FLEXIBLE, ALGEBRAICALLY UNREALIZABLE CURVES: REHABILITATION OF HILBERT-ROHN-GUDKOV APPROACH

S. Yu. Orevkov and E. I. Shustin

> Ordnung angeht, so habe ich mich — freilich auf einem recht umständlichen Wege — davon überzeugt, daß die 11 Züge, die sie nach Harnack haben kann, keinesfalls sämtlich außerhalb von einander verlaufen dürfen, sondern daß ein Zug existieren muß, in dessen Innerem *ein* Zug und in dessen Äußerem *neun* Züge verlaufen oder umgekehrt.
>
> D.Hilbert. *"Mathematische Probleme"*

Introduction

The question on topology of real algebraic varieties, raised by Hilbert in his 16th problem, still is far from any complete answer and remains a point of a beautiful interaction of topology, differential and algebraic geometry, arithmetic, singularity theory, which reveals interesting geometric phenomena.

In the present paper we classify real plane curves of degree 8 with a singular point of quadratic tangency of four smooth branches (equivalently, curves of degree 8 on the quadratic cone, or curves of bidegree $(8,4)$ of the Hirzebruch surface $\mathcal{F}_2$). Our main result consists in the two isotopy classifications, a classification of real algebraic curves, and a classification of real pseudo-holomorphic curves, which show that *there exist real pseudo-holomorphic curves which are not isotopic to any real algebraic curve (in the corresponding class)*. Real pseudo-holomorphic curves form a subclass of *flexible curves* introduced by Viro in [21]. He also posed a problem to distinguish between flexible and algebraic curves, which we answer in the considered case.

The pseudo-holomorphic classification is based on the braid group technique developed by the first author [13] and Gromov's theory of pseudo-holomorphic curves [7]. To prohibit non-algebraic pseudo-holomorphic curves we apply the Hilbert-Rohn-Gudkov method.

The detailed formulation of results is given in Section 1, the restrictions valid in the pseudo-holomorphic case are presented in Section 2, the Hilbert-Rohn-Gudkov method is applied in Section 3, at last, constructions are found in section 4. In section 3.5 below we give a historical comment.

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Note, that another method of prohibition which may work in the case when a pseudo-holomorphic curve exists, was suggested by S. Fiedler-LeTouzé (see [4]). It employs an auxiliary pencil of cubics.

The Hilbert-Rohn-Gudkov approach in the context of modern real algebraic geometry.

Thesis. Hilbert and Rohn [16, 17] suggested a method to prove topological restrictions for plane real algebraic curves. This method was developed by Gudkov who succeeded to complete the classification of real plane nonsingular projective curves of degree 6 in this way [10]. Later this method (called *Hilbert-Rohn method* in [10, 9]) was applied by Gudkov's school to other problems: Polotovskii [15] studied reducible curves of degree 6 in general position and the second author studied smoothings (dissipations) of four tangent branched (singularity of type X_{21}).[1]. In particular, the prohibition of an algebraic curve as shown in Figure 3 was announced in [19], but the proof has not been published (here we present the proof in all details).

Antithesis. Hilbert-Rohn method requires a consideration of a large tree of possibilities. This may cause mistakes. For instance, both Hilbert and Gudkov erroneously claimed that they prohibited the M-sextic $\langle 5 \sqcup 1\langle 5\rangle\rangle$ (later Gudkov found the mistake and constructed this curve).

Very soon after Gudkov's paper [10], Arnold [1] suggested a topological approach for the study of plane real algebraic curves. This approach allowed to Rokhlin to prove a congruence for M-curves of any even degree. Kharlamov, Gudkov and Krahnov modified this congruence for $(M-1)$-curves. The Gudkov's result originally proved by use of Hilbert-Rohn method appeared to be an immediate consequence of the congruences. This is why Hilbert-Rohn-Gudkov approach has been almost forgotten.

Using a modification of Arnold's topological method (proposed by the first author in [13]), Polotovskii and Gushchin (private communication) reproved all the prohibitions which were proved by Hilbert-Rohn method in [15]. Almost all the prohibitions for M-smoothings proved in [19] (a single exception is the smoothing in Figure 3) can be reproved in the same way. Thus, a natural question arises: "Do we need the Hilbert-Rohn method?".

Synthesis. Analizing the properties of real algebraic curves used in the topological methods, Viro introduced in [21] flexible curves (2-surfaces in $\mathbf{CP}^2$ satisfying these properties). Suppose one proved that a certain arrangement of ovals is algebraically unrealizable. If auxiliary lines, conics and pencils of lines are not used in the proof then one proves automatically that the arrangement is unrealizable by a flexible curve (see [21]). But even if auxiliary lines, conics and pencils of lines are used in the proof then the proof is still valid for real pseudo-holomorphic curves (see Remark 1 below).

Here we present a proof of the prohibition of the curve in Figure 3 using Hilbert-Rohn-Gudkov method and we show that this curve is pseudo-holomorphically realizable which implies that no topological method (even combined with the usage of auxiliary lines and conics) can prohibit it. Thus, the answer to the above question is "Yes, we still need it".

[1]The latter application of the Hilbert-Rohn method was suggested by Viro, who motivated his suggestion by the fact that all these examples are related to K3 surfaces.

1. Statement of results

1.1. M-smoothings of four tangent branches.

Theorem 1. *Let*

$$f(x,y) = \sum_{k,l\geq 0;\ k+2l\leq 8} a_{kl}x^k y^l \tag{1}$$

be a polynomial with real coefficients which has four distinct real parabolic asymptotes at infinity, i.e. $\sum_{k+2l=8} a_{kl}x^k y^l = \prod_{j=1}^4 (y-a_j x^2)$ *with distinct real* $a_1,\dots,a_4$*. Suppose that the real affine curve* $f=0$ *is non-singular, has* 9 *ovals (bounded components), and its unbounded branches are arranged on* $\mathbf{R}^2$ *as in Figure 1 (up to an isotopy which does not mix the branches at infinity). Then:*

1) The arrangement of the ovals with respect to the pencil of vertical lines is one of those depicted in Figure 2 (up to a rotation by 180°*);*

2) (Cp. Viro [22; Sect. 4.7]) *All the possibilities in Figure 2 are realizable.*

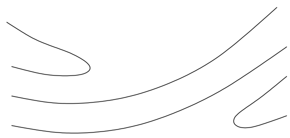


Figure 1

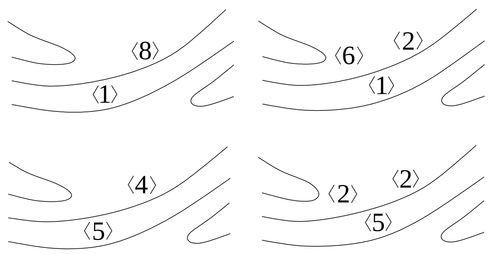


Figure 2

In the statement of Theorem 1, the *arrangement of a curve with respect to the pencil of vertical lines* means a connected component in the space of differentiable curves which have at most four intersection points with each real vertical line.

Let $z = x+iy$, $w = u+iv$, $i=\sqrt{-1}$ be coordinates and $\omega = dx\wedge dy + du\wedge dv$ the standard symplectic form in $\mathbf{C}^2$. Let J be *an almost complex structure* on $\mathbf{C}^2$, i.e. a smooth mapping $\mathbf{C}^2 \to \mathrm{End}_{\mathbf{R}}(\mathbf{C}^2)$ such that $J_p^2 = -I$ for all $p\in\mathbf{C}^2$. A real 2-plane L is called a *J-tangent at p* if $J_p(L)=L$. Following [7], we say that J is *tame* if $\omega|_L > 0$ for any J-tangent L. A real smooth 2-surface $C\subset\mathbf{C}^2$ is called *J-holomorphic* if T_pC is a J-tangent at p for any $p\in C$.

Let $\mathrm{Conj}:\mathbf{C}^2\to\mathbf{C}^2$ be the complex conjugation $(z,w)\mapsto(\bar z,\bar w)$. We say that $C\in\mathbf{C}^2$ is *an affine real pseudo-holomorphic curve in* $\mathbf{C}^2$ if $\mathrm{Conj}(C)=C$ and C is a J-holomorphic curve for some tame almost complex structure J such that J is standard outside a sufficiently large 4-ball and Conj-anti-invariant everywhere, i.e. $\mathrm{Conj}\circ J_p = J_{\bar p}^{-1}\circ\mathrm{Conj}$ for any $p\in\mathbf{C}^2$.

Theorem 2. *Let* $C\subset\mathbf{C}^2$ *be an affine real pseudo-holomorphic curve which coincides with* $\prod_{j=1}^4(y-a_jx^2)$ *outside a sufficiently large* 4*-ball where* $a_1,\dots,a_4$ *are distinct real numbers. Suppose that* $C\cap\mathbf{R}$ *has* 9 *ovals, and its unbounded branches are as in Figure 1 (as in the hypothesis of Theorem 1). Then:*

1) The arrangement of the ovals with respect to the pencil of vertical real pseudo-holomorphic lines (see Remark 1) is either one of those depicted in Figure 2 (up to a rotation by 180°*) or the arrangement depicted in Figure 3;*

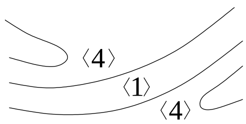


FIGURE 3

2) All the possibilities in Figures 2 and 3 are realizable.

Remark 1. Due to Gromov's results [7], if an almost complex structure J on $\mathbf{CP}^2$ is tame then there exists a unique J-holomorphic line passing through any two given points. If J is Conj-anti-invariant and the points are real then the line is also real (otherwise its conjugate would be another line through the same two points). In the context of Theorem 2, each vertical (i.e., passing through $(0:1:0) \in \mathbf{CP}^2$) pseudo-holomorphic line meets C at 4 points because the number of intersections is a homological invariant (all intersections of pseudo-holomorphic curves are positive).

Remark 2. In the situation of Theorems 1 and 2, one cannot have more than 9 ovals by Harnack's inequality.

Let us give some reformulations of Theorem 1 (each time, Theorem 2 can be reformulated analogously).

