* is a two-fold unbranched covering of $\mathbf{C}^2$, which is a contradiction.

Suppose that $N = 3$. Then by Lemma 4.2 three cases may occur: 1) L_F consists of two components and $\mu_x f^* = 1$ for all $x \in \pi(L_F) - \{\infty\}$; 2) L_F is irreducible and $\mu_x f^* = 1$ for all $x \in \pi(L_F) - \{\infty, x_0\}$, where $x_0 \neq \infty$; or 3) L_F is irreducible and $\mu_x f^* = 2$ for all $x \in \pi(L_F) - \{\infty\}$. However, the first two cases are impossible because otherwise f^* would be an unbranched three-fold covering of $\mathbf{C}^2$ or of $\mathbf{C}^2 - f^*(x_0)$ which contradicts the fact that these varieties are simply connected.

So we assume that the case 3) holds. Then by Lemma 5.3 the curve $f(L_F) \cap \mathbf{C}^2$ is isomorphic to $\mathbf{C}^1$ and we may apply the following theorem of Abhyankar and Moh [8]: *If l is a curve in $\mathbf{C}^2$ which is isomorphic to $\mathbf{C}^1$ then there exists a polynomial automorphism $\alpha : \mathbf{C}^2 \to \mathbf{C}^2$ such that $\alpha(l)$ is a line.* Therefore, there space $M = \mathbf{C}^2 - f(L_F)$ is homotopically equivalent to a circle, and this implies that there exists a unique (up to equivalence) three-fold unbranched covering of M. It may be uniquely extended up to a branched covering of $\mathbf{C}^2$. It is easy to see that the branching order of this covering along $f(L_F)$ is equal to three. However, we assumed that $\mu_l f^* = 2$. This contradiction completes the proof of the theorem.

Remark. Repeating the above arguments, one can prove the following assertion. *If $f : \mathbf{C}^2 \to \mathbf{C}^2$ is a polynomial mapping satisfying condition (1), then the curve L_F (see §2) cannot contain an irreducible component l such that $\mu_l f^* = N - 1$.*

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