Theorem 1A. *All the dissipations in the sense of* [22][2] *of* X_{21} *with* 9 *ovals and the non-compact branch as in Figure 1, are those depicted in Figure 2.*

Let C be the curve $f = 0$ where f satisfies the hypothesis of Theorem 1. A minimal smooth compactification of $\mathbf{C}^2$ where the closure of C is non-singular, is the Hirzebruch surface $\mathcal{F}_2$. It may be defined by four coordinate charts, each one isomorphic to $\mathbf{C}^2$, with coordinates respectively, (x, y), $(x_1, y_1) = (1/x, y/x^2)$, $(x_2, y_2) = (x, 1/y)$, and $(x_3, y_3) = (1/x, x^2/y)$. where (x, y) are the coordinates from Theorem 1. Let $\mathrm{pr} : \mathcal{F}_2 \to \mathbf{P}^1$ be the fibration $(x, y) \mapsto (x : 1)$ (resp. $(x_1, y_1) \mapsto (1 : x_1)$, $(x_2, y_2) \mapsto (x_2 : 1)$, $(x_3, y_3) \mapsto (1 : x_3)$). One has $\mathrm{Pic}(\mathcal{F}_2) = H^2(\mathcal{F}_2) = \mathbf{Z} \otimes \mathbf{Z}$. As generators, we choose the fiber $F = \{x_3 = 0\}$ and the exceptional section $E = \{y_2 = 0\}$ of pr. Their intersections are: $E^2 = -2$, $F^2 = 0$, $EF = 1$. Let $H = \{y = 0\}$. Then $H \sim E + 2F$ and $HE = 0$, $HF = 1$, $H^2 = 2$. The linear system $|4H|$ can be naturally identified with the space of all polynomials of the form (1), in particular, $C \sim 4H$. Let $K_{\mathcal{F}_2}$ be the canonical class of $\mathcal{F}_2$. Since $dx \wedge dy = (dx_3 \wedge dy_3)/(x_3^4 y_3^2)$, we have $K_{\mathcal{F}_2} \sim -2E - 4F \sim -2H$ and by adjunction formula, the genus of C is $C(C + K_{\mathcal{F}_2})/2 + 1 = 4H(4H - 2H)/2 + 1 = 4H^2 + 1 = 9$. Hence, the curve C is an M-curve of the linear system $|4H|$ on $\mathcal{F}_2$ (this proves the statement of Remark 2).

The set of real points of $\mathcal{F}_2$ is diffeomorphic to a torus. We present it by a rectangle whose vertical (resp. horizontal) sides give F (resp. E) after the identification. A smoothly embedded circle O is called an *oval* if it cuts $\mathbf{R}\mathcal{F}_2$ into two parts. If an oval O does not meet E then the component of $\mathcal{F}_2 \setminus O$ which contains E is called the *exterior of* O. The other component is called the *interior of* O.

Theorem 1B. *Let* C *be an* M*-curve of the linear system* $|4H|$ *on* $\mathcal{F}_2$*. Suppose* C *has an oval* O *such that* $O \cup F \cup E$ *is arranged as in Figure 10 up to an isotopy. Then the conclusion is the same as in Theorem 1.*

[2] In general, not any non-singular deformation is equivalent to a dissipation

Let us say that a curve in $\mathbf{RP}^2$ is *symmetric* if it is invariant under the involution $\xi : (u_0 : u_1 : u_2) \mapsto (u_0 : u_1 : -u_2)$. The space of symmetric curves of degree $2k$ is in 1-1 correspondence with $|kH|$: a function f of the form (1) corresponds to $u_0^8 f(u_1/u_0, u_2^2/u_0)$. Topologically this means that we replace the lower half of the rectangle by the mirror image of the upper half (double covering), and then contract E into a single point (blow down).

Corollary. *Let C be a real pseudo-holomorphic symmetric M-curve of degree 8 on $\mathbf{RP}^2$. Suppose, C has a nest of the depth three (three ovals one inside another). Then:*

1) The real scheme of C is either one of

$$\langle 1 \sqcup 1\langle 2 \sqcup 1\langle 17\rangle\rangle\rangle,\ \langle 1 \sqcup 1\langle 10 \sqcup 1\langle 9\rangle\rangle\rangle,\ \langle 9 \sqcup 1\langle 10 \sqcup 1\langle 1\rangle\rangle\rangle,\ \langle 17 \sqcup 1\langle 2 \sqcup 1\langle 1\rangle\rangle\rangle, \quad (2)$$

or

$$\langle 9 \sqcup 1\langle 2 \sqcup 1\langle 9\rangle\rangle\rangle. \quad (3)$$

2) If C is algebraic then the real scheme (3) is not possible.

3) The real schemes (2) are realizable by symmetric algebraic curves.

4) The real scheme (3) is realizable by a symmetric real pseudo-holomorphic curve.

1.2. $(M-1)$-smoothings of four tangent branches.

Theorem 3. *Under the hypotheses of Theorem 1 assume that the curve $f = 0$ has 8 ovals. Then*

(1) *the arrangement of the ovals with respect to the pencil of vertical lines is either one obtained from the arrangements, shown in Figure 2 by removing one oval, or is as shown in Figure 4;*
(2) *all the arrangements, except for those shown in Figure 4(b,d), are realizable by algebraic curves.*

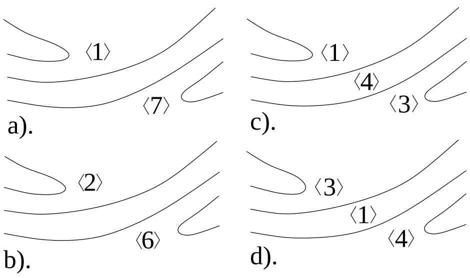


FIGURE 4

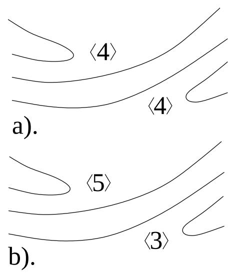


FIGURE 5

Remark. The existence of algebraic curves realizing arrangements shown in Figure 4(b,d) is not known.

Theorem 4. *Under the hypotheses of Theorem 2 assume that the real pseudo-holomorphic curve C has 8 ovals. Then the arrangement of its ovals with respect to the pencil of vertical lines is*

(1) *either one of those listed in Theorem 3,*
(2) *or one of the shown in Figure 5.*

All the above arrangements are realizable.

1.3. Affine M-sextic.

The next statement concerns a different class of curves, real affine sextics, but its proof has much in common with the proof of Theorems 1-4. Note that this provides one more example of a non-algebraic real pseudo-holomorphic affine sextic (cf. [4]).

Theorem 5. *(1) There exists a real pseudo-holomorphic affine curve of degree 6, having six non-closed branches and 9 ovals, located as shown in Figure 6.*

(2) There is no real algebraic affine curve of degree 6 isotopic to the curve shown in Figure 6.

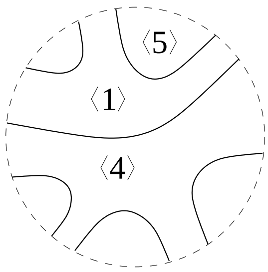


FIGURE 6

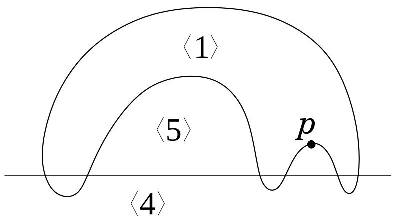


FIGURE 7

This theorem will be proved in Section 3.6. It is clear that Theorem 5(2) is equivalent to the fact that there does not exist a line and an M-sextic arranged on $\mathbf{RP}^2$ as in Figure 7. Theorem 5(1) is equivalent to the realizability of Figure 7 by a J-holomorphic line and a J-holomorphic M-sextic where J is a Conj-anti-invariant almost complex structure on $\mathbf{CP}^2$ tamed by the Fubini-Studi symplectic form. Such an arrangement was constructed in [13, Section 7.2, $A_4(1,4,5)$].

A classification up to isotopy of affine M-sextics is obtained in [13]. In view of the results in [4, 12, 13] Theorem 5 leaves only one real affine pseudo-holomorphic M-sextic, whose algebraic realizability is unknown.

Remark. The pseudo-holomorphic sextic in Figure 7 can be degenerated into the singular sextic in Figure 8 which is transformed into the curve in Figure 9 by a birational mapping $\mathbf{P}^2 \to \mathcal{F}_2$. This was the hint to look for a proof of Theorem 5(2) similar to that of Theorem 1(2).

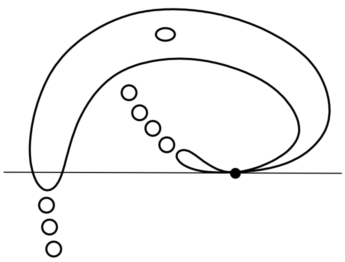


FIGURE 8

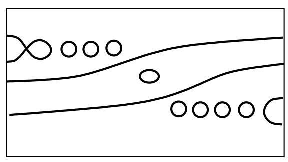


FIGURE 9

2. Restrictions which are valid in pseudo-holomorphic case

We shall expose the proof of Theorem 2 in the case when C is algebraic. However, by Gromov's theory of rational pseudo-holomorphic curves, all the arguments can be translated into the pseudo-holomorphic language.

2.1. Restrictions coming from Bezout's theorem.
Let C be a curve satisfying the hypothesis of Theorem 1B. Denote the empty ovals by $O_1, \dots, O_9$ (from the left to the right) and let $j_i \in \{1,2,3\}$ be the number indicating the region in Figure 10 which contains the oval O_i.

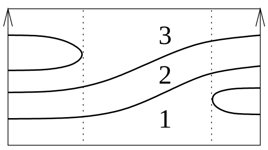


Figure 10

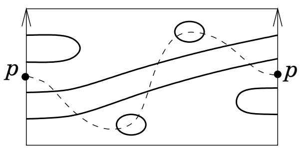


Figure 11

Lemma 2.1. *The sequence* $i_1 \dots i_9$ *cannot contain:*
a). $\dots 1 \dots 3 \dots$; *b).* $\dots 2 \dots 3 \dots 1 \dots$ *or* $\dots 3 \dots 1 \dots 2 \dots$.

Proof. a). Suppose that $j < k$, $i_j = 1$, and $i_k = 3$. There exists a curve in the linear system $|H|$ (a parabola $y = ax^2 + bx + c$ in the coordinates x, y) passing through p, O_j, and O_k (see Figure 11). It meets C in ≥ 10 points while $HC = 4H^2 = 8$.
b). The curve in the linear system $|H|$ passing through the three ovals meets C in ≥ 10 points. □

Let us call the empty ovals in the region 2 *interior ovals* and those in 1 and 3 *exterior ovals*.

Lemma 2.2. *Two vertical lines through interior ovals cannot separate exterior ovals. In other words, the sequence* $i_1 \dots i_9$ *cannot contain* $\dots 2 \dots k_1 \dots 2 \dots k_2 \dots$ *nor* $\dots k_1 \dots 2 \dots k_2 \dots 2 \dots$ *where* $k_1, k_2 \in \{1,3\}$

Proof. Suppose p_1, p_3 are inside interior ovals, p_2, p_4 are inside exterior ovals, and the points $p_1, \dots, p_4$ are numbered from the left to the right. Let N be a curve from the linear system $|H + F|$ (of the form $y = (b_3x^3 + b_2x^2 + b_1x + b_0)/(a_1x + a_0)$ in the coordinates x, y) passing through $p_1, \dots, p_4$ and one more empty oval. N is rational (hence, connected) and $CN = 4H(H+F) = 12$. Each fiber of pr cuts N once, hence N meets $p_1, \dots, p_4$ in this order and has ≥ 14 intersections with C (at least one intersection on each arc p_1p_2, p_2p_3, p_3p_4, p_4p_1, and 10 intersections with the 5 empty ovals). □

Corollary 2.3. *Up to a rotation by* $180°$*, the sequence* $i_1 \dots i_9$ *either has the form*

$$\underbrace{3\dots3}_{a_1}\ \underbrace{2\dots2}_{a_2}\ \underbrace{3\dots3}_{a_3}, \tag{4}$$

or has the form

$$\underbrace{3\dots3}_{a_1}\ \underbrace{2\dots2}_{a_2}\ \underbrace{1\dots1}_{a_3} \tag{5}$$

where in the both cases $a_j \geq 0 \quad (j = 1, \dots, 4)$.

2.2. Congruence.

As usual, we say that an oval is even (odd) if it is surrounded by an even (odd) number of other ovals. We have $1 + a_1 + a_3$ even ovals and a_2 odd ones in the notation of (4) and (5). The congruence modulo 8 in our case takes the following form.

Proposition 2.4. *If $a_1 + a_2 + a_3 = 9$ (M-curve) then*

$$a_2 \equiv 1 \mod 4.$$

If $a_1 + a_2 + a_3 = 8$ ($M-1$-curve) then

$$a_2 \equiv 0,\ 1 \mod 4.$$

Proof. Let $C \subset \mathcal{F}_2$ be a real nonsingular curve in the linear system $|4H|$. Take the double covering $\pi : X \to \mathcal{F}_2$ ramified along C. The surface X is simply connected, since $\mathcal{F}_2$ is. Hence

$$b_*(X, \mathbf{Z}/2\mathbf{Z}) = \chi(X) = 2\chi(\mathcal{F}_2) - \chi(C) = 24\ .$$

By the Hirzebruch formula [11]

$$\sigma(X) = 2\sigma(\mathcal{F}_2) - \frac{C^2}{2} = -16\ .$$

On the other hand, X possesses an antiholomorphic involution Conj_X such that $\pi \circ \mathrm{Conj}_X = \mathrm{Conj} \circ \pi$ and $\mathrm{Fix}(\mathrm{Conj}_X)$ covers via π the components of $\mathbf{R}\mathcal{F}_2$ bounded from inside by even ovals and from outside by odd ovals. So, one can easily compute that

$$b_*(\mathrm{Fix}(\mathrm{Conj}_X), \mathbf{Z}/2\mathbf{Z}) = 6 + 2(a_1 + a_2 + a_3)$$
$$= \begin{cases} b_*(X, \mathbf{Z}/2\mathbf{Z}), & \text{if} \quad a_1 + a_2 + a_3 = 9, \\ b_*(X, \mathbf{Z}/2\mathbf{Z}) - 2, & \text{if} \quad a_1 + a_2 + a_3 = 8, \end{cases}$$

hence by Rokhlin and Gudkov-Krahnov-Kharlamov congruences (see details in [23])

$$-2(1 + a_1 + a_3 - a_2) = \chi(\mathrm{Fix}(\mathrm{Conj}_X))$$
$$\equiv \begin{cases} \sigma(X) = -16 \mod 16, & \text{if} \quad a_1 + a_2 + a_3 = 9, \\ \sigma(X) \pm 2 = -16 \pm 2, & \text{if} \quad a_1 + a_2 + a_3 = 8, \end{cases}$$

and we are done.

2.3. Restrictions coming from the pencil of vertical lines. Now we shall apply the link-theoretical approach proposed in [13].

Let $y = F(x)$ be an algebraic m-valued function without poles. Let $x = \gamma(t)$, $t \in [0, 1]$ be a closed path avoiding the branch points of F. Then one can consider the braid with m strings $F \circ \gamma$ (we think of braids as of multivalued functions on a segment such that the values are distinct at any point). If γ is a simple closed path then the braid is *quasipositive* (see [18]), i.e. it has the form $\prod_j a_j \sigma_1 a_j^{-1}$, $a_j \in B_m$ where B_m denotes the group of braids with m strings and $\sigma_1, \ldots, \sigma_{m-1}$ denote the standard generators of B_m (note that all σ_j are conjugated to each other).

For a braid $b = \prod_j \sigma_{i_j}^{k_j}$, let us denote the *exponent sum* $\sum_j k_j$ by $e(b)$ and let $\hat{b}$ be the link in the sphere S^3 which is the closure of b. If L is a link and V a Seifert matrix of L corresponding to a connected Seifert surface then we denote $\det L = \det(V + V^T)$. Note that $\det L = \det G$ where G is a *Goeritz matrix* of a link L (see [5]). The *Alexander polynomial* of the L is $\det(V - tV^T)$. Let $\zeta \in \mathbf{C}$, $|\zeta| = 1$, $\zeta \neq 1$. The Tristram signature and nullity $\operatorname{sign}_\zeta(L)$ and $\operatorname{null}_\zeta(L)$ are defined as the signature and the nullity of the Hermitian matrix $(1-\zeta)V + (1-\bar{\zeta})V^T$.

The Murasugi-Tristram inequality provides the following quasipositivity tests (see details in [13]).

Proposition 2.5. *Let b be a quasipositive braid with m strings. Then for any $\zeta \in \mathbf{C}$, $|\zeta| = 1$, $\zeta \neq 1$, one has*

$$1 + \operatorname{null}_\zeta(\hat{b}) \geq |\operatorname{sign}_\zeta(\hat{b})| + m - e(b)$$

Corollary 2.6. *Let b be a quasipositive braid with m strings. If $m - e(b) > 1$ then* $\det \hat{b} = 0$.

Corollary 2.7. *Let b be a quasipositive braid with m strings. If $m - e(b) = 1$ then the Alexander polynomial of $\hat{b}$ has no simple root on the unit circle.*

Let $f(x,y)$ be as in (1) and let $y = F(x)$ be the 4-valued algebraic function implicitly defined by $f(x,y) = 0$. Let γ be the boundary of a half-disc $\{x \mid \operatorname{Im} x \geq \varepsilon,\ |x| \leq R\}$, $0 < \varepsilon \ll 1 \ll R$. Let $b = b_{\mathbf{R}} b_\infty$ be the corresponding braid where $b_{\mathbf{R}}$ corresponds to the part of γ with $\operatorname{Im} x = \varepsilon$ and b_∞ corresponds to the part of γ with $|x| = R$. Then $b_\infty = \Delta^2$ where $\Delta = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$ is the Garside element of the braid group B_4. It is easy to check that $b_{\mathbf{R}}$ is determined by the arrangement of the curve $f(x,y) = 0$ on the real plane (see [13]).

Proof of Theorem 2(1). Let the notation be as in Sect. 1.1. We consider the 2 cases (4) and (5) allowed by Corollary 2.3. Let us denote the braid corresponding to the curve C by b. In both cases we have $m = 4$, $e(b) = 2$, hence, by Corollary 2.6, we must have $\det \hat{b} = 0$.

Case 1. The sequence $i_1 \dots i_9$ has the form (4). Applying the procedure described in [13; Section 3.4], we obtain the braid (see Figure 12)[3]

$$b = \sigma_3^{-a_1}\sigma_2\sigma_3^{-1}\sigma_2^{-a_2}\sigma_3\sigma_2^{-1}\sigma_3^{-a_3}\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\Delta^2 \tag{6}$$

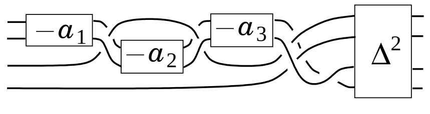


FIGURE 12

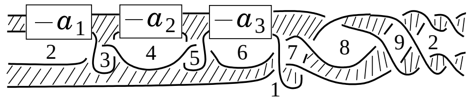


FIGURE 13

[3]More precisely, [13; Section 3.4] applied to the mirror image of C yields the mirror image of $b_{\mathbf{R}}$.

Let us transform $\hat{b}$ as in Figure 13. Then the Goeritz matrix (see [5]) is

$$\begin{pmatrix} -5 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 1 \\ 2 & a_1-2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & a_2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & a_3 & -1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

The numbers $1,\dots,9$ in Figure 13 indicate the order of rows and columns. The determinant of this matrix is

$$d = a_2(-4a_1a_3 + 6a_1 - 2a_3 + 7) + a_1 + a_3 + 1$$

One can check that the only integral non-negative solutions (a_1, a_2, a_3) of the equation $d = 0$ with $a_1 + a_2 + a_3 = 9$, are $(0, 1, 8)$, $(6, 1, 2)$, $(0, 5, 4)$, and $(2, 5, 2)$.

Case 2. The sequence $i_1 \dots i_9$ has the form (5). Similarly, we have (see Fig. 14)

$$b = \sigma_3^{-a_1}\sigma_2\sigma_3^{-1}\sigma_2^{-a_2}\sigma_1\sigma_2^{-1}\sigma_1^{-a_3-1}\Delta^2, \tag{7}$$

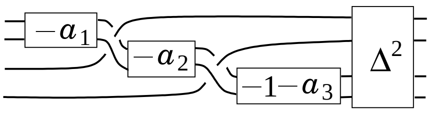


FIGURE 14

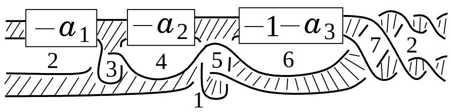


FIGURE 15

The Goeritz matrix (see Fig. 15) and its determinant are

$$\begin{pmatrix} -5 & 2 & 0 & 0 & 2 & 0 & 1 \\ 2 & a_1-2 & -1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & a_2-2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & a_3+3 & -1 \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$d = a_2(4a_1a_3 - 6a_1 - 6a_3 - 7) - a_1 - a_3 - 1$$

and the only integral non-negative solution of $d = 0$ with $a_1 + a_2 + a_3 = 9$ is $a_1 = 4$, $a_2 = 1$, $a_3 = 4$. Theorem 2(1) is proved.

Lemma 2.8. *Let C_0 be a real pseudo-holomorphic curve in $\mathcal{F}_2$ representing the homology class $[4H]$ in $H_2(\mathcal{F}_2)$. Suppose C_0 is as in Fig. 16 with $n_1 + n_2 + n_3 = 8$. Then (n_1, n_2, n_3) is one of $(2, 0, 6)$, $(2, 6, 0)$, $(5, 0, 3)$, $(5, 3, 0)$, $(6, 1, 1)$.*

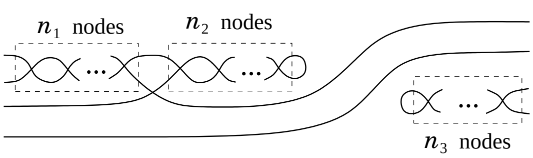


FIGURE 16

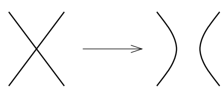


FIGURE 17

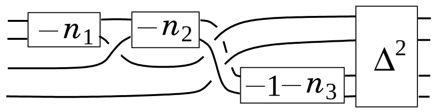


FIGURE 18

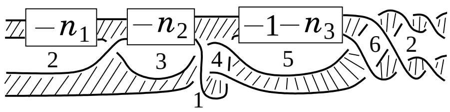


FIGURE 19

Proof. Like in the proof of Theorem 2(1), we have $m = 4$, $e(b) = 2$, hence, by Corollary 2.6, the existence of the flexible curve implies $\det \hat{b} = 0$. The braid is (see Fig. 18) $b = \sigma_3^{-n_1}\sigma_2^{-1}\sigma_3^{-n_2}\sigma_2\sigma_3^{-1}\sigma_1\sigma_2^{-1}\sigma_1^{-n_3-1}\Delta^2$. The Goeritz matrix (see Fig. 19) and its determinant are

$$\begin{pmatrix} -5 & 2 & 0 & 2 & 0 & 1 \\ 2 & n_1-4 & 1 & 0 & 0 & -1 \\ 0 & 1 & n_2 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & n_3+3 & -1 \\ 1 & -1 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$d = n_1n_2 + n_1n_3 + 4n_2n_3 - 2n_1 - n_2 - n_3 - 2$$

The only integral non-negative solutions (n_1, n_2, n_3) of the equation $d = 0$ with $n_1 + n_2 + n_3 = 8$, are $(2, 0, 6)$, $(2, 6, 0)$, $(5, 0, 3)$, $(5, 3, 0)$, and $(6, 1, 1)$.

Note, that in the cases $(n_1, n_2, n_3) = (5, 0, 3)$ or $(5, 3, 0)$, the normalization of the curve has two real components and the curve has 9 nodes. Thus, by genus formula, the curve is reducible: $4H = 3H + H$.

Corollary 2.9. *Let C_0 be a real pseudo-holomorphic curve in $\mathcal{F}_2$ representing the homology class $[4H]$ in $H_2(\mathcal{F}_2)$. Suppose C_0 is obtained from Fig. 16 with $n_1+n_2+n_3 = 8$ either by adding another oval or by smoothing of any number of nodes as in Figure 17. Then (n_1, n_2, n_3) is one of $(2, 0, 6)$, $(2, 6, 0)$, $(5, 0, 3)$, $(5, 3, 0)$, $(6, 1, 1)$.*

Proof. Smoothings as in Figure 17 do not change the braid. Adding another oval, we replace a braid b by a braid $b' = b_1\sigma_j^{-1}b_2$ where $b_1b_2 = b$. Clearly, the quasipositivity of b' implies the quasipositivity of b. □

Table 1. The braid b of the form (7)

a_1	a_2	a_3	$p(t)$	$q(u)$
7	1	0	$\Phi_6(t)\cdot\ldots$	
6	1	1	$\Phi_6(t)\cdot\ldots$	
5	1	2	$t^6 - 3t^5 + 3t^4 - 3t^3 + \ldots$	$u^6 + 3u^4 - 45u^2 + 17$
4	4	0	$\Phi_6(t)\cdot\ldots$	
2	4	2	$2t^6 - 8t^5 + 14t^4 - 17t^3 + \ldots$	$u^6 + 3u^4 - 61u^2 + 65$
3	5	0	$\Phi_6(t)\cdot\ldots$	
2	5	1	$t^8 - 5t^7 + 12t^6 - 17t^5 + 19t^4 - \ldots$	$u^8 - 4u^6 + 14u^4 - 4u^2 + 89$

Proof of Theorem 4. As in the proof of Theorem 2(1), we consider the 2 cases (4) and (5) allowed by Corollary 2.3 for the sequence $i_1 \ldots i_8$. The corresponding braids are (6) and (7) respectively with $a_1 + a_2 + a_3 = 8$. By Proposition 2.4, we have $a_2 = 0, 1, 4, 5$, or 8 in the both cases. We have $m - e(b) = 4 - 3 = 1$.

Case 1. The sequence (4). Here we suppose $a_3 \neq 0$. The only values of (a_1, a_2, a_3) which are allowed by Proposition 2.4 and are not realizable by flexible curves are $(1,1,6)$, $(2,1,5)$, $(3,1,4)$, $(4,1,3)$, and $(1,4,3)$. In all these cases, the braid b of the form (6) has $\operatorname{sign}_{-1}(b) = -2$, $\operatorname{null}_{-1}(b) = 0$ which contradicts Proposition 2.5.

Case 2. The sequence (5). By symmetry, we may suppose $a_1 \geq a_3$. For any braid b of the form (7) with $a_1 + a_2 + a_3 = 8$, $a_2 = 0, 1, 4, 5$ we have $\operatorname{sign}_{-1}(\hat{b}) = \operatorname{null}_{-1}(b) = 0$. For these braids (except the cases $(a_1, 0, a_3)$, $(4,1,3)$, and $(3,4,1)$ where a flexible curve exists) we compute the factorization over $\mathbf{Z}$ of the Alexander polynomial $p(t)$ of $\hat{b}$ and obtain a contradiction with Corollary 2.7. To check that $p(t)$ has a simple root on the unit circle (when there is no cyclotomic factor $\Phi_k(t)$ of $p(t)$), we substitute $t = (u+i)/(u-i)$. Then $p(t) = q(u)/(u-i)^n$ where $\deg q = n := \deg p$ and the real roots of q correspond to the roots of p on the unit circle. The results of the computations are listed in Table 1. □

Lemma 2.10. *Let C_0 be a real pseudo-holomorphic curve in $\mathcal{F}_2$ representing the homology class $[4H]$ in $H_2(\mathcal{F}_2)$. Suppose C_0 has a single singularity which is an ordinary double point. Then the set of real points of C_0 cannot be obtained from Fig. 4(d) by a contraction of one of the dashed segments in Fig. 20 as it is shown Fig. 21.*

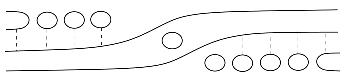


FIGURE 20

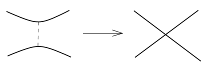


FIGURE 21

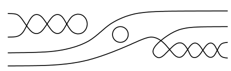


FIGURE 22

Remark. The transformation in Fig. 21 applied to the 4th oval provides the arrangement in Fig. 22. This arrangement is realizable by a pseudo-holomorphic curve.

Proof. The proof is similar to that of Theorem 2(1) and Lemma 2.8. The braid is $b = \sigma_3^{-n_1}\sigma_2^{-1}\sigma_3^{-n_2}\sigma_2\sigma_3^{-1}\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1^{-a_3-1}\Delta^2$. with $n_1 + n_2 = a_1$, and $(a_1, a_2) = (3,4)$ or $(4,3)$. The determinant of the Göritz matrix is

$$d = -7 - 3n_2 - 5n_1 + 7n_2n_1 - 3a_3 - 4a_3n_2n_1 + 16a_3n_2 + 3a_3n_1.$$

The only solution of $d = 0$ is $(n_1, n_2, a_3) = (4, 0, 3)$. It corresponds to Figure 22. □

2.4. Trigonal curves.

Proposition 2.11. *Let C be a real pseudo-holomorphic curve in $\mathcal{F}_2$ representing the homology class $[3H + 2F]$ in $H_2(\mathcal{F}_2)$ which has a real connected component cutting E twice as in Figure 23 and 8 ovals in $\mathbf{R}\mathcal{F}_2 \setminus E$. Then the ovals are arranged with respect to the pencil of vertical lines as in Figure 23 with $(a_1, a_2, a_3) = (1,0,7)$, $(1,4,3)$, $(3,4,1)$, or $(7,0,1)$.*

All these arrangements are realizable by algebraic curves.

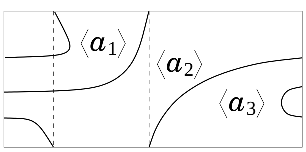


FIGURE 23

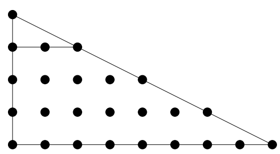


FIGURE 25

This fact can be proved by methods of [14]. The details are to be published in a forthcoming paper of the first author.

Corollary 2.12. *There exist algebraic curves on $\mathcal{F}_2$ arranged as in Figure 4(a,c).*

Proof. We use Viro's method cutting the triangle $(0,0)$-$(8,0)$-$(0,4)$ along the segments $(0,3)$-$(2,3)$ (see Figure 25). The quadrangle $(0,0)$-$(8,0)$-$(2,3)$-$(0,3)$ corresponds to the curve in Figure 23. □

2.5. Affine sextic.

Let C be a real pseudo-holomorphic curve of degree 6 in $\mathbf{CP}^2$ arranged with respect to a line L as in Figure 7. Let q be a point inside the empty oval which is the most remote from L among $\langle 5\rangle$. Denote the pencil of (pseudo-holomorphic) lines through q by $\mathcal{L}_q$. It is shown in [13; Section 5.4] that C and L are arranged with respect to $\mathcal{L}_q$ as in Figure 24 where $\mathcal{L}_q$ is the pencil of vertical lines and the double points should be smoothed as in Figure 17.

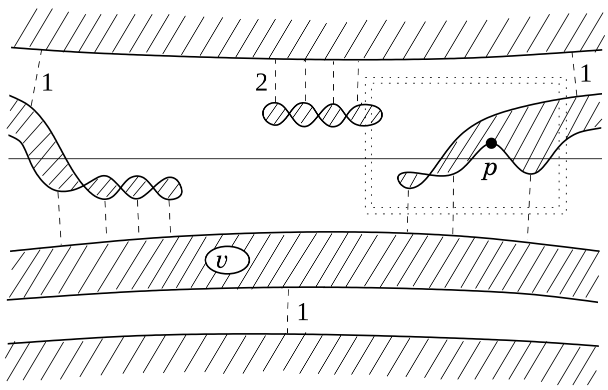


FIGURE 24

Lemma 2.13. *Let C be a real pseudo-holomorphic curve of degree* 6 *on* $\mathbf{CP}^2$ *whose arrangement with respect to a line L and the pencil of vertical lines is obtained from Figure 24 by*

- *a contraction of one of the dashed segments in Figure 24 as it is shown in Figure 21.*

and then, maybe, followed by

- *removing the interior empty oval (the oval v in Figure 24);*
- *smoothing of any number of nodes as in Figure 17.*

Then the only dashed segment which can be contracted is indicated by 2 *in Figure 24.*

Proof. We shall show that if the contracted dashed segment is not 2 then the braid constructed as in Section 2.3 is not quasipositive. Note that the smoothing in Figure 17 does not change the braid. We may consider only the case when the oval v is removed (if we insert this oval then σ_3^{-1} is inserted somewhere inside the braid word).

For each dashed segment we write the braid b. A computation shows that for all of them except the segment 2 we have $e(b) = 2$ and $\det \hat{b} \neq 0$ which contradicts Corollary 2.6. □

Remarks. **1.** All the segments indicated by 1 yield the same braid.

2. If the contracted dashed segment is outside the indicated rectangle then the piece of $C \cup L$ inside the rectangle can be replaced by three tangent branches. Hence, in these cases Lemma 2.13 follows from Lemma 2.8 (see Remark in Section 1.3).

3. The Hilbert-Rohn-Gudkov approach

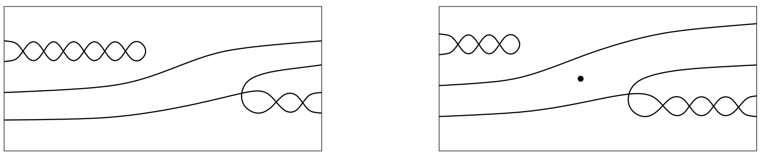
The restriction part of Theorems 1 and 3 follows from the restriction part of Theorems 2 and 4 (proved in Section 1) and Lemmas 3.1 and 3.2 below.

Lemma 3.1. *The M-curve shown in Figure 3 cannot be realized by algebraic curves (1).*

Lemma 3.2. *The $(M-1)$-curves shown in Figure 5 cannot be realized by algebraic curves (1).*

As we mentioned in Introduction, the algebraic realizability of the curves in Figures 4(b) and 4(d) is open. However we reduce the problem to the existence of certain rational curves we hope are simpler to be excluded.

Lemma 3.3. *If there exists an $(M-1)$-curve shown in Figure 4(b) (resp. 4(d)) then there exists a rational real curve in the linear system $|4H|$ having 9 ordinary nodes and shown in Figure 26(a) (resp 26(b)).*



a). b).

Figure 26

The proof is highly inspired by the Hilbert-Rohn-Gudkov method [16, 17, 10].

The strategy is as follows. Assume that an algebraic curve (1) as depicted in Figure 3, 4, or 5 does exist. Then, moving along straight lines in $|4H| = \mathbf{P}^{24}$, we degenerate the original curve in a certain curve with 8 nodes, then deform it, moving along a straight line in $|4H|$ so that some characteristic of a curve changes in a strongly monotone way, thus, we cannot come back without some degeneration. In Lemmas 3.1 and 3.2, the results of the previous section show that no degeneration can occur, which leads to the contradiction. In Lemma 3.3, the only degenerations allowed by the results of the Section 2 is shown in Figure 26.

Remark. An isotopy classification of algebraic M- and $(M-1)$-curves (1) was announced in [19]. The proof, based purely on the Hilbert-Rohn-Gudkov method, has never been published. The present paper reproves all the results announced in [19], concerning the series $A(a, b, c)$ (in the notation of [19]), with the two exceptions:

(1) the arrangement shown in Figure 4(c), erroneously claimed to be prohibited, is, in fact, realizable by algebraic curves (Korchagin, unpublished), here we provide a construction in section 4;

(2) the existence of an algebraic curve with the arrangement shown in Figure 4(d), mentioned in Lemma 3.3, is open.

3.1. Preparatory lemmas.

Lemma 3.4. *Let $C \in |4H|$ be an irreducible curve with $r \geq 1$ nodes as its only singularities. Then by a small equisingular variation of C in $|4H|$ one can move prescribed $m = \min\{r, 7\}$ nodes in general position.*

Proof. Applying the argument of Theorems 2, 3 [8], or Theorem 1 [20] to curves on $\mathcal{F}_2$, one reduces the statement to the non-specialty of the linear series on C cut out by the linear system of curves in $|4H|$ having singularities at the chosen m nodes of C and passing through the rest of singular points of C, and the required non-specialty in turn follows from $2m < -C \cdot K_{\mathcal{F}_2}$, where m is the number of arbitrarily movable nodes, $K_{\mathcal{F}_2}$ is a canonical divisor of $\mathcal{F}_2$. Here $m \leq 7$, $K_{\mathcal{F}_2} = -2H$, hence $2m \leq 14 < -C \cdot K_{\mathcal{F}_2} = 8H^2 = 16$.

Lemma 3.5. *(1) Let $D \sim kF + lE$ be an effective divisor on $\mathcal{F}_2$ which does not contain E. Then $k \geq 2l \geq 0$ and*

$$\dim |D| = \frac{D^2 - DK_{\mathcal{F}_2}}{2} = kl - l^2 + k \ .$$

A reduced element of $|D|$ has at most $kl - l^2 - l$ singular points in $\mathcal{F}_2 \backslash E$.

(2) Under the above assumption, let $z_1, ..., z_m$, $z_{m+1}, ..., z_{m+r}$ ($m \geq 0$, $r \geq 0$) be fixed generic points in $\mathcal{F}_2$. If $m \leq kl - l^2 - l$ then the dimension of the linear system $|D|_{m,r}$ of curves $C \in |D|$ having singularities at $z_1, ..., z_m$ and passing through $z_{m+1}, ..., z_{m+r}$ is $\min\{\dim |D| - 3m - r, \ -1\}$.

(3) Under the above assumption, the set of irreducible curves $C \in |D|_{m,r}$ with $n \geq m$ singular points and the total Tjurina number τ is empty or has codimension $\geq \min\{\tau - m, \ 2k - 1 + n - m\}$.

Proof. (1) First, $l = FD \geq 0$. Since $D \not\supset E$, $DE = k - 2l \geq 0$. The linear systems $|F|$ and $|H|$ have no basepoint, hence $|D| = |(k - 2l)F + lH|$ has no basepoint, and by Bertini's theorem a generic member of $|D|$, say D itself, in non-singular. Since $DK_{\mathcal{F}_2} = 2k > 0$, we have $D^2 > D^2 + DK_{\mathcal{F}_2} = 2g(D) - 2$, hence $h^1(D, \mathcal{O}_D \otimes \mathcal{O}_{\mathcal{F}_2}(D)) = 0$, which implies by Riemann-Roch

$$\dim |D| = h^0(\mathcal{F}_2, \mathcal{O}_{\mathcal{F}_2}(D)) - 1 = h^0(D, \mathcal{O}_D \otimes \mathcal{O}_{\mathcal{F}_2}(D))$$

$$= D^2 - g(D) + 1 = \frac{D^2 - DK_{\mathcal{F}_2}}{2} = kl - l^2 + k \ .$$

At last, it is easy to see that the maximal possible number of singular points has a (reduced) curve $C \in |D|$ consisting of l distinct generic irreducible curves from $|H|$ and $k-2l$ distinct generic fibers, and this curve has in general $(k-2)l + 2 \cdot l(l-1)/2 = kl - l^2 - l$ nodes.

(2) The statement for $r > 0$ follows easily from that for $r = 0$, so we assume that $r = 0$. Note also that $\dim |D|_{m,0} \geq \dim |D| - 3m$, because m singular points impose $3m$ linear conditions on elements of $|D|$.

If $D \in |H|$, then $\dim |D| = 3$ and, given a point $z_1 \in \mathcal{F}_2 \backslash E$, the only curve $C \in |D|$, singular at z_1, is that consisting of E and the double fiber through z_1.

Consider the nodal curve $C \in |D|$ constructed above. Since $C'K_{\mathcal{F}_2} < 0$ for any component C' of C, it is classically known (see, for example, [6] for a modern exposition) that one can smooth out some nodes of C and obtain a curve $C_1 \in |D|$ having exactly m nodes in $\mathcal{F}_2 \backslash E$. By the same argument the germ V_m at C_1 of the set of curves in D, having m nodes, is smooth of dimension $\dim |D| - m$. Looking at the natural projection $V_m \to (\mathcal{F}_2)^m$, we obtain that the dimension of $|D|_{m,0}$ (if nonempty) is at most $\dim |D| - 3m$, hence is equal to $\dim |D| - 3m$ as noticed above.

(3) Without loss or generality suppose that $r = 0$. Let $w_1, ..., w_n$ be singular points of C and let $V \subset |D|$ be a germ at C of the set of curves with n singular points analytically equivalent to the respective singular points of C. The Zariski tangent space to V at D is $H^0(C, \mathcal{J}_{X/C} \otimes \mathcal{O}_{\mathcal{F}_2}(D))$, where $\mathcal{J}_{X/C}$ is the ideal sheaf of the following zero-dimensional scheme $X \subset C$: X is concentrated at $w_1, ..., w_n$ and defined at each point w_i by an ideal $\langle f, f_x, f_y \rangle \subset \mathcal{O}_{D,w_i}$ with $f(x, y) = 0$ being an equation of C in local coordinates x, y. The degree of X is the total Tjurina number τ. By [6], for any subscheme $X' \subset X$ containing all the points $w_1, ..., w_n$ and satisfying $\deg X' - n < -CK_{\mathcal{F}_2} = 2k$, it holds that

$$h^1(D, \mathcal{J}_{X'/C} \otimes \mathcal{O}_{\mathcal{F}_2}(D)) = 0 \ . \tag{8}$$

Take such a subscheme X' of degree $\min\{\tau,\ 2k - 1 + n\}$. One has

$$\dim V \leq h^0(C, \mathcal{J}_{X/C} \otimes \mathcal{O}_{\mathcal{F}_2}(D)) \leq h^0(C, \mathcal{J}_{X'/C} \otimes \mathcal{O}_{\mathcal{F}_2}(D)) \ ,$$

then (8) and the exact sequence

$$0 \to \mathcal{J}_{X'/C} \otimes \mathcal{O}_{\mathcal{F}_2}(D) \to \mathcal{O}_C \otimes \mathcal{O}_{\mathcal{F}_2}(D) \to \mathcal{O}_{X'} \to 0$$

imply

$$h^0(C, \mathcal{J}_{X'/C} \otimes \mathcal{O}_{\mathcal{F}_2}(D)) = h^0(C, \mathcal{O}_C \otimes \mathcal{O}_{\mathcal{F}_2}(D)) - \deg X'$$

$$= \dim |D| - \min\{\tau,\ 2k - 1 + n\} \ ,$$

hence $\dim V \leq \dim |D| - \min\{\tau,\ 2k - 1 + n\}$. Projecting V to $(\mathcal{F}_2)^m$ by $C \mapsto (w_1, ..., w_m)$, we obtain that

$$\dim(V \cap |D|_{m,0}) \leq \dim |D| - \min\{\tau,\ 2k - 1 + n\} - 2m$$

$$= \dim |D|_{m,0} - \min\{\tau - m,\ 2k - 1 + n - m\} \ .$$

Remark. $\mathcal{F}_2$ can be considered as the toric surface associated to the fan generated by the vectors $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(-2, -1)$. The corresponding coordinate charts are described in Introduction. The linear system $|D|$ can be identified with the set the set of all polynomials of the form $\sum a_{ij} x^i y^j$ with (i, j) belonging to the convex hull of the points $(0, 0)$, $(k, 0)$, $(k - 2l, l)$, $(0, l)$. The fact that $E \not\subset D$ means that $a_{i,l} \neq 0$ for some i (this is another proof of the part (1) of Lemma 3.5).

Lemma 3.6. *In the notation of Lemma 3.5, let $D = 4H$ and $m \le 7$. Consider curves in the linear system $|4H|_{m,0}$. Then*

1. *a generic member of $|4H|_{m,0}$ is irreducible curve with nodes at $z_1, ..., z_m$ as its only singularities;*
2. *the set of irreducible curves with the total Tjurina number $\ge m + s$, has codimension $\ge s$ for every $s = 1, 2, 3$;*
3. *the set of non-reduced curves has codimension ≥ 3;*
4. *the set of reducible curves which do not contain E has codimension ≥ 3, except for the following case: $m = 6$ or 7, $V \subset |4H|_{m,0}$ consists of curves $C' + C''$, where $C', C'' \in |4F + 2E|$ pass through $z_1, ..., z_m$, and $\dim V = \dim |4H|_{m,0} - (8 - m)$;*
5. *for any $m = 0, ..., 7$, the set of curves containing E has codimension 1, a generic member of this set is $E + G$, where $G \in |8F + 3E|$ is an irreducible curve having nodes at $z_1, ..., z_m$ as its only singularities and intersecting E transversally at two points; the set of curves $E + G \in |4H|_{m,0}$ with reducible G has codimension ≥ 3 in $|4H|_{m,0}$.*

Proof. A curve $C \in |4H|$ which is the union of generic members of $|H|$ has 12 nodes. Smoothing out $12 - m$ of these nodes (which is possible according to [6], as noticed in the proof of Lemma 3.5), one obtains an irreducible curve in $|4H|$ with m nodes in $\mathcal{F}_2 \backslash E$ as its only singularities, hence a generic member of $|4H|_{m,0}$ is irreducible with m nodes as its only singularities.

The second statement follows immediately from Lemma 3.5(3).

Let $C \in |4H|_{m,0}$ be a generic curve of some component V of the set of non-integral curves. Then $C = 2C_1 + C_2 + lE$, where $C_1 \in |k_1 F + l_1 E|$, $C_2 \in |k_2 F + l_2 E|$ are reduced curves with no component in common, $k_1 \ge 2l_1$, $k_2 \ge l_2$, $l_3 \ge 0$, $2k_1 + k_2 = 8$, $2l_1 + l_2 + l_3 = 4$. Assume that C_1 passes through r points among $z_1, ..., z_m$. Then C_2 is singular at the other $m - r$ points among $z_1, ..., z_m$. Then by Lemma 3.5(2)

$$\dim V = \dim |k_1 F + l_1 E|_{0,r} + \dim |k_2 F + l_2 E|_{m-r,0}$$

$$= k_1 l_1 - l_1^2 + k_1 - r + k_2 l_2 - l_2^2 + k_2 - 3(m - r) .$$

A computation gives $\dim V \le \dim |4H|_{m,0} - 3 = 22 - 3m$ in all the cases, but two ones indicated in the assertion of Lemma, when the codimension is ≤ 2. For a generic curve $C = C_2 + E \in |4H|_{m,0}$, the curve $C_2 \in |8F + 3E|_{m,0}$ is reduced, and as above one can show that C_2 is irreducible with nodes at $z_1, ..., z_m$ as its only singularities. Moreover, the construction in the proof of Lemma 3.5 gives C_2 intersecting E transversally at two distinct points.

3.2. Proof of Lemma 3.1. Assume that there exists a real nonsingular curve $C \in |4H|$ with 10 ovals located as shown in Figure 3. We shall construct a continuous deformations of the curve C into a nodal curve depicted in Figure 27, adding nodes one-by-one. Then we prove that the latter curve does not exist.

Denote by Δ the closure of the component of $\mathbf{R}\mathcal{F}_2 \backslash C$ bounded by the nonempty oval of C and the empty oval, lying inside the nonempty one. If $C(t)$, $t \in \mathbf{R}$, is a deformation of $C = C(0)$ and the curves $C(t)$ are isotopic to C then by $\Delta(t)$ we denote the corresponding deformation of the domain Δ. If $C(t^*)$ is a degeneration of $C(t)$, $t \to t^*$, then by $\Delta(t^*)$ we denote the limit of $\Delta(t)$ as $t \to t^*$.

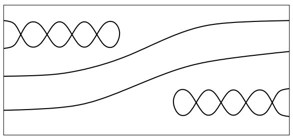


FIGURE 27

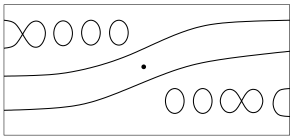


FIGURE 28

Given a real nodal curve, by *pseudo-ovals* we call its subsets homeomorphic to circle.

Step 1. Let $0 \le m \le 7$ and $C^{(m)} \in |4H|$ be a real irreducible curve with m nodes and $10 - m$ connected components and such that by an appropriate smoothing of the nodes one can obtain a curve C with 10 ovals located in $\mathbf{R}\mathcal{F}_2$ as shown in Figure 3. Suppose also that $C^{(m)}$ is *tame* in the following sense:

(1) if there is a node inside $\Delta^{(m)}$ then it is a *solitary point* (i.e. an isolated point of $\mathbf{R}C^{(m)}$);
(2) the nodes outside $\operatorname{Int}\Delta^{(m)}$ are intersections of two real local branches, and they join two empty pseudo-ovals outside $\Delta^{(m)}$, or an empty pseudo-oval outside $\Delta^{(m)}$ with the pseudo-ovals $\Delta^{(m)}$ in the way shown in Figure 28.

Clearly, $C = C^{(0)}$ satisfies the above conditions.

By Lemma 3.4 we can suppose that the nodes $z_1, ..., z_m$ of $C^{(m)}$ are in general position in $\mathcal{F}_2$. Since $\dim|2H| = 8$, there exists a curve in $|2H|$ passing through $z_1, ..., z_m$. Take a generic curve $G^{(m)} \in |2H|_{0,m}$ and consider the real straight line Λ_m in $|4H|$ through $C^{(m)}$ and $2G^{(m)}$. Curves in Λ_m which are close to $C^{(m)}$ have nodes at $z_1, ..., z_m$ as well and they are isotopic to $C^{(m)}$. Let t be a coordinate on $\Lambda_m \backslash (2G^{(m)})$ such that $C(0) = C^{(m)}$, $\lim_{t\to\infty} C(t) = 2G^{(m)}$, and the domain $\Delta(t)$ (bounded from outside by the nonempty pseudo-oval of $C(t)$) grows as t grows from 0 to the minimal positive t^* such that $C(t^*)$ is not isotopic to $C^{(m)}$ in $\mathbf{R}\mathcal{F}_2$.

We claim that $t^* < \infty$. Indeed, assume that all the curves $C(t)$, $t > 0$, are isotopic. Take a generic fiber $\widetilde{F}$ which crosses the nonempty pseudo-oval of $C^{(m)}$ at four points. The assumption made means that pairs of these four points approach two point $G^{(m)} \cap \widetilde{F} \subset \widetilde{F} \backslash E$. However, since $\Delta(t)$ grows, two of the points in $C(t) \cap \widetilde{F}$ approach each other, but the other two approach E from the opposite sides, contradicting the last assumption.

By a small variation of $C^{(m)}$ in $|4H|_{m,0}$ we can obtain that $C(t^*)$ is a generic element of a family in $|4H|_{m,0}$ of codimension 1. By Lemma 3.6 one has four possibilities, either $C(t^*)$ is reduced irreducible with $m + 1$ real nodes as its only singularities, or $m \ge 1$ and $C(t^*)$ is irreducible with nodes at all $z_1, ..., z_m$ but one, where $C(t^*)$ has a cusp, or $C(t^*) = C_2 + E$, $C_2 \in |8F + 3E|$ as described in Lemma 3.6, or $m = 7$ and $C(t^*) = G' + G''$, where $G', G'' \in |2H|_{0,7}$.

The case $C(t^*) = C_2 + E$ is impossible. Indeed, if $C(t^*) = C_2 + E$ then exactly one of the pseudo-ovals of $C(t)$, $t \to t^* - 0$, approaches E (if there were two such pseudo-ovals we would have ≥ 4 intersection points of C_2 with E), and this pseudo-oval is the nonempty one, that we can easily see from the behavior of the intersection points of $C(t)$ with $\widetilde{F}$. Hence C_2 should have at least $10-m$ connected components and m nodes what is impossible because by Harnack's inequality the number of connected components does not exceed $g(C_2) + 1 = (C_2^2 + C_2 K_{\mathcal{F}_2})/2 + 2 - m = 9 - m$.

The case $m = 7$ and $C(t^*) = G' + G''$, $G', G'' \in |2H|_{0,7}$, is impossible as well.

Indeed, $\dim|2H|_{0,7} = 1$, hence the curves of type $G' + G''$, $G', G'' \in |2H|_{0,7}$, form a plane in the three-space $|4H|_{7,0}$ passing through $2G_1$, $2G_2$, $G_1 + G_2$, where $G_1 \neq G_2 \in |2H|_{0,7}$. However, $\Lambda_7 \backslash \{2G^{(7)}\}$ does not meet this plane, because $C^{(m)}$ is not there.

At last, for $m \geq 1$, a cusp at z_i, $1 \leq i \leq 7$, cannot appear, since no any loop of $C(t)$ contracts to z_i as t grows.

The remaining case is $C(t^*) = C^{(m+1)}$, an irreducible curve with $m + 1$ real nodes, which must be tame by Corollary 2.9.

Step 2. The procedure performed in Step 1 gives a real irreducible tame curve $C^{(8)}$ with 8 real nodes $z_1, ..., z_8$ and two connected components, which can be smoothed into a nonsingular curve depicted in Figure 3. By Lemma 3.4, the nodes $z_1, ..., z_7$ can be moved in general position, thus, as explained in the proof of Lemma 3.6, the set V_8 of curves with 8 nodes is a smooth two-dimensional variety in the three-space $|4H|_{7,0}$. We assume $C^{(8)}$ to be a generic element of this variety. Due to $\dim|2H| = 8$, there exists $G^{(8)} \in |2H|$ passing through $z_1, ..., z_8$. As in the previous step, we introduce the line $\Lambda_8 \subset |4H|_{7,0}$ through $C^{(8)}$ and $2G^{(8)}$, an appropriate coordinate t on this line, and consider the curve $C(t^*)$ which is not isotopic to $C^{(8)}$ and corresponds to the minimal such positive t^*.

Since V_8 is two-dimensional and almost all elements of Λ_8 belong to V_8, the curve $C(t^*)$ must be a generic element of an one- or two-dimensional family in $|4H|_{7,0}$. By Lemma 3.6 we have four possibilities, either $C(t^*)$ is reduced irreducible with 9 real nodes as its only singularities, or $C(t^*)$ is irreducible with nodes at all $z_1, ..., z_8$ but one, where $C(t^*)$ has a cusp, or $C(t^*) = C_2 + E$, where $C_2 \in |8F + 3E|$ is reduced irreducible, or $C(t^*) = G' + G''$, where $G', G'' \in |2H|_{0,7}$.

As in Step 1, one prohibits the latter three possibilities. Hence $C(t^*)$ is irreducible with 9 nodes and is tame by Corollary 2.9. Moving a little along the straight line in $|4H|$ spanned by $C(t^*)$ and $2G$, where $G \in |2H|$ passes through all 8 non-isolated nodes of $C(t^*)$, we deform $C(t^*)$ so that the single point inside $\Delta(t^*)$ disappear, whereas all other nodes persist, thus, we obtain a curve $\widetilde{C}$ with 8 nodes as shown in Figure 27.

Step 3. Let $z_1, ..., z_8$ be the nodes of $\widetilde{C}$, seven of them in general position, and let $\widetilde{C}$ be a generic element of the variety $V_8 \subset |4H|_{7,0}$ of curves with 8 nodes. Take a curve $G \in |2H|$ through $z_1, ..., z_8$. It is irreducible by Lemma 3.6. Consider the straight line $\Lambda \subset |4H|_{7,0}$ through $\widetilde{C}$ and $2G$ and choose a coordinate t on $\Lambda \backslash \{2G\}$ as it was done in the previous steps.

As above one shows that the curves $C(t)$, $t > 0$, cannot be isotopic to each other, hence there exists the minimal positive t^* such that $C(t^*)$ is not isotopic to $\widetilde{C}$. The argument used in Step 2 leaves two possibilities for $C(t^*)$, either an irreducible curve with 9 nodes, or a curve of type $C_2 + E$, where $C_2 \in |8F + 3E|$ is reduced irreducible.

The case $C(t^*) = C_2 + E$ is impossible by Proposition 2.11.

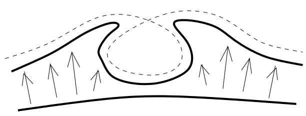


FIGURE 29

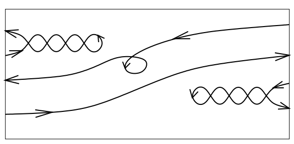


FIGURE 30

The case of an irreducible curve $C(t^*)$ is impossible as well. Indeed, Bezout's theorem allows three possibilities for the 9th node

(1) either this node is non-isolated and joins two pseudo-ovals as $t \to t^* - 0$, but such a situation is prohibited in Corollary 2.9;
(2) or this node is a single point outside $\Delta(t^*)$, which is impossible, because smoothing all the nodes, we can obtain an M-curve in $|4H|$ with 10 ovals outside each other what contradicts Proposition 2.4;
(3) or this node is non-isolated and appears as in Figure 29. This is impossible as well by the following reason. Being rational, such a curve has a complex orientation (coming from one of the components of $C(t^*) \setminus \mathbf{R}C(t^*)$), say, as shown in Figure 30. By [2], smoothing out the nodes in accordance with the complex orientation, we obtain an M-curve shown in Figure 3 with the induced complex orientation which contradicts Fiedler's theorem on alternation of complex orientations in the pencil of fibers [3]: the induced orientation of the empty odd oval does not agree with the orientation of neighboring empty even ovals.

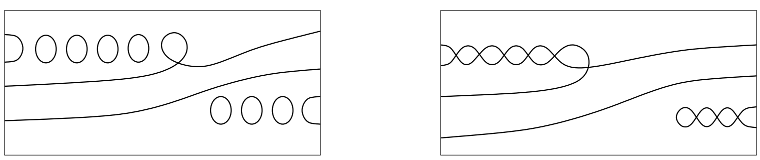
The proof is completed.

3.3. Proof of Lemma 3.2.

(1) Assume that there exists a real nonsingular curve $C \in |4H|$ with 9 ovals located as shown in Figure 5(a). We perform the procedure described in Step 1 in section 3.2. We claim that as result one necessarily obtains an irreducible curve with 8 nodes as its only singularities, shown in Figure 27.

Indeed, on each stage of the procedure performed one obtains a degenerate curve $C(t^*)$, and then one excludes all degenerations but $C(t^*) = C^{(m+1)}$. The degeneration $C(t^*) = C_2 + E$, $C_2 \in |8F + 3E|$, is forbidden by Proposition 2.11, degenerations to $C' + C''$, $C', C'' \in |2H|$, or to a cuspidal curve can be prohibited as described in Step 1, section 3.2. Now we notice that the nodes which appear in $C^{(8)}$ cannot join a pseudo-oval contained in the boundary of the domain Δ as shown in Figure 30, because such a curve is deformed into an M-curve prohibited in section 3.2. By Corollary 2.9 the curve $C^{(8)}$ must be tame which leaves the only possibility shown in Figure 27, but such a curve is prohibited in Step 3, section 3.2.

(2) Assume that there exists a real nonsingular curve $C \in |4H|$ with 9 ovals located as shown in Figure 5(b), and then perform the procedure of Step 1, section 3.2. Analyzing possible degenerations as is done above, and using the fact that there is no M-curve obtained from that in Figure 5(b) by adding oval, and using Proposition 2.11, Corollary 2.9, one obtains an irreducible curve $C^{(8)}$ with 8 nodes, which connect pseudo-ovals outside domain Δ in the way shown in Figure 28, ore connect the pseudo-oval, bounding Δ, with another pseudo-oval shown in Figure 31(a).



a). b).

FIGURE 31

Then we deform $C^{(8)}$ along the line $\Lambda \subset |4H|_{7,0}$ through $\widetilde{C}$ and $2G$, where $G \in |2H|$ passes through all the 8 nodes of $C^{(8)}$, as done in Step 3, section 3.2. Reasoning as in Step 3, section 3.2, one shows that there must be a degeneration $C^{(9)}$. The only possible shape for $C^{(9)}$ is depicted in Figure 31(b), according to Corollary 2.9. However, the curve shown in Figure 31(b) has 2 global real branches and 9 nodes which exceeds $10 = p_a + 1$, where p_a is the arithmetic genus of $C^{(9)}$. Hence $C^{(9)}$ is reducible and splits as $C^{(9)} = C_1 + C_2$, $C_1 \in |H|$, $C_2 \in |3H|$, what is forbidden by Lemma 3.6: by construction $C^{(9)}$ is a generic element of a family of codimension 2 in $|4H|_{7,0}$, whereas $C_1 + C_2$ belongs to a family of codimension ≥ 3.

O. ne can derive the latter claim as follows. The curve $C_1 \in |H|$ passes through 6 nodes of $C^{(9)}$, among them 5 nodes are in general position by construction (see Step 2, section 3.2), which leads to contradiction, since $\dim |H| = 3$.

3.4. Proof of Lemma 3.3.

Assume that there exists a real nonsingular curve $C \in |4H|$ with 9 ovals located as shown in Figure 4(b). We perform the procedure described in Step 1 in section 3.2 and, according to the argument in 3.2, 3.3, obtain a curve $C^{(8)}$ with 8 nodes. Then we deform $C^{(8)}$ along the line $\Lambda \subset |4H|_{7,0}$ through $\widetilde{C}$ and $2G$, where $G \in |2H|$ passes through all the 8 nodes of $C^{(8)}$, as done in Step 3, section 3.2. Reasoning as in Step 3, section 3.2, one shows that there must be a degeneration $C^{(9)}$. The only possible shape for $C^{(9)}$ is depicted in Figure 26(a), according to Corollary 2.9.

If we assume that there exists a real nonsingular curve $C \in |4H|$ with 9 ovals located as shown in Figure 4(d), then the same reasoning leads to the rational curve depicted in Figure 26(b), when one uses Proposition 2.11 instead of Corollary 2.9.

3.5. Comment.

The argument presented in sections 3.2, 3.3, 3.4 reveals the main idea of the Hilbert-Rohn-Gudkov approach, which can be traced back to Hilbert and Rohn [16, 17]. Assuming the existence a hypothetical real curve, one tries to degenerate it and then shows that a degeneration coming necessarily on a certain stage cannot exist by some reason. Namely, first, one accumulates ordinary double points moving along segments of straight lines in the space of curves (see [16, 17, 10] with detailed setting for sextic curves, and section 3.2 above). Then one deforms the curve obtained along a one-dimensional equisingular stratum so that some quantitative characteristic of of the curve changes monotonically, hence one cannot move through a whole component of the equisingular stratum (which is homeomorphic to a circle) without further degenerations of the curve. The final step consists in an analysis of possible degenerations on the last stage and their prohibitions.

We should point out that Hilbert and Rohn did not give a strong justification for that final step, and this was Gudkov, who classified all degenerations and gave rigorous proofs of their nonexistence. This requires some development of the local singularity theory, but the basic thing is the following. The presence of specific singular points on an algebraic curve imposes a certain number of conditions, which is called the *virtual (expected) codimension* of the corresponding equisingular stratum in the space of curves considered. What Gudkov did for curves of degree 6 [8, 10] and what we did in section 3.1 was to show that the *true codimension* of the equisingular strata, which appear in the degeneration process, coincides with the virtual one, and that any degeneration has codimension 1 with respect to the previous one. This heavily relies upon the *general position* of singularities (see, for example, Lemma 3.4 as starting point in our reasoning), and as result reduces the amount of possible degenerations drastically.

In this connection we would like to comment the prohibition of the curve shown in Figure 5(b) (see section 3.3). The crucial point of the proof is the fact that such a curve cannot degenerate to $C_1 + C_2$, $C_1 \in |H|$, $C_2 \in |3H|$ (note that this degeneration does exist for real pseudo-holomorphic curves).

Among existing M- and $(M-1)$-curves C there are those obtained by smoothing of $C_1 + C_2$, $C_1 \in |H|$, $C_2 \in |3H|$. What happens if we apply the above argument to these curves? If we perform the degeneration process, presented in sections 3.2, 3.3, 3.4, to such an existing curve C, we cannot obtain the degeneration $C_1 + C_2$, because we keep the most of the nodes in general position, which is not the case for $C_1 + C_2$. So, the curve C necessarily leads to another degeneration, say, $E + C_2$, $C_2 \in |8F + 3E|$. But these degenerations are impossible for the curves considered in section 2 by topological reasons.

3.6. Application of the Hilbert-Rohn-Gudkov method to sextic curves. Our aim is to complete the proof of Theorem 5. The following statement accumulates some results of Gudkov [10]. It can also be proven similarly to Lemmas 3.4-3.6.

Lemma 3.8. *(1) Let $C \subset \mathbf{P}^2$ be an irreducible curve of degree 6 with $r \geq 1$ nodes as its only singularities. Then by a small equisingular variation of C in the space $\mathbf{P}^{27}$ of projective plane sextics one can move prescribed $m = \min\{r, 8\}$ nodes in general position.*

(2) The linear system $\mathbf{P}^{27}_m$ of sextics having singularities at generic points $z_1, ..., z_m \in \mathbf{P}^2$, $m \leq 8$, has dimension $27 - 3m$. Such a system satisfies the conditions:

- a generic member of $\mathbf{P}^{27}_m$ is an irreducible curve with nodes at $z_1, ..., z_m$ as its only singularities;

- the set of irreducible curves in $\mathbf{P}^{27}_m$ with the total Tjurina number $\geq m + s$, has codimension $\geq s$ for every $s = 1, 2, 3$;

- the set of non-reduced curves in $\mathbf{P}^{27}_m$ has codimension ≥ 3;

- the set of reducible curves which do not contain E has codimension ≥ 3, except for the following case: $m = 7$ or 8, $V \subset \mathbf{P}^{27}_{m,0}$ consists of curves $C'C''$, where C', C'' are cubics passing through $z_1, ..., z_m$, and $\dim V = \dim \mathbf{P}^{27}_{m,0} - (9 - m)$.

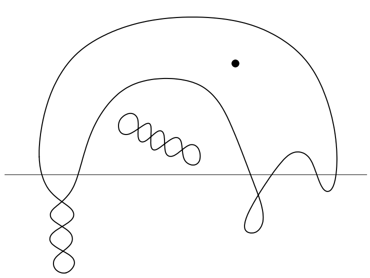


FIGURE 32

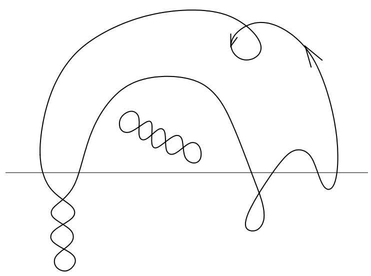


FIGURE 33

Now assume that there exists a real projective nonsingular sextic curve C with 11 ovals arranged with respect to the straight line L as shown in Figure 7. Fix a point p on the oval crossing L (see Figure 7). Next we apply the degeneration procedure similar to that in Step 1, Section 3.2. Namely, having an irreducible sextic $C^{(m)}$ with $m \le 8$ nodes, which can be smoothed into the curve in Figure 7, we consider the family of curves $C^{(m)} + tC_3^2$, $t \ge 0$, where C_3 is a cubic passing through p and the nodes of $C^{(m)}$, and we assume that the domain Δ bounded by the pseudo-oval crossing L grows as t increases. As in Section 3.2 one shows that there must be a degenerate curve $C(t^*)$ which is irreducible with $m+1$ nodes. In such a way one degenerates the original curve C into a curve $C^{(9)}$ with 9 nodes. The results of Section 2.6 leave only one possible shape for $C^{(9)}$ which is shown in Figure 32. Now we consider the family $C^{(9)} + tC_3^2$, $t \ge 0$ where C_3 is a cubic passing through p and all the nodes of $C^{(9)}$ but the solitary point. The solitary point disappears when $t > 0$, the domain Δ grows, hence one cannot come back to $C^{(9)}$ without degeneration. The only degeneration allowed by Lemma 3.8 is an irreducible curve with 9 nodes. This node cannot be a solitary point outside Δ, since such a curve would deform in a forbidden nonsingular sextic with 11 ovals outside each other. Also there cannot appear a pseudo-oval on the boundary of Δ as shown in Figure 29, since such a curve would deform into a sextic with forbidden complex orientation (see Figure 33). At last, the appearance of a node joining pseudo-ovals of the curve in Figure 32 is forbidden by the results of Section 2.6. So, we come to contradiction.

4. Constructions

4.1. Constructions of the algebraic curves.

Let us prove Part 2 of Theorem 1. The two arrangements in the right hand side of Figure 2 are realized in [22; Sect. 4.7]. The construction of the two curves in the left hand side of Figure 2 is shown in Figure 34. We use Viro's method cutting the triangle $(0,0)$-$(8,0)$-$(0,4)$ along the segments $(0,3)$-$(6,1)$ and $(0,3)$-$(6,0)$.

Note that the curves in the right hand side of Figure 2 can be also constructed as in Figure 34 if one chooses another chart in the quadrangle $(0,3)$-$(6,1)$-$(8,0)$-$(6,0)$.

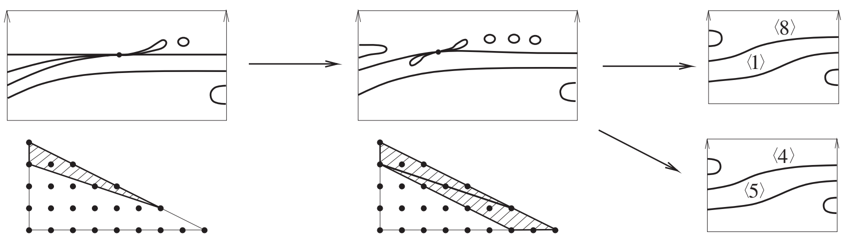


FIGURE 34

4.2. Constructions of the pseudo-holomorphic curve. It is easy to check (see [4; Sect. 4.1] for details) that an arrangement of a curve with respect to a pencil of lines is pseudo-holomorphically realizable iff the braid described in Sect. 1.2 is quasipositive. Thus, to realize Figure 3, it suffices to check that

$$\sigma_3^{-5}\sigma_2^{-1}\sigma_3\sigma_2^{-2}\sigma_1^{-1}\sigma_2\sigma_1^{-4}\Delta^2 = (\alpha_1^{-1}\sigma_3\alpha_1)(\alpha_2^{-1}\sigma_1\alpha_2)$$

where $\alpha_2 = \sigma_3\sigma_2\sigma_3^4$ and $\alpha_1 = \sigma_2\sigma_1^{-2}\sigma_3^{-2}\sigma_2\alpha_2$ (as in Sect. 1.2, we denote $\Delta = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$).

The curve in Figure 5(a) is obtained from that in Figure 3 by removing an oval. Hence, the corresponding braid being obtained from a quasipositive braid by removing a negative generator, is also quasipositive.

All the curves in Figures 4(a,b) and 5(a,b) define the braid conjugate to each other (the arrows in Figure 35 indicate how to move a twist from one group to the other one).

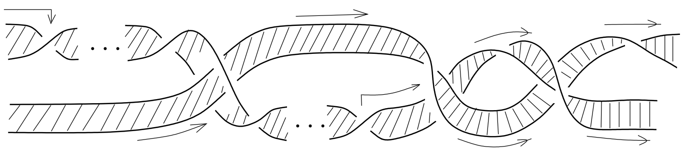


FIGURE 35

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Laboratoire E. Picard, UFR MIG, Université Paul Sabatier, 118 route de Narbonne, 31062, Toulouse, France
E-mail address: orevkov@picard.ups-tlse.fr

School of Mathematical Sciences, Tel Aviv University, Ramat Aviv, 69978 Tel Aviv, Israel
E-mail address: shustin@math.tau.ac.